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ON SHEAR DEFORMABLE BEAM THEORIES: THE FREQUENCY AND NORMAL MODE EQUATIONS OF THE HOMOGENEOUS ORTHOTROPIC BICKFORD BEAM

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This paper provides a step forwards the construction and documentation of the frequency equations and the characteristic functions of a general three-degrees-of-freedom theory that describes the plane motion of shear deformable elastic beams. The governing equations of this shear deformable beam theory (G3DOFBT) involve a general shape function of the transverse beam co-ordinate parameter, the *a posteriori* choice of which specifies the distribution of the transverse shear strain or stress along the beam thickness. Different choices of this shape function produce, as particular cases, the corresponding governing equations of different beam theories. These include the differential equations of the Euler–Bernoulli beam theory as well as the corresponding equations of the shear deformable theories due to Timoshenko and Bickford. Other examples can also be found by considering the shear deformable beam theories produced as one-dimensional versions of relevant refined plate theories. Since corresponding developments of the Timoshenko beam theory are already available in the literature, the Bickford theory is considered as the pilot beam theory in this study. The frequency equations, the characteristic functions and the orthogonality conditions of this theory, which assumes a through-thickness parabolic distribution of the transverse shear strain or stress, are constructed analytically for all the classical sets of boundary conditions applied at the beam ends. Some preliminary numerical results are also presented and discussed for beams having both their ends simply supported or clamped.

1. INTRODUCTION

The characteristic functions (normal modes) of a homogeneous finite beam that vibrates in accordance with the hypotheses of the Euler–Bernoulli beam theory are well documented in the literature and adequately classified for a variety of end boundary conditions. Reference [1], for instance, gives detailed expressions as well as the most important properties of these sets of orthogonal functions for beams subjected to 10 different types of end supports. These sets of characteristic functions have played an important role in the understanding of the dynamic behaviour of thin elastic beams made up of homogeneous isotropic material. Apart from this fundamental role, they are also of particular importance in the static and dynamic analysis of thin elastic structural elements, the mechanical behaviour of which is adequately described by the governing partial differential equations of classical plate or shell theories (see, for instance, references [2–5]).

As far as the Timoshenko beam theory [6, 7] is concerned, an adequate construction of its frequency equations and characteristic functions has already been presented by Huang [8], for six different types of end boundary conditions, whereas an additional, pure shear mode, for simply–simply supported Timoshenko beams that vibrate without transverse deflection, was identified in reference [9]. It should, however, be noted that apart from a rather straightforward recent application [10], the Timoshenko characteristic functions have not been used in the literature to the same extent as the Euler–Bernoulli beam functions. It is noted in this respect that, after a slight approximation was applied to the Timoshenko beam ordinary differential operator, a simpler form of relevant sets of characteristic functions was presented by Abramovich [11].

This paper is considered as a step towards the generalization of a certain but basic class of refined theories that describe the plane flexural motion of shear deformable elastic beams as well as the construction and the documentation of their frequency equations and characteristic functions. The proposed developments are consistent with a general three-degrees-of-freedom beam theory (G3DOFBT), the governing equations of which can be obtained by specializing the equations of the corresponding shear deformable plate theory [12] in one dimension. For application purposes, however, only flexural beam motions are considered in this paper. Hence, a further specialization of those one-dimensional equations is considered, which is consistent with the broad but special class of structural elements that involve homogeneous or symmetric laminated beams. It should be noted in this respect that frequency equations and characteristic functions of homogeneous beams only are derived in this study. As far as more complicated material arrangements are concerned, corresponding results will be reported in future publications.

For the purpose of completeness and self-sufficiency, all the equations and the relevant quotations required for an adequate use of the G3DOFBT are initially outlined in this paper as briefly as possible. They involve a general shape function of the beam transverse co-ordinate parameter, the *a posteriori* choice of which specifies the distribution of the transverse shear strain or stress across the beam thickness. Different choices of this shape function produce, as particular cases, the corresponding governing equations of different beam theories. These include, for example, the differential equations of the Euler–Bernoulli theory or the shear deformable theories due to Timoshenko [6, 7] and Bickford [13].

Contrary to the Euler–Bernoulli or the Timoshenko beam theories, which are quoted by choosing a constant (zero) or a linear shape function in the G3DOFBT, respectively, any other (non-linear) choice of that shape function converts the fourth order operator involved into a sixth order differential operator. The Bickford beam theory [13] is the earliest and perhaps the simplest relevant example and is therefore considered as the pilot beam theory in this study. Other examples can, however, also be found by considering the shear deformable beam theories produced as one-dimensional versions of relevant refined plate theories (see, for instance, references [14–16]). As far as Bickford's theory is concerned, the frequency equations as well as the corresponding characteristic functions are constructed analytically for all the 10 classical sets of boundary conditions that are usually applied at the beam ends. Some preliminary numerical results are also presented and discussed for beams having both their ends simply supported or clamped.

2. GENERALIZATION OF SHEAR DEFORMABLE BEAM THEORIES THAT USE THREE DEGREES OF FREEDOM (G3DOFBT)

Consider an elastic beam of length L and thickness h and assume that its middle-axis coincides with the Ox-axis of a Cartesian co-ordinate system $Oxyz$ (the positive Oz-axis is directed upwards). Consider also, for convenience, that the cross-section of the beam has a rectangular shape with unit width. Assuming further that the beam is deformed in the Oxz plane only, the G3DOFBT formulation begins with the following displacement approximation:

$$\begin{aligned} U(x, y, z, t) &= u(x, t) - zw_{,x} + \phi(z)u_1(x, t), \\ W(x, y, z, t) &= w(x, t). \end{aligned} \tag{1}$$

In the usual manner, this displacement approximation assumes plane strain conditions and therefore neglects the Poisson ratio effects in the normal to the x–z plane direction. At this stage, no particular form is assigned to the function $\phi(z)$, though it is assumed to have dimensions of length. Moreover, by enforcing u to represent the in-plane displacement of the beam middle axis and u_1 to be the value of the transverse shear strain on the plate middle axis, further constraints might be imposed on $\phi(z)$. Although both of these are only potential requirements, and as such might well be ignored, they impose the following constraints on $\phi(z)$ and its derivative [see equations (3) and (4)]:

$$\phi(0) = 0, \quad \left.\frac{\mathrm{d}\phi}{\mathrm{d}z}\right|_{z=0} = 1. \tag{2}$$

The displacement approximation (1) yields the following non-zero strain components:

$$\varepsilon_x = e_x^c + zk_x^c + \phi(z)k_x^a, \quad \gamma_{xz} = \phi'(z)e_{xz}^a, \tag{3}$$

where a prime denotes ordinary differentiation and

$$e_x^c = u_{,x}, \quad k_x^c = -w_{,xx}, \quad k_x^a = u_{1,x}, \quad e_{xz}^a = u_1. \tag{4}$$

Among the two kinds of the appearing middle-plane strain components, the in-plane component, denoted by a superscript "c" is identical with its Euler–Bernoulli theory counterpart. The additional component denoted with a superscript "a" is purely due to shear deformation effects and, after the choice $\phi'(0) = 1$, yields the value of the transverse shear strain, γ_{xz}, on the beam middle axis. It then becomes evident that the derivative of the general function $\phi(z)$ dictates the "shape" of the transverse shear strains along the plate thickness. Similarly, two kinds of middle-axis curvatures occur; the one denoted with a superscript "c" is again identical to its Euler–Bernoulli theory counterpart and the additional one is again due to purely shear deformation effects.

G3DOFBT still makes use of the conventional force and moment resultants,

$$(N^c, M^c) = \int_{-h/2}^{h/2} \sigma_x(1, z)\,\mathrm{d}z, \tag{5}$$

but, on the basis of the variationally consistent vectorial procedure described in reference [12], it associates to them the following additional force and moment resultants:

$$Q^a = \int_{-h/2}^{h/2} \tau_{xz}\phi'(z)\,\mathrm{d}z, \quad M^a = \int_{-h/2}^{h/2} \sigma_x\phi(z)\,\mathrm{d}z. \tag{6}$$

The equations of motion can then be obtained either variationally or vectorially [12]. Assuming that the beam lateral boundaries ($z = \pm h/2$) are free of external tractions, these equations are given, in terms of force and moment resultants, as follows:

$$N^c_{,x} = (\rho_0 u - \rho_1 w_{,x} + \rho_{01} u_1)_{,tt},$$
$$M^c_{,xx} = (\rho_0 w - \rho_2 w_{,xx} + \rho_1 u_{,x} + \rho_{11} u_{1,x})_{,tt}, \tag{7}$$
$$M^a_{,x} - Q^a = (\rho_{01} u - \rho_{11} w_{,x} + \rho_{02} u_1)_{,tt},$$

where the inertia coefficients which appear are defined as follows:

$$\rho_i = \int_{-h/2}^{h/2} \rho z^i \, \mathrm{d}z, \quad \rho_{ij} = \int_{-h/2}^{h/2} \rho z^i [\phi(z)]^j \, \mathrm{d}z. \tag{8}$$

Additional to the equations of motion (7), the variational approach (Hamilton's principle) also yields all possible sets of variationally consistent boundary conditions that an be applied on the beam ends ($x = 0, L$), namely,

$$u \text{ or } N^c \text{ prescribed,} \tag{9a}$$

$$w \text{ or } M^c_{,x} + \rho_2 w_{,xtt} - \rho_{11} u_{1,tt} - \rho_1 u_{,tt} \text{ prescribed,} \tag{9b}$$

$$w_{,x} \text{ or } M^c \text{ prescribed,} \tag{9c}$$

$$u_1 \text{ or } M^a \text{ prescribed.} \tag{9d}$$

Note should be made of the fact that the application of Hamilton's principle reveals three extra rotatory inertia terms in the natural boundary condition (9b). Since such inertia terms do not occur during the variational formulation of the equations of motion and the corresponding boundary conditions of three-dimensional elasticity, the physical meaning of their appearance in the natural boundary condition (9b) is not obvious. Hence, no evident reason could suggest the existence of boundary inertial terms, during the traditional formulation of an approximate beam theory by means of an engineering vectorial approach, when the transverse movement restriction is removed at an end of the beam ($w \neq 0$). The variational formulation (Hamilton's principle) of a beam model reveals, however, that consideration of these inertial terms is essential in making the natural boundary condition (9b) dynamically equivalent to its corresponding three-dimensional counterpart (τ_{xz} prescribed). It should be noted, in this connection, that several kinds of such discrepancies between the vectorial and the variational formulation of one-dimensional beam or two-dimensional plate and shell mathematical models have been revealed in recent years. Some of them have already been resolved [12, 17], though the one described in this investigation is among the discrepancies (e.g., reference [18]) that may need further consideration and study.

As far as homogeneous isotropic beams are concerned, the afore-mentioned boundary inertia terms were also obtained by Bickford [13] during the variational development of his so-called parabolic shear deformable beam theory. Moreover, the first of these terms was also obtained during the variational formulation of the Euler–Bernoulli theory [19], in which case the second term that includes u_1 is obviously zero whereas the third term is nullified by virtue definition (8a). The generality of the present formulation further reveals, however, that such boundary inertia terms are also associated with all refined shear deformable beam theories. However, as was suggested by one of the referees and is hence shown in Appendix A, these terms can be removed from the equations of the Timoshenko beam theory.

After the constitution of the elastic beam is decided, the three partial differential equations (7) can be expressed in terms of the same number of main unknown displacement functions (u, w and u_1). With this purpose in mind, it can next be assumed that the elastic beam considered is most generally composed of an arbitrary number, N, of linearly elastic specially orthotropic layers, which are perfectly bonded together. Hooke's law then suggests that the following stress–strain relations hold in the kth layer ($k = 1, 2, \ldots, N$):

$$\sigma_x = Q_{11}^{(k)} \varepsilon_x, \quad \tau_{xz} = Q_{55}^{(k)} \gamma_{xz}, \tag{10}$$

where Q's are the well-known reduced elastic stiffnesses [20]. Hence, the introduction of equations (10) and (3) into equations (5) and (6) yield the following constitutive equations:

$$\begin{pmatrix} N^c \\ M^c \\ M^a \end{pmatrix} = \begin{pmatrix} A_{11}^c & B_{11}^c & B_{11}^a \\ B_{11}^c & D_{11}^c & D_{11}^a \\ B_{11}^a & D_{11}^a & D_{11}^{aa} \end{pmatrix} \begin{pmatrix} e_x^c \\ k_x^c \\ k_x^a \end{pmatrix}, \quad Q^a = A_{55}^a e_{xz}^a, \tag{11}$$

where the rigidities which appear are defined as follows:

$$(A_{11}^c, B_{11}^c, B_{11}^a, D_{11}^c, D_{11}^a, D_{11}^{aa}) = \int_{-h/2}^{h/2} Q_{11}^{(k)} (1, z, \phi(z), z^2, z\phi(z), \phi^2(z))\, \mathrm{d}z,$$

$$A_{55}^a = \int_{-h/2}^{h/2} Q_{55}^{(k)} [\phi'(z)]^2\, \mathrm{d}z. \tag{12}$$

Finally, upon inserting equations (4) and (11) into equation (7), the following set of simultaneous partial differential equations are obtained:

$$\begin{aligned} A_{11}^c u_{,xx} - B_{11}^c w_{,xxx} + B_{11}^a u_{1,xx} &= (\rho_0 u - \rho_1 w_{,x} + \rho_{01} u_1)_{,tt}, \\ B_{11}^c u_{,xxx} - D_{11}^c w_{,xxxx} + D_{11}^a u_{1,xxx} &= (\rho_0 w - \rho_2 w_{,xx} + \rho_1 u_{,x} + \rho_{11} u_{1,x})_{,tt}, \\ B_{11}^a u_{,xx} - D_{11}^a w_{,xxx} + D_{11}^{aa} u_{1,xx} - A_{55}^a u_1 &= (\rho_{01} u - \rho_{11} w_{,x} + \rho_{02} u_1)_{,tt}, \end{aligned} \tag{13}$$

for the three unknown degrees of freedom, u, u_1 and w. It can be verified (by setting $v = w_{,x}$) that, in accordance with the number (four) of the end boundary conditions (9), equation (13) form an eighth order set of simultaneous differential equations, with respect to the spatial co-ordinate parameter.

It should be finally noted that, in the case of the laminated version of the Euler–Bernoulli theory ($\phi = u_1 = 0$), the last of equation (7) becomes an identity. This means that the last of equation (13) should be dropped, thus leaving only two differential equations for the same number of main unknown functions (u and w). The last of the boundary conditions