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FINITE ELEMENT VIBRATION ANALYSIS OF ROTATING TIMOSHENKO BEAMS

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The stiffness and mass matrices of a rotating twisted and tapered beam element are derived. The angle of twist, breadth and depth are assumed to vary linearly along the length of beam. The effects of shear deformation and rotary inertia are also considered in deriving the elemental matrices. The first four natural frequencies and mode shapes in bending–bending mode are calculated for cantilever beams. The effects of twist, offset, speed of rotation and variation of depth and breadth taper ratios are studied.

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1. INTRODUCTION

Generally, turbomachine blades are idealized as tapered cantilever beams. In order to refine the analysis the effects of pre-twist and rotation are also included. As the blades are short in some of the designs and may vibrate in higher-frequency ranges, the effects of shear deformation and rotary inertia may be of considerable magnitude. Various investigators in the field of turbine-blade vibrations have solved the differential equations of motion, by taking into account one or more of the above-mentioned effects.

The analysis of tapered beams has been made by many investigators using different techniques. Rao [1] used the Galerkin method, Housner and Keightley [2] applied the Myklestad procedure, Rao and Carnegie [3] used the finite differences approach, Martin [4] adopted a perturbation technique and Mabie and Rogers [5] solved the differential equations using Bessel functions to find the natural frequencies of vibration of tapered cantilever beams.

In analysing the pre-twisted beams, various approaches have been used by various investigators. Mendelson and Gendler [6] used the station functions, Rosard [7] applied the Myklestad method, Di Prima and Handleman [8] adopted the Rayleigh–Ritz procedure, Carnegie and Thomas [9] used the finite difference techniques and Rao [10] considered the Galerkin method for the analysis of pre-twisted beams. The bending vibration of a twisted rotating beam was considered by Targoff [11].

The rotation effect in beam analysis has also been considered using different formulation and solution procedures. Targoff applied the matrix method, Yntema [12] applied the Rayleigh–Ritz procedure, Isakson and Eisley [13] used the Rayleigh–Southwell procedure, Banerjee and Rao [14] applied the Galerkin procedure, Carnegie *et al.* [15] applied the finite difference scheme, Krupka and Baumanis [16] used the Myklestad method and Rao

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and Carnegie [17] used the extended Holzer method to find the vibration characteristics of rotating cantilever beams.

The finite element technique has also been applied by many investigators, mostly for the vibration analysis of beams of uniform cross-section. All these investigations differ from one another in the nodal degrees of freedom taken for deriving the elemental stiffness and mass matrices. McCalley [18] derived the consistent mass and stiffness matrices by selecting the total deflection and total slope as nodal co-ordinates. Kapur [19] took bending deflection, shear deflection, bending slope and shear slope as nodal degrees of freedom and derived the elemental matrices of beams with linearly varying inertia. Carnegie *et al.* [20] analyzed uniform beams by taking few internal nodes in it. Nickel and Sector [21] used total deflection, total slope and bending slope of the two nodes and the bending slope at the mid-point of the beam as the degrees of freedom to derive the elemental stiffness and the mass matrices of order seven. Thomas and Abbas [22] analyzed uniform Timoshenko beams by taking total deflection, total slope, bending slope and the derivative of the bending slope as nodal degrees of freedom.

The finite element analysis of vibrations of twisted blades based on beam theory was considered by Sisto and Chang [23]. Sabuncu and Thomas [24] studied the vibration characteristics of pre-twisted aerofoil cross-section blade packets under rotating conditions. An improved two-node Timoshenko beam finite element was derived by Friedman and Kosmatka [25]. The vibration of Timoshenko beams with discontinuities in cross-section was investigated by Farghaly and Gadelab [26, 27]. Corn *et al.* [28] derived finite element models through Guyan condensation method for the transverse vibration of short beams. Gupta and Rao [29] considered the finite element analysis of tapered and twisted Timoshenko beams.

In this work, the finite element technique is applied to find the natural frequencies and mode shapes of beams in the bending–bending mode of vibration by taking into account the taper, the pre-twist and the rotation simultaneously. The coupling that exists between the flexural and torsional vibration is not considered. The taper and the angle of twist are assumed to vary linearly along the length of the beam. The element stiffness and mass matrices are derived and the effects of offset, rotation, pre-twist, depth and breadth taper ratios and rotary inertia and shear deformation on the natural frequencies are studied. Various special cases of beam vibration can be obtained from the general equations derived.

2. ELEMENT STIFFNESS AND MASS MATRICES

2.1. DISPLACEMENT MODEL

Figure 1(a) shows a doubly tapered, twisted beam element of length l with the nodes as 1 and 2. The breadth, depth and the twist of the element are assumed to be linearly varying along its length. The breadth and depth at the two nodal points are shown as b_1, h_1 and b_2, h_2 respectively. The pre-twist at the two nodes is denoted by θ_1 and θ_2. Figure 1(b) shows the nodal degrees of freedom of the element where bending deflection, bending slope, shear deflections and shear slope in the two planes are taken as the nodal degrees of freedom. Figure 1(c) shows the angle of twist θ at any section z. The beam is assumed to rotate about the x–x-axis at a speed of Ω rad/s.

The total deflections of the element in the y and x directions at a distance z from node 1, $w(z)$ and $v(z)$, are taken as

$$w(z) = w_b(z) + w_s(z), \qquad v(z) = v_b(z) + v_s(z), \tag{1}$$

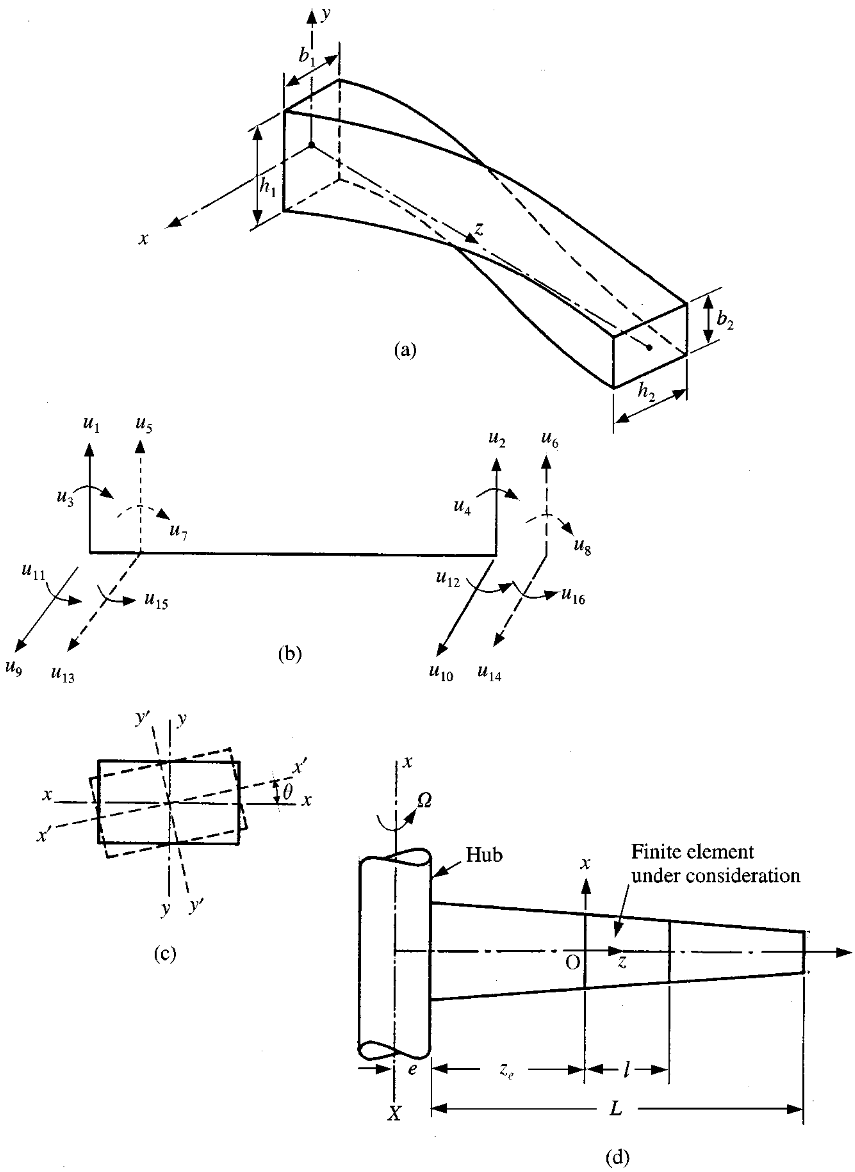

Figure 1. (a) An element of tapered and twisted beam, (b) degrees of freedom of an element, (c) angle of twist θ, (d) rotation of tapered beam.

where $w_b(z)$ and $v_b(z)$ are the deflections due to bending in the yz and xz planes respectively, and $w_s(z)$ and $v_s(z)$ are the deflections due to shear in the corresponding planes.

The displacement models for $w_b(z)$, $w_s(z)$, $v_b(z)$ and $v_s(z)$ are assumed to be polynomials of third degree. They are similar in nature except for the nodal constants. These expressions are given by

$$w_b(z) = \frac{u_1}{l^3}(2z^3 - 3lz^2 + l^3) + \frac{u_2}{l^3}(3lz^2 - 2z^3) - \frac{u_3}{l^2}(z^3 - 2lz^2 + l^2 z) - \frac{u_4}{l^2}(z^3 - lz^2),$$

$$w_s(z) = \frac{u_5}{l^3}(2z^3 - 3lz^2 - l^3) + \frac{u_6}{l^3}(3lz^2 - 2z^3) - \frac{u_7}{l^2}(z^3 - 2lz^2 + l^2 z) - \frac{u_8}{l^2}(z^3 - lz^2),$$

$$v_b(z) = \frac{u_9}{l^3}(2z^3 - 3lz^2 - l^3) + \frac{u_{10}}{l^3}(3lz^2 - 2z^3) - \frac{u_{11}}{l^2}(z^3 - 2lz^2 + l^2 z)$$
$$- \frac{u_{12}}{l^2}(z^3 - lz^2), \tag{2}$$

$$v_s(z) = \frac{u_{13}}{l^3}(2z^3 - 3lz^2 - l^3) + \frac{u_{14}}{l^3}(3lz^2 - 2z^3) - \frac{u_{15}}{l^2}(z^3 - 2lz^2 + l^2 z)$$
$$- \frac{u_{16}}{l^2}(z^3 - lz^2),$$

where u_1, u_2, u_3 and u_4 represent the bending degrees of freedom and u_5, u_6, u_7 and u_8 are the shear degrees of freedom in the yz plane; u_9, u_{10}, u_{11} and u_{12} represent the bending degrees of freedom and u_{13}, u_{14}, u_{15} and u_{16} shear degrees of freedom in the xz plane.

2.2. ELEMENT STIFFNESS MATRIX

The total strain energy U of a beam of length l, due to bending and shear deformation including rotary inertia and rotation effects is given by

$$U = \int_0^l \left[\left\{ \frac{EI_{xx}}{2} \left(\frac{\partial^2 w_b}{\partial z^2} \right)^2 + EI_{xy} \frac{\partial^2 w_b}{\partial z^2} \frac{\partial^2 v_b}{\partial z^2} + \frac{EI_{yy}}{2} \left(\frac{\partial^2 v_b}{\partial z^2} \right)^2 \right\} \right.$$
$$\left. + \frac{\mu AG}{2} \left\{ \left(\frac{\partial w_s}{\partial z} \right)^2 + \left(\frac{\partial v_s}{\partial z} \right)^2 \right\} \right] \mathrm{d}z + \frac{1}{2} \int_0^l P(z) \left(\frac{\partial w_b}{\partial z} + \frac{\partial w_s}{\partial z} \right)^2 \mathrm{d}z \tag{3}$$
$$+ \frac{1}{2} \int_0^l P(z) \left(\frac{\partial v_b}{\partial z} + \frac{\partial v_s}{\partial z} \right)^2 \mathrm{d}z - \int_0^l p_w(z)(w_b + w_s)\,\mathrm{d}z - \int_0^l p_v(z)(v_b + v_s)\,\mathrm{d}z.$$

where

$$P(z) = \int_{e+z_e+z}^{L+e} m\Omega^2 \xi\,\mathrm{d}\xi \approx \frac{\rho A \Omega^2}{2g}\left[(L+e)^2 - (e + z_e + z)^2\right]$$
$$= \frac{\rho A \Omega^2}{g}\left[\left(\mathrm{e}L + \frac{1}{2}L^2 - ez_e - \frac{1}{2}z_e^2\right) - (e + z_e)z - \frac{1}{2}z^2\right], \tag{4}$$

$$p_w(z) = \frac{\rho A \Omega^2}{g}(w_b + w_s), \qquad p_v(z) = \frac{\rho A \Omega^2}{g}(v_b + v_s), \tag{5, 6}$$

where e is the offset and z_e is the distance of the first node of the element from the root of the beam as shown in Figure 1(d), and $P(z)$ is the axial force acting at section z.

As the cross-section of the element changes with z and as the element is twisted, the cross-sectional area A, and the moments of inertia I_{xx}, I_{yy} and I_{xy} will be functions of z:

$$\begin{aligned} A(z) = b(z)h(z) &= \left\{b_1 + (b_2 - b_1)\frac{z}{l}\right\}\left\{h_1 + (h_2 - h_1)\frac{z}{l}\right\} \\ &= \frac{1}{l^2}(c_1 z^2 + c_2 lz + c_3 l^2), \end{aligned} \tag{7}$$

where

$$c_1 = (b_2 - b_1)(h_2 - h_1), \quad c_2 = b_1(h_2 - h_1) + h_1(b_2 - b_1), \quad c_3 = b_1 h_1, \tag{8}$$

$$\begin{aligned} &I_{xx}(z) = I_{x'x'}\cos^2\theta + I_{y'y'}\sin^2\theta, \qquad I_{yy}(z) = I_{y'y'}\cos^2\theta + I_{x'x'}\sin^2\theta, \\ &I_{xy}(z) = (I_{x'x'} - I_{y'y'})\frac{\sin 2\theta}{2}, \end{aligned} \tag{9}$$

where $x'x'$ and $y'y'$ are the axes inclined at an angle θ, the angle of twist, at any point in the element, to the original axes xx and yy as shown in Figure 1(c). The value of $I_{x'y'} = 0$ and the values of $I_{x'x'}$ and $I_{y'y'}$ can be computed as

$$I_{x'x'}(z) = \frac{b(z)h^3(z)}{12} = \frac{1}{12l^4}[a_1 z^4 + a_2 lz^3 + a_3 l^2 z^2 + a_4 l^3 z + a_5 l^4], \tag{10}$$

where

$$\begin{aligned} &a_1 = (b_2 - b_1)(h_2 - h_1)^3, \quad a_2 = b_1(h_2 - h_1)^3 + 3(b_2 - b_1)(h_2 - h_1)^2 h_1, \\ &a_3 = 3\{b_1 h_1 (h_2 - h_1)^3 + (b_2 - b_1)(h_2 - h_1)h_1^2, \\ &a_4 = 3b_1 h_1^2 (h_2 - h_1) + (b_2 - b_1)h_1^3, \quad a_5 = b_1 h_1^3, \end{aligned} \tag{11}$$

$$I_{y'y'}(z) = \frac{h(z)b^3(z)}{12} = \frac{1}{12l^4}[d_1 z^4 + d_2 lz^3 + d_3 l^2 z^2 + d_4 l^3 z + d_5 l^4], \tag{12}$$

where

$$\begin{aligned} &d_1 = (h_2 - h_1)(b_2 - b_1)^3, \quad d_2 = h_1(b_2 - b_1)^3 + 3(h_2 - h_1)(b_2 - b_1)^2 b_1, \\ &d_3 = 3\{h_1 b_1 (b_2 - b_1)^3 + (h_2 - h_1)(b_2 - b_1)b_1^2, \\ &d_4 = 3h_1 b_1^2 (b_2 - b_1) + (h_2 - h_1)b_1^3, \quad d_5 = h_1 b_1^3. \end{aligned} \tag{13}$$

By substituting the expressions of w_b, w_s, v_b, v_s, A, I_{xx}, I_{xy} and I_{yy} from equations (2), (7) and (9) into equation (3), the strain energy U can be expressed as

$$U = \tfrac{1}{2}\mathbf{u}^{\mathrm{T}}[K]\mathbf{u}, \tag{14}$$

where **u** is the vector of nodal displacements $u_1, u_2, \ldots, u_{16}$, and $[K]$ is the elemental stiffness matrix of order 16. Denoting the integrals

$$\int_0^l EI_{xx}\left(\frac{\partial^2 w_b}{\partial z^2}\right)^2 \mathrm{d}z = [u_1 \;\; u_2 \;\; u_3 \;\; u_4]^{\mathrm{T}}[AK][u_1 \;\; u_2 \;\; u_3 \;\; u_4], \tag{15}$$

$$\int_0^l EI_{yy}\left(\frac{\partial^2 v_b}{\partial z^2}\right)^2 \mathrm{d}z = [u_9 \;\; u_{10} \;\; u_{11} \;\; u_{12}]^{\mathrm{T}}[BK][u_9 \;\; u_{10} \;\; u_{11} \;\; u_{12}], \tag{16}$$

$$\int_0^l \mu AG\left(\frac{\partial w_s}{\partial z}\right)^2 \mathrm{d}z = [u_5 \;\; u_6 \;\; u_7 \;\; u_8]^{\mathrm{T}}[CK][u_5 \;\; u_6 \;\; u_7 \;\; u_8], \tag{17}$$

$$\int_0^l EI_{xy}\left(\frac{\partial^2 w_b}{\partial z^2}\right)\left(\frac{\partial^2 v_b}{\partial z^2}\right) \mathrm{d}z = [u_1 \;\; u_2 \;\; u_3 \;\; u_4]^{\mathrm{T}}[DK][u_9 \;\; u_{10} \;\; u_{11} \;\; u_{12}], \tag{18}$$

$$\int_0^l P(z)\left(\frac{\partial w_b}{\partial z}\right)^2 \mathrm{d}z = [u_1 \;\; u_2 \;\; u_3 \;\; u_4]^{\mathrm{T}}[EK][u_1 \;\; u_2 \;\; u_3 \;\; u_4] \tag{19}$$

and

$$\int_0^l \frac{2\rho A\Omega^2}{g}(w_b^2)\,\mathrm{d}z = [u_1 \;\; u_2 \;\; u_3 \;\; u_4]^{\mathrm{T}}[FK][u_1 \;\; u_2 \;\; u_3 \;\; u_4], \tag{20}$$

the element stiffness matrix can be expressed as

$$[K] =$$

$$\begin{bmatrix} [AK]+[EK]-[FK] & [EK]-[FK] & [DK] & [0] \\ [EK]-[FK] & [CK]+[EK]-[FK] & [0] & [0] \\ [DK] & [0] & [BK]+[EK]-[FK] & [EK]-[FK] \\ [0] & [0] & [EK]-[FK] & [CK]+[EK]-[FK] \end{bmatrix}, \tag{21}$$

where $[AK]$, $[BK]$, $[CK]$, $[DK]$, $\{EK]$ and $[FK]$ are symmetric matrices of order 4 and their elements are formulated in Appendix B. [0] is a null matrix of order 4.

2.3. ELEMENT MASS MATRIX

The kinetic energy of the element T including the effects of shear deformation and rotary inertia is given by

$$\begin{aligned} T = \int_0^l \Bigg[& \frac{\rho A}{2g}\left(\frac{\partial w_b}{\partial t} + \frac{\partial w_s}{\partial t}\right)^2 + \frac{\rho A}{2g}\left(\frac{\partial v_b}{\partial t} + \frac{\partial v_s}{\partial t}\right)^2 + \frac{\rho I_{yy}}{2g}\left(\frac{\partial^2 v_b}{\partial z\,\partial t}\right) \\ & + \frac{\rho I_{xy}}{g}\left(\frac{\partial^2 w_b}{\partial z\,\partial t}\right)\left(\frac{\partial^2 v_b}{\partial z\,\partial t}\right) + \frac{\rho I_{xx}}{2g}\left(\frac{\partial^2 w_b}{\partial z\,\partial t}\right)^2 \Bigg] \mathrm{d}z. \end{aligned} \tag{22}$$

By defining

$$\int_0^l \frac{\rho A}{g}\left(\frac{\partial w_b}{\partial t}\right)^2 \mathrm{d}z = [\dot{u}_1\ \ \dot{u}_2\ \ \dot{u}_3\ \ \dot{u}_4]^{\mathrm{T}}[AM][\dot{u}_1\ \ \dot{u}_2\ \ \dot{u}_3\ \ \dot{u}_4], \tag{23}$$

$$\int_0^l \frac{\rho I_{xx}}{g}\left(\frac{\partial^2 w_b}{\partial z\,\partial t}\right)^2 \mathrm{d}z = [\dot{u}_1\ \ \dot{u}_2\ \ \dot{u}_3\ \ \dot{u}_4]^{\mathrm{T}}[BM][\dot{u}_1\ \ \dot{u}_2\ \ \dot{u}_3\ \ \dot{u}_4], \tag{24}$$

$$\int_0^l \frac{\rho I_{yy}}{g}\left(\frac{\partial^2 v_b}{\partial z\,\partial t}\right)^2 \mathrm{d}z = [\dot{u}_9\ \ \dot{u}_{10}\ \ \dot{u}_{11}\ \ \dot{u}_{12}]^{\mathrm{T}}[CM][\dot{u}_9\ \ \dot{u}_{10}\ \ \dot{u}_{11}\ \ \dot{u}_{12}], \tag{25}$$

and

$$\int_0^l \frac{\rho I_{xy}}{g}\left(\frac{\partial^2 w_b}{\partial z\,\partial t}\right)\left(\frac{\partial^2 v_b}{\partial z\,\partial t}\right) \mathrm{d}z = [\dot{u}_1\ \ \dot{u}_2\ \ \dot{u}_3\ \ \dot{u}_4]^{\mathrm{T}}[DM][\dot{u}_9\ \ \dot{u}_{10}\ \ \dot{u}_{11}\ \ \dot{u}_{12}], \tag{26}$$

where $\dot{u}_l$ denotes the time derivative of the nodal displacement u_i, $i = 1, 2, \ldots, 16$, the kinetic energy of the element can be expressed as

$$T = \tfrac{1}{2}\dot{\mathbf{u}}[M]\dot{\mathbf{u}}, \tag{27}$$

where $[M]$ is the mass matrix given by

$$\underset{16\times 16}{[M]} = \begin{bmatrix} [AM]+[BM] & [AM] & [DM] & [0] \\ [AM] & [AM] & [AM] & [0] \\ [DM] & [AM] & [AM]+[CM] & [AM] \\ [0] & [0] & [AM] & [AM] \end{bmatrix} \tag{28}$$

and $[AM]$, $[BM]$ $[CM]$ and $[DM]$ are symmetric matrices of order 4 whose elements are defined in Appendix A.

2.4. BOUNDARY CONDITIONS

The following boundary conditions are to be applied depending on the type of end conditions.

$$\text{Free end}\quad \frac{\partial w_s}{\partial z} = 0,\quad \frac{\partial v_s}{\partial z} = 0,\quad \frac{\partial^2 v_b}{\partial z^2} = 0,\quad \frac{\partial^2 w_b}{\partial z^2} = 0. \tag{29}$$

$$\text{Clamped end}\quad w_s = 0,\quad w_b = 0,\quad v_s = 0,\quad v_b = 0,\quad \frac{\partial w_b}{\partial z} = 0,\quad \frac{\partial v_b}{\partial z} = 0. \tag{30}$$

$$\text{Hinged end}\quad w_s = 0,\quad w_b = 0,\quad v_s = 0,\quad v_b = 0.\quad \frac{\partial^2 v_b}{\partial z^2} = 0,\quad \frac{\partial^2 w_b}{\partial z^2} = 0. \tag{31}$$

TABLE 1

Natural frequencies (Hz)

No of elements	First mode	Second mode	Third mode	Fourth mode
1	304.3	1191·66	2327·07	4552·23
2	296·22	1161·70	1779·50	4106·83
3	295·03	1155·97	1746·55	3747·04
4	294·85	1154·94	1741·39	3697·05
5	294·78	1154·67	1739·99	3689·82
6	294·78	1154·58	1739·40	3683·98
7	294·78	1154·54	1739·25	3683·91
8	294·78	1154·50	1739·10	3683·85

Data: length of beam = 0·1524 m, breadth at root = 0·0254 m, depth at root = 0·0046 m, depth taper ratio = 2·29, breadth taper ratio = 2·56, twist angle = 45°, shear coefficient = 0·833, $E = 2{\cdot}07 \times 10^{11}$ N/m^2, $G = E/2{\cdot}6$, offset = 0, mass density = 800 kg/m^3.

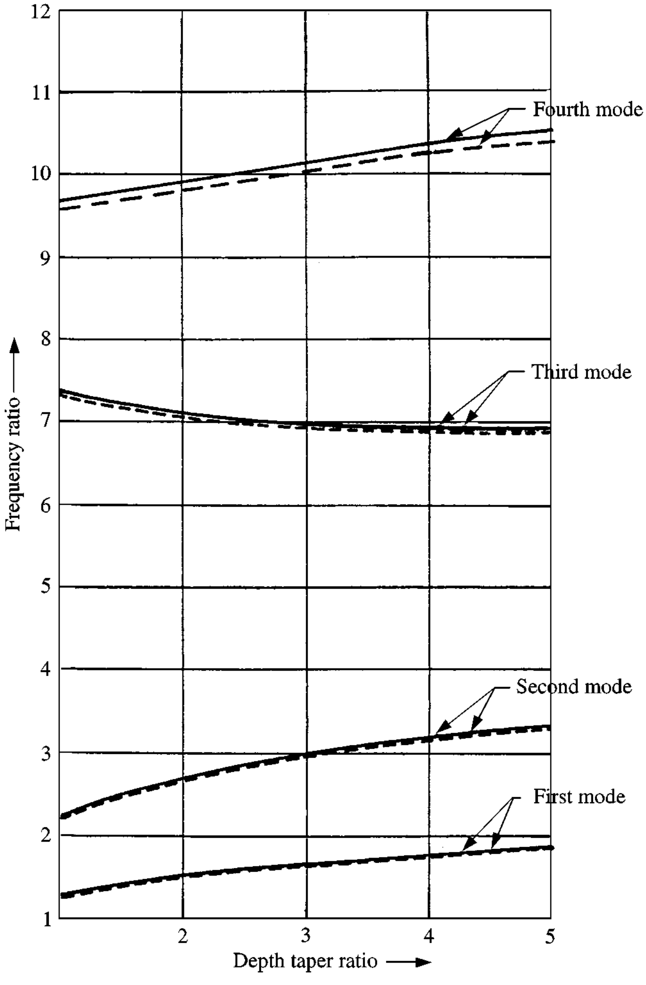

Figure 2. Effect of depth taper ratio and shear deformation on frequency ratio of rotating beam: ——, without shear deformation; – – – –, with shear deformation.

3. SPECIAL CASES

The various special cases of the beam vibration problem can be solved by applying one or more of the following four conditions:

(a) For non-rotating beams: $\Omega = 0$ which results in $[EK] = [FK] = [0]$. (32)

(b) For uniform beams: by setting $b_2 = b_1$ and $h_2 = h_1$, one obtains

$$c_1 = c_2 = 0 \text{ and } c_3 = b_1 h_1,\ a_1 = a_2 = a_3 = a_4 = 0 \quad \text{and} \quad a_5 = b_1 h_1^3, \tag{33}$$

$$d_1 = d_2 = d_3 = d_4 = 0 \quad \text{and} \quad d_5 = h_1 b_1^3.$$

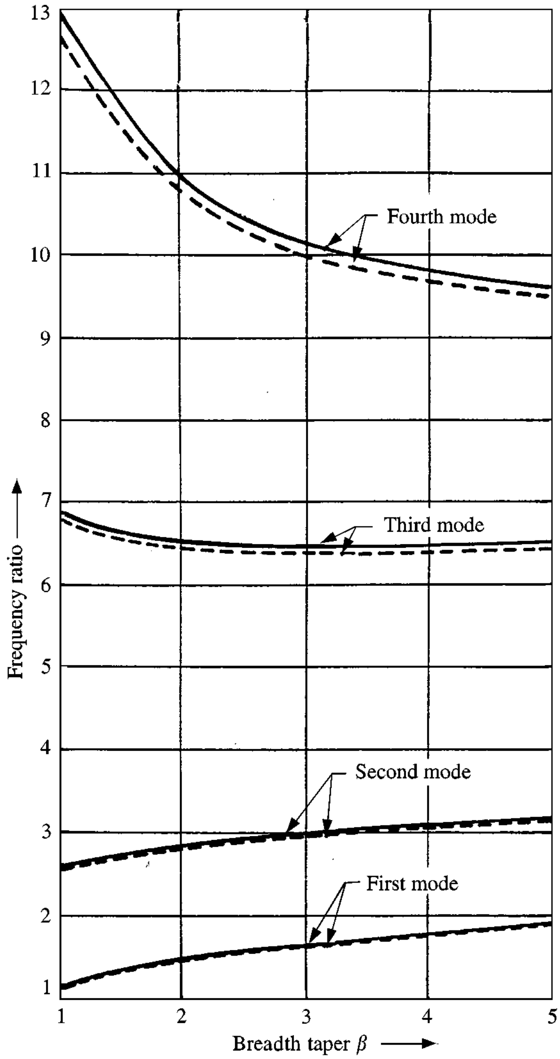

Figure 3. Effect of breadth taper ratio and shear deformation on frequency ratio of rotating beam: ——, with shear deformation; – – – –, without shear deformation.

(c) For neglecting the effect of shear deformation: $w_s = v_s = 0$ so that equations (1) and (2) become

$$w(z) = w_b(z) \quad \text{and} \quad v(z) = v_b(z). \tag{34}$$

Due to this, the order of $[K]$ and $[M]$ matrices reduces from 16 to 8.

(d) For beams without pre-twist: in this case, there will be no coupling between the moment of inertia terms and one obtains

$$I_{xx} = I_{x'x'}, \quad I_{yy} = I_{y'y'}, \quad I_{xy} = 0. \tag{35}$$

For vibration in the yz plane, $v_b = v_s = 0$; for vibration in the xz plane, $w_b = w_s = 0$. This condition further reduces the size of matrices $[K]$ and $[M]$ by half. Thus, for the general case (with shear deformation) the matrices will be of order 8 and, if coupled with condition (c) these will be of order 4.

If the conditions (a), (b) and (d) are applied one gets the following expressions for $[K]$ and $[M]$ for non-rotating uniform beams without pre-twist but with a consideration of rotary inertia and shear deformation effects (for vibration in the y–z plane):

$$[K] = \frac{EI_{xx}}{l^3}\begin{bmatrix} 12 & -12 & -6l & -6l & 0 & 0 & 0 & 0 \\ & 12 & 6l & 6l & 0 & 0 & 0 & 0 \\ & & 4l^2 & 2l^2 & 0 & 0 & 0 & 0 \\ & & & 4l^2 & 0 & 0 & 0 & 0 \\ & & & & 36J & -36J & -3lJ & -3lJ \\ & \mathit{Symmetric} & & & & 36J & 3lJ & 3lJ \\ & & & & & & 4l^2J & -1^2J \\ & & & & & & & 4l^2J \end{bmatrix}, \tag{36}$$

$$[M] = \frac{\rho AI}{420g}$$

$$\begin{bmatrix} 156+36\bar{P} & 54-36\bar{P} & 22l-3l\bar{P} & 13l-3l\bar{P} & 156 & 54 & 22l & 13l \\ & 156+36\bar{P} & -13l+3l\bar{P} & 22l+3l\bar{P} & 54 & 156 & -13l & 22l \\ & & 4l^2+4l^2\bar{P} & -3l^2-l^2\bar{P} & 22l & -13l & 4l^2 & -3l^2 \\ & & & 4l^2-4l^2\bar{P} & 13l & 22l & -3l^2 & 4l^2 \\ & & & & 156 & 54 & 22l & 13l \\ & & & & & 156 & -13l & 22l \\ & & & & & & 4l^2 & -3l^2 \\ & & & & & & & 4l^2 \end{bmatrix}, \tag{37}$$

where $J = \mu G b_1 h_1 l^2/(30EI_{xx})$, and $\bar{P} = 14I_{xx}/(b_1 h_1 l^2)$, Equations (36) and (37) further reduce to the following well-known equations if the effects of shear deformation and rotary inertia are neglected:

$$[K] = \frac{EI_{xx}}{l^3}\begin{bmatrix} 12 & -12 & -6l & -6l \\ & 12 & 6l & 6l \\ & Symmetric & 4l^2 & 2l^2 \\ & & & 4l^2 \end{bmatrix}, \tag{38}$$

$$[M] = \frac{\rho b_1 h_1}{420g}\begin{bmatrix} 156 & 54 & -22l & 13l \\ & 156 & -13l & 22l \\ & Symmetric & 4l^2 & -3l \\ & & & 4l^2 \end{bmatrix}. \tag{39}$$

4. NUMERICAL RESULTS

The element stiffness and mass matrices developed are used for the dynamic analysis of cantilever beams. By using the standard procedures of structural analysis, the eigenvalue

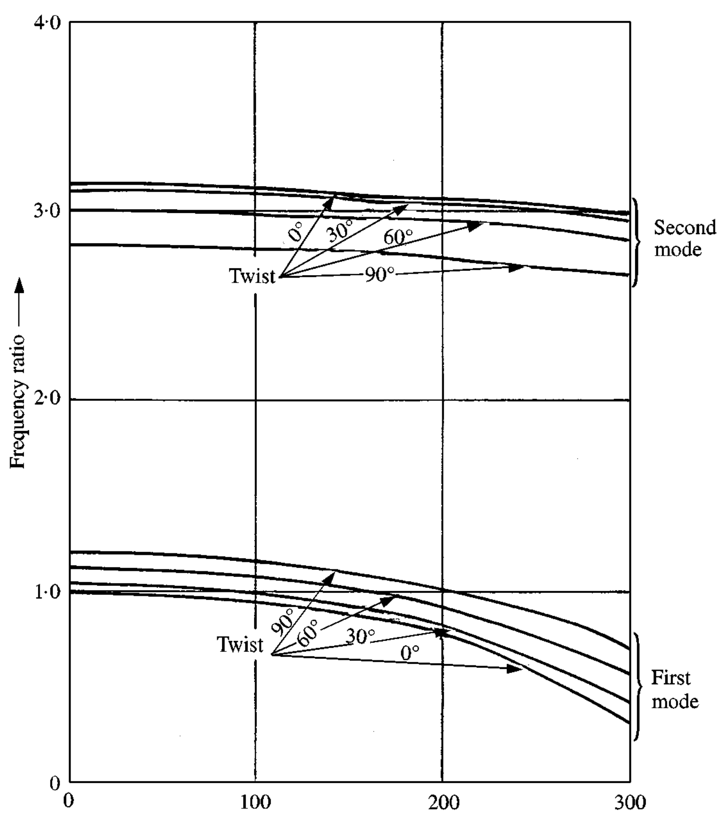

Figure 4. Effect of rotation and twist on first and second natural frequencies.

problem can be stated as

$$([\underset{\sim}{\mathbf{K}}] - \omega^2[\underset{\sim}{M}])\underset{\sim}{\mathbf{U}} = \mathbf{0}, \tag{40}$$

where $[\underset{\sim}{\mathbf{K}}]$ and $[\underset{\sim}{M}]$ denote the stiffness and mass matrices of the structure, respectively, $\underset{\sim}{\mathbf{U}}$ indicates nodal displacement vector of the structure, and ω is the natural frequency of vibration.

A study of the convergence properties of the general element developed has been made and the results are given in Table 1. It is seen that the results converge well even with only four elements. The effects of shear deformation and depth taper ratio on the natural frequencies of a rotating twisted beam are shown in Figure 2 for a beam of length 0·254 m, offset zero, depth at root 0·00865 m, breadth at root 0·0173 m, twist 45°, rotation 100 r.p.s.

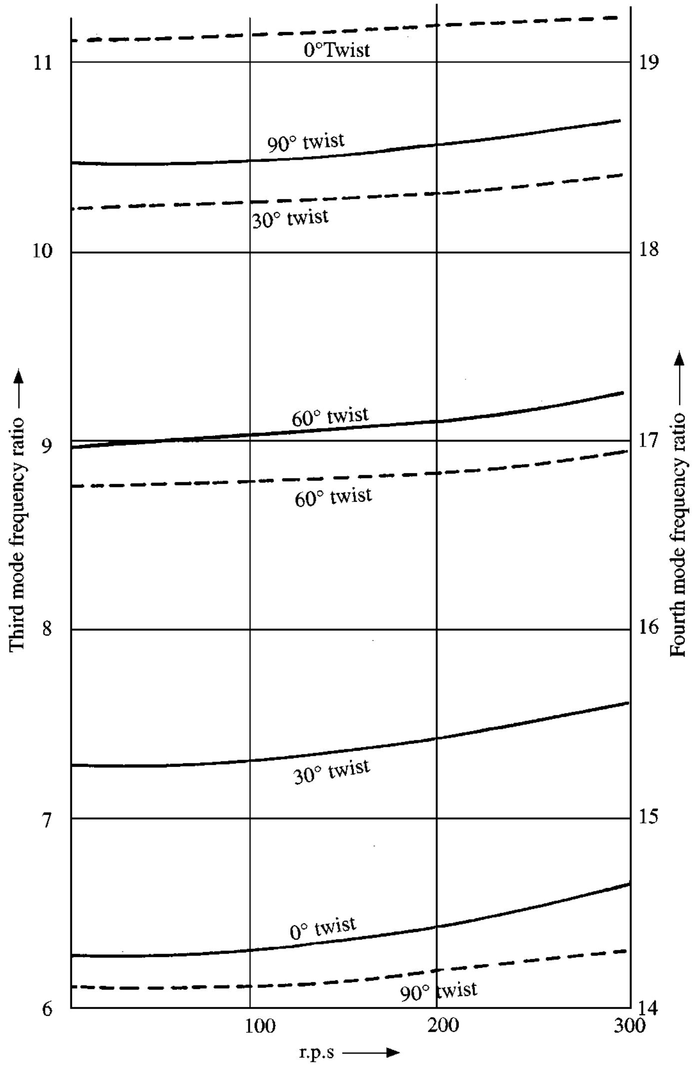

Figure 5. Effect of rotation and twist on third and fourth natural frequencies: ——, third mode; – – –, fourth mode.

and breadth taper ratio 3. The material properties of the beam are taken the same as those given in Table 1. The effect of shear deformation is found to reduce the frequencies at higher modes while at lower modes the results are nearly unaffected. There is an increase in the frequencies of vibration with an increase in the depth taper ratio in the first, second and fourth modes while a decrease has been observed in the case of third mode (vibration in a perpendicular plane). Figure 3 shows the variation of natural frequencies with breadth taper ratio. In this case the data are the same as in the case of Figure 2.

In Figures 4 and 5, the variation of frequency ratio with rotation and pre-twist is studied. It can be seen that the frequency ratio changes slightly with the rotation but appreciably with the twist. At higher modes (in Figure 5) the effect of twist can be seen to be more pronounced. It is also observed that the frequency ratio increases with an increase in the twist in the case of first and third modes while it decreases with an increase of the twist in the case of second and fourth modes of vibration. In Figure 6, the effect of offset is studied for a twisted blade having 60° twist with the other data the same as that of Figure 4. It is observed that an increase in offset changes the frequency ratio more at higher values of rotation. The frequency ratio has been found to increase with an increase in the offset.

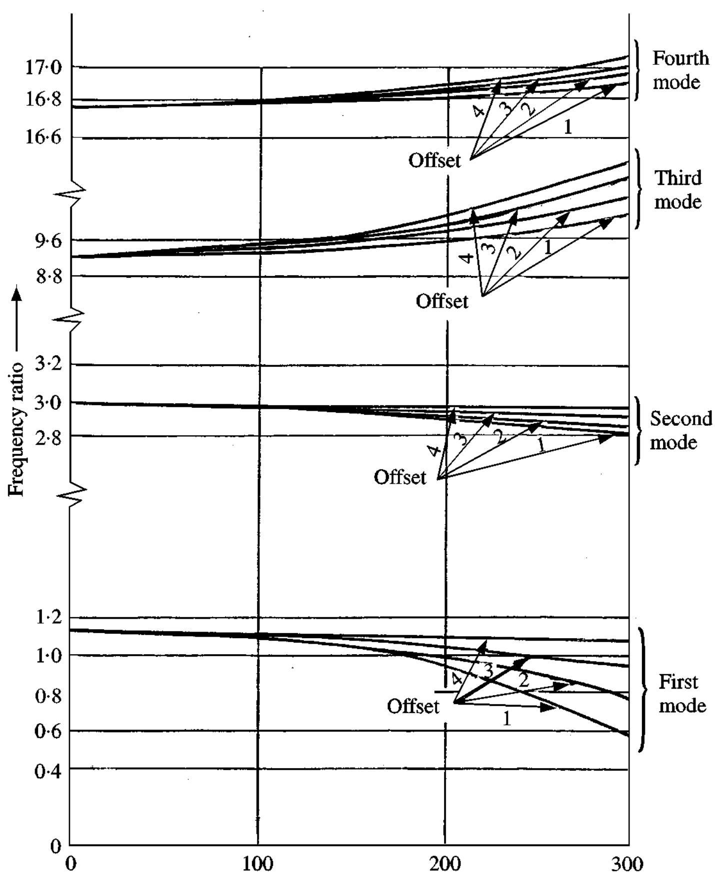

Figure 6. Effect of offset and rotation on frequency ratio. Offset 1: $e = 0$ m; offset 2: $e = 0.0254$ m; offset 3: $e = 0.0508$ m; offset 4: $e = 0.0762$ m.

5. CONCLUSION

The mass and stiffness matrices of a thick rotating beam element with taper and twist are developed for the eigenvalue analysis of rotating, doubly tapered and twisted Timoshenko beams. The element has been found to give reasonably accurate results even with four finite elements. The effects of breadth and depth taper ratios, twist angle, shear deformation, offset and rotation on the natural frequencies of vibration of cantilever beams are found. The consideration of shear deformation is found to reduce the values of the higher natural frequencies of vibration of the beam. An increase in the breadth and depth taper ratios is found to increase the first two modes of vibration. The frequency ratio is found to change only slightly with rotation but appreciably with the twist of the beam.

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APPENDIX A: EXPRESSIONS FOR STRAIN AND KINETIC ENERGIES

A.1. EXPRESSION FOR STRAIN ENERGY (U)

A.1.1. Strain energy due to bending

If the bending deflections in yz and xz planes of a beam are w_b and v_b, respectively, the axial strain and stress induced due to w_b and v_b are given by ε_{xx} due to $w_b = -y(\partial^2 w_b/\partial z^2)$; ε_{xx} due to $v_b = -x\,\partial^2 v_b/\partial z^2$ and σ_{xx} due to $w_b = -Ey\,\partial^2 w_b/\partial z^2$; σ_{xx} due to $v_b = -Ex\,\partial^2 v_b/\partial z^2$.

The strain energy stored in the beam due to bending is given by: U due to bending

$$= \frac{1}{2}\int_V \varepsilon_{xx}\sigma_{xx}\,\mathrm{d}V$$

$$= \frac{1}{2}\int_V (\varepsilon_{xx} \text{ due to } w_b + \varepsilon_{xx} \text{ due to } v_b)(\sigma_{xx} \text{ due to } w_b + \sigma_{xx} \text{ due to } v_b)\,\mathrm{d}V$$

$$= \frac{E}{2}\int_0^l \mathrm{d}z \int_A \left(y\frac{\partial^2 w_b}{\partial z^2} + x\frac{\partial^2 v_b}{\partial z^2}\right)^2 \mathrm{d}A$$

$$= \frac{E}{2}\int_0^l \mathrm{d}z\left[\left(\frac{\partial^2 w_b}{\partial z^2}\right)^2 \int_A y^2\,\mathrm{d}A + \left(\frac{\partial^2 v_b}{\partial z^2}\right)^2 \int_A x^2\,\mathrm{d}A + 2\frac{\partial^2 w_b}{\partial z^2}\frac{\partial^2 v_b}{\partial z^2}\int_A xy\,\mathrm{d}A\right]$$

$$= \int_0^l \left[\frac{EI_{xx}}{2}\left(\frac{\partial^2 w_b}{\partial z^2}\right)^2 + EI_{xy}\frac{\partial^2 w_b}{\partial z^2}\frac{\partial^2 v_b}{\partial z^2} + \frac{EI_{yy}}{2}\left(\frac{\partial^2 v_b}{\partial z^2}\right)^2\right]\mathrm{d}z, \tag{A.1}$$

where V is the volume, l is the length and A is the cross-sectional area of the beam.

A.1.2. Strain energy due to shearing

Let F_x and F_y be the shear forces that produce the shear deflections $\mathrm{d}v_s$ and $\mathrm{d}w_s$ in an element of length $\mathrm{d}z$ respectively. Then the strain energy of the beam due to shearing is given by

$$U_{due\ to\ shearing} = \frac{1}{2}\int_0^l \left(F_x \frac{\mathrm{d}v_s}{\mathrm{d}z} + F_y \frac{\mathrm{d}w_s}{\mathrm{d}z}\right)\mathrm{d}z.$$

By substituting $AG\mu\ \mathrm{d}v_s/\mathrm{d}z$ and $AG\mu\ \mathrm{d}w_s/\mathrm{d}z$ for F_x and F_y, respectively, one obtains

$$U_{due\ to\ shearing} = \int_0^l \frac{\mu AG}{2}\left[\left(\frac{\mathrm{d}w_s}{\mathrm{d}z}\right)^2 + \left(\frac{\mathrm{d}v_s}{\mathrm{d}z}\right)^2\right]\mathrm{d}z. \qquad \text{(A.2)}$$

A.1.3. Strain energy due to rotation

The rotation of a beam induces an axial force P in the beam due to centrifugal action. If the beam is bending in the yz plane (Figure A1), the change in the horizontal projection of an element of length $\mathrm{d}s$ is given by

$$\mathrm{d}s - \mathrm{d}z = \left\{(\mathrm{d}z)^2 + \left(\frac{\partial w}{\partial z}\mathrm{d}z\right)^2\right\}^{1/2} - \mathrm{d}z \approx \frac{1}{2}\left(\frac{\partial w}{\partial z}\right)^2 \mathrm{d}z.$$

Since the axial force P acts against the changes in the horizontal projection, the work done by P is given by

$$U_{due\ to\ P\ and\ w} = -\frac{1}{2}\int_0^l P(z)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{d}z. \qquad \text{(A.3)}$$

The work done by the transverse distributed force $p_w(z)$ can be written as

$$U_{due\ to\ p_w} = \int_0^l p_w(z)\, w\, \mathrm{d}z. \qquad \text{(A.4)}$$

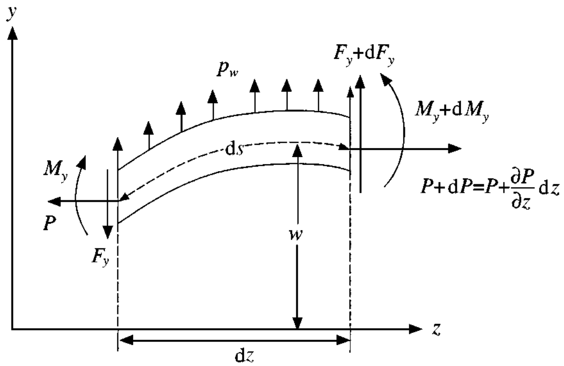

Figure A.1. An element of the beam in equilibrium.

The expressions corresponding to the bending of the beam in the xz plane can be obtained similarly as

$$U_{due\ to\ P\ and\ v} = \frac{1}{2}\int_0^l P(z)\left(\frac{\mathrm{d}v}{\mathrm{d}z}\right)^2 \mathrm{d}z, \qquad U_{due\ to\ p_v} = \int_0^l p_v(z)\, v\, \mathrm{d}z. \qquad (A.5, 6)$$

The total strain energy of the beam can be obtained as given in equation (3) by combining equations (A.1)–(A.6).

A.2. EXPRESSION FOR KINETIC ENERGY (T)

Consider a small element of area dA and length dz at a point in the cross-section having co-ordinates (x, y) with respect to x- and y-axes. The kinetic energy of this element is given by

$$\frac{\rho}{2g}\left[(\dot{w}^2 + \dot{v}^2) + (y\dot{\phi}_x + x\dot{\phi}_y)^2\right] \mathrm{d}z\, \mathrm{d}A,$$

where ϕ_x and ϕ_y denote the bending slopes, $\partial v_b/\partial z$ and $\partial w_b/\partial z$, respectively, and a dot over a symbol represents derivative with respect to time. Integrating this equation over the beam cross-section, the kinetic energy of an element of length dz can be obtained as

$$\mathrm{d}T = \frac{\rho}{2g}\left[A(\dot{w}^2 + \dot{v}^2) + (I_{xx}\dot{\phi}_y^2 + 2I_{xy}\dot{\phi}_x\dot{\phi}_y + I_{yy}\dot{\phi}_x^2)\right] \mathrm{d}z.$$

The kinetic energy of the entire beam (T) can be expressed as

$$T = \int_0^l \left[\frac{\rho A}{2g}(\dot{w}^2 + \dot{v}^2) + \frac{\rho}{2g}(I_{xx}\dot{\phi}_y^2 + 2I_{xy}\dot{\phi}_x\dot{\phi}_y + I_{yy}\dot{\phi}_x^2)\right] \mathrm{d}z, \qquad (A.7)$$

which can be seen to be the same as in equation (22).

APPENDIX B: EXPRESSIONS FOR [AK], {BK], … , [DM]

The following notation is used for convenience:

$$U_i = \int_0^l z^{i-1}\, \mathrm{d}z, \qquad L_i = l^{i-1}, \quad i = 1, 2, \ldots, n, \qquad (B.1, 2)$$

$$V_i = \int_0^l z^{i-1}\cos^2\left[(\theta_2 - \theta_1)\frac{z}{l} + \theta_1\right]\mathrm{d}z, \quad i = 1, 2, \ldots, n, \qquad (B.3)$$

$$S_i = \int_0^l z^{i-1}\sin 2\left[(\theta_2 - \theta_1)\frac{z}{l} + \theta_1\right]\mathrm{d}z, \quad i = 1, 2, \ldots, n, \qquad (B.4)$$

where θ_1 and θ_2 denote the values of pre-twist at nodes 1 and 2, respectively, of the element.

As the nature of w_b, w_s, v_b and v_s is the same except for their positions in the stiffness and mass matrices, one can use $\bar{w}$ to denote any one of the quantities w_b, w_s, v_b or v_s and in

TABLE B1

Values of $H_{i,j}$, $R_{i,j,k}$, $Q_{i,j,k}$

			$R_{i,j,k}$			$Q_{i,j,k}$				
i	j	$H_{i,j}$	$k=1$	$k=2$	$k=3$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
1	1	0	144·0	−144·0	36·0	36·0	−72·0	36·0	0·0	0·0
1	2	0	−144·0	144·0	−36·0	−36·0	72·0	−36·0	0·0	0·0
1	3	1	−72·0	84·0	−24·0	−18·0	42·0	−30·0	6·0	0·0
1	4	1	−72·0	60·0	−12·0	−18·0	30·0	−12·0	0·0	0·0
2	2	0	144·0	−144·0	36·0	36·0	−72·0	36·0	0·0	0·0
2	3	1	72·0	−84·0	24·0	18·0	−42·0	30·0	−6·0	0·0
2	4	1	72·0	−60·0	12·0	18·0	−30·0	12·0	0·0	0·0
3	3	2	36·0	−48·0	16·0	9·0	−24·0	22·0	−8·0	1·0
3	4	2	36·0	−36·0	8·0	9·0	−18·0	11·0	−2·0	0·0
4	4	2	36·0	−24·0	4·0	9·0	−12·0	4·0	0·0	0·0

a similar manner the set $(\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4)$ can be used to represent any one of the sets (u_1, u_2, u_3, u_4), (u_5, u_6, u_7, u_8), $(u_9, u_{10}, u_{11}, u_{12})$ or $(u_{13}, u_{14}, u_{15}, u_{16})$. Thus,

$$\bar{w}(z) = \frac{\bar{u}_1}{l^3}(2z^3 - 3lz^2 + l^2) - \frac{\bar{u}_3}{l^2}(z^3 - 2lz^2 + l^2 z) + \frac{\bar{u}_2}{l^3}(3lz^2 - 2z^3) - \frac{\bar{u}_4}{l^2}(z^3 - lz^2), \quad \text{(B.5)}$$

$$\frac{d\bar{w}}{dz} = \frac{\bar{u}_1}{l^3}(6z^2 - 6lz) - \frac{\bar{u}_3}{l^2}(3z^2 - 4lz + l^2) + \frac{\bar{u}_2}{l^3}(6lz - 6z^2) - \frac{\bar{u}_4}{l^2}(3z^2 - 2lz), \quad \text{(B.6)}$$

$$\frac{d^2\bar{w}}{dz^2} = \frac{\bar{u}_1}{l^3}(12z - 6l) - \frac{\bar{u}_3}{l^2}(6z - 4l) + \frac{\bar{u}_2}{l^3}(6l - 12z) - \frac{\bar{u}_4}{l^2}(6z - 2l). \quad \text{(B.7)}$$

By letting $P_{i,j,k}$ $(i = 1, \ldots, 4; j = 1, \ldots, 4; k = 1, \ldots, 7)$ denote the coefficient of $z^{k-1}\, l^{7-k}$ for the $\bar{u}_i\bar{u}_j$ term in the expression of $\bar{w}^2$, $Q_{i,j,k}$ $(i = 1, \ldots, 4;\ j = 1, \ldots, 4;\ k = 1, \ldots, 5)$, the coefficient of $z^{k-1}\, l^{5-k}$ for the $\bar{u}_i\bar{u}_j$ term in the expression of $(d\bar{w}/dz)^2$, $R_{i,j,k}$ $(i = 1, \ldots, 4;$ $j = 1, \ldots, 4;\ k = 1, \ldots, 3)$, the coefficient of $z^{k-1}\, l^{3-k}$ for the $\bar{u}_i\bar{u}_j$ term in the expression of $(d^2\bar{w}/dz^2)^2$, $H_{i,j}$ $(i = 1, \ldots, 4;\ j = 1, \ldots, 4)$, the index coefficient of l to account for the difference in the index of l due to multiplication of rotational degrees of freedom $\bar{u}_1$ and $\bar{u}_2$ and the displacement degrees of freedom $\bar{u}_3$ and $\bar{u}_4$, the values of $P_{i,j,k}$, $Q_{i,j,k}$, $R_{i,j,k}$ and $H_{i,j}$ can be obtained as shown in Tables B1 and B2.

B.1. EVALUATION OF [BK]

As the procedure for the derivation of $[AK], [BK], \ldots, [DM]$ is the same, the expression for $[BK]$ is derived here as an illustration:

$$\begin{aligned} [u_9\ u_{10}\ u_{11}\ u_{12}]^{\mathrm{T}}[BK][u_9\ u_{10}\ u_{11}\ u_{12}] &= \int_0^l EI_{yy}\left(\frac{\partial^2 v_b}{\partial z^2}\right)^2 dz \\ &= \int_0^l EI_{yy}\left(\frac{\partial^2 \bar{w}}{\partial z^2}\right)^2 dz, \end{aligned} \quad \text{(B.8)}$$

TABLE B2

Values of $P_{i,j,k}$

		$P_{i,j,k}$						
i	j	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
1	1	4·0	−12·0	9·0	4·0	−6·0	0·0	1·0
1	2	−4·0	12·0	−9·0	−2·0	3·0	0·0	0·0
1	3	−2·0	7·0	−8·0	2·0	2·0	−1·0	0·0
1	4	−2·0	5·0	−3·0	−1·0	1·0	0·0	0·0
2	2	4·0	−12·0	9·0	0·0	0·0	0·0	0·0
2	3	2·0	−7·0	8·0	−3·0	0·0	0·0	0·0
2	4	2·0	−5·0	3·0	0·0	0·0	0·0	0·0
3	3	1·0	−4·0	6·0	−4·0	1·0	0·0	0·0
3	4	1·0	−3·0	3·0	−1·0	0·0	0·0	0·0
4	4	1·0	−2·0	1·0	0·0	0·0	0·0	0·0

where

$$\bar{w} = v_b \quad \text{and} \quad \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \bar{u}_4 \end{Bmatrix} = \begin{Bmatrix} u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{Bmatrix}.$$

Substituting the values of I_{yy} and $\bar{w}$ into equation (B.8),

$$\int_0^l EI_{yy}\left(\frac{\partial^2 \bar{w}}{\partial z^2}\right)^2 \mathrm{d}z = \int_0^l E\left[I_{x'x'} + (I_{y'y'} - I_{x'x'})\cos^2\left\{(\theta_2 - \theta_1)\frac{z}{l} + \theta_1\right\}\right] \tag{B.9}$$

$$\left[\frac{\bar{u}_1}{l^3}(12z - 6l) - \frac{\bar{u}_3}{l^2}(6z - 4l) + \frac{\bar{u}_2}{l^3}(6l - 12z) - \frac{\bar{u}_4}{l^2}(6z - 2l)\right]^2 \mathrm{d}z$$

with

$$BK_{1,1} = \text{coefficient of } \bar{u}_1\bar{u}_1 = \text{coefficient of } u_9 u_9$$

$$= \frac{E}{12l^{10}}\, l^{H_{i,j}} \int_0^l \Bigg[\Bigg\langle (a_1 z^4 + a_2 l z^3 + a_3 l^2 z^2 + a_4 l^3 z + a_5 l^4)$$

$$+ \{(d_1 z^4 + d_2 l z^3 + d_3 l^2 z^2 + d_4 l^3 z + d_5 l^4)$$

$$- (a_1 z^4 + a_2 l z^3 + a_3 l^2 z^2 + a_4 l^3 z + a_5 l^4)\}\cos^2\left\{(\theta_2 - \theta_1)\frac{z}{l} + \theta_1\right\}\Bigg\rangle$$

$$\times \{R_{1,1,1} z^2 + R_{1,1,2} l z + R_{1,1,3} l^2\}\Bigg]\mathrm{d}z$$

$$= \frac{E}{12l^{10}} L_{(H_{1,1+1})} \sum_{i=1}^{5} a_i [R_{1,1,1} U_{8-i} + L_{i+1} R_{1,1,2} U_{7-i} + L_{i+2} R_{1,1,3} V_{6-i}]$$

$$(d_i - a_i)[L_i R_{1,1,1} V_{8-i} + L_{i+1} R_{1,1,2} V_{7-i} + L_{i+2} R_{1,1,3} V_{6-i}] \tag{B.10}$$

$$= \frac{E}{12l^{10}} L_{(H_{1,1+1})} \sum_{i=1}^{5} \sum_{j=1}^{3} \{a_i L_{i+j-1} U_{9-i-j} R_{1,1,j}\} + (d_i - a_i)\{L_{i+j-1} V_{9-i-j} R_{1,1,j}\}.$$

This relation can be generalized as

$$BK_{I,J} = \frac{E}{12l^{10}} L_{(H_{1,J+1})} \sum_{i=1}^{5} \sum_{j=1}^{3} \{a_i L_{(i+j-1)} U_{(9-i-j)} R_{I,J,j}\}$$
$$+ (d_i - a_i)\{L_{(i+j-1)} V_{(9-i-j)} R_{I,J,j}\}, \quad I = 1, \ldots, 4, \quad J = I, \ldots, 4,$$
$$= \frac{E}{12l^{10}} \sum_{i=1}^{5} \sum_{j=1}^{3} [\{a_i L_{(i+j+H_{I,J})} U_{(9-i-j)} R_{I,J,j}\}$$
$$+ (d_i - a_i)\{L_{(i+j+H_{I,J})} V_{(9-i-j)} R_{I,J,j}\}], \quad I = 1, \ldots, 4, \quad J = 1, \ldots, 4. \tag{B.11}$$

Similarly,

$$AK_{I,J} = \frac{E}{12l^{10}} \sum_{i=1}^{5} \sum_{j=1}^{3} [\{d_i L_{(i+j+H_{I,J})} U_{(9-i-j)} R_{I,J,j}\}$$
$$+ (a_i - d_i)\{L_{(i+j+H_{I,J})} V_{(9-i-j)} R_{I,J,j}\}], \quad I = 1, \ldots, 4, \quad J = 1, \ldots, 4, \tag{B.12}$$

$$CK_{I,J} = \frac{\mu G}{l^8} \sum_{i=1}^{5} \sum_{j=1}^{3} [c_i L_{(i+j+H_{I,J})} U_{(9-i-j)} Q_{I,J,j}], \quad I = 1, \ldots, 4, \quad J = 1, \ldots, 4, \tag{B.13}$$

$$DK_{I,J} = \frac{E}{12l^{10}} \sum_{i=1}^{5} \sum_{j=1}^{3} [(a_i - d_i) L_{(i+j+H_{I,J})} S_{(9-i-j)} R_{I,J,j}], \quad I = 1, \ldots, 4, \quad J = 1, \ldots, 4, \tag{B.14}$$

$$AM_{I,J} = \frac{\rho}{gl^8} \sum_{i=1}^{3} \sum_{j=1}^{7} [c_i L_{(i+j+H_{I,J})} U_{(11-i-j)} P_{I,J,j}], \quad I = 1, \ldots, 4, \quad J = 1, \ldots, 4, \tag{B.15}$$

$$BM_{I,J} = \frac{\rho}{12gl^{10}} \sum_{i=1}^{5} \sum_{j=1}^{5} [\{d_i L_{(i+j+H_{I,J})} U_{(11-i-j)} Q_{I,J,j}\}$$
$$+ (a_i - d_i)\{L_{(i+j+H_{I,J})} V_{(11-i-j)} Q_{I,J,j}\}], \quad I = 1, \ldots, 4, \quad J = 1, \ldots, 4, \tag{B.16}$$

$$CM_{I,J} = \frac{\rho}{12gl^{10}} \sum_{i=1}^{5} \sum_{j=1}^{5} [\{a_i L_{(i+j+H_{I,J})} U_{(11-i-j)} Q_{I,J,j}\}$$
$$+ (d_i - a_i)\{L_{(i+j+H_{I,J})} V_{(11-i-j)} Q_{I,J,j}\}], \quad I = 1, \ldots, 4, \quad J = 1, \ldots, 4, \tag{B.17}$$

$$DM_{I,J} = \frac{\rho}{12gl^{10}} \sum_{i=1}^{5} \sum_{j=1}^{5} [(a_i - d_i) L_{(i+j+H_{I,J})} S_{(11-i-j)} Q_{I,J,j}],$$
$$I = 1, \ldots, 4, \quad J = 1, \ldots, 4. \tag{B.18}$$

B.2. EVALUATION OF [EK] AND [FK]

$$[u_1 \;\; u_2 \;\; u_3 \;\; u_4]^{\mathrm{T}} [EK] \; [u_1 \;\; u_2 \;\; u_3 \;\; u_4] = \int_0^l P(z) \left(\frac{\partial w_b}{\partial z}\right)^2 \mathrm{d}z$$
$$= \int_0^l \frac{\rho A \Omega^2}{g} \left[\left(eL + \frac{1}{2} L^2 - ez_e - z_e^2\right) - (e + z_e) z - \frac{1}{2} z^2\right] \left(\frac{\partial w_b}{\partial z}\right)^2 \mathrm{d}z$$
$$= \int_0^l \frac{\rho A \Omega^2}{g} \left(eL + \frac{1}{2} L^2 - ez_e - z_e^2\right) \left(\frac{\partial w_b}{\partial z}\right)^2 \mathrm{d}z$$
$$- \int_0^l \frac{\rho A \Omega^2}{g} (e + z_e) z \left(\frac{\partial w_b}{\partial z}\right)^2 \mathrm{d}z - \int_0^l \frac{1}{2} \frac{\rho A \Omega^2}{g} z^2 \left(\frac{\partial w_b}{\partial z}\right)^2 \mathrm{d}z. \tag{B.19}$$

Thus,

$$\mathrm{EK}_{I,J} = \frac{\rho \Omega^2}{gl^8} \left(eL + \frac{1}{2} L^2 - ez_e - z_e^2\right) \sum_{i=1}^{3} \sum_{j=1}^{5} [C_i L_{(i+j+H_{I,J})} U_{(9-i-j)} Q_{I,J,j}]$$
$$- \frac{\rho \Omega^2}{gl^8} (e + z_e) \sum_{i=1}^{3} \sum_{j=1}^{5} [C_i L_{(i+j+H_{I,J})} U_{(10-i-j)} Q_{I,J,j}]$$
$$- \frac{\rho \Omega^2}{2gl^8} \sum_{i=1}^{3} \sum_{j=1}^{5} [C_i L_{(i+j+H_{I,J})} U_{(11-i-j)} Q_{I,J,j}], \quad I = 1, \ldots, 4, \quad J = 1, \ldots, 4. \tag{B.20}$$

Similarly,

$$FK_{I,J} = \frac{2\rho \Omega^2}{gl^8} \sum_{i=1}^{3} \sum_{j=1}^{7} [C_i L_{(i+j+H_{I,J})} U_{(11-i-j)} P_{I,J,j}], \quad I = 1, \ldots, 4, \quad J = I, \ldots, 4. \tag{B.21}$$

APPENDIX C: NOMENCLATURE

A	area of cross-section
b	breadth of beam
e	offset
E	Young's modulus
g	acceleration due to gravity
G	shear modulus
h	depth of beam
I_{xx}, I_{yy}, I_{xy}	moment of inertia of beam cross-section about xx-, yy- and xy-axis
$[K]$	element stiffness matrix
l	length of an element
L	length of total beam
$[M]$	element mass matrix
t	time parameter
u	nodal degrees of freedom

U	strain energy
v	displacement in xz plane
w	displacement in yz plane
x, y	co-ordinate axes
z	co-ordinate axis and length parameter
z_e	distance of the first node of the element from the root of the beam
frequency ratio	ratio of modal frequency to frequency of fundamental mode of uniform beam with the same root cross-section and without shear deformation effects
α	depth taper ratio $= h_1/h_2$
β	breadth taper ratio $= b_1/b_2$
θ	angle of twist
ρ	weight density
μ	shear coefficient
Ω	rotational speed of the beam (rad/s)

Subscripts

b	bending
s	shear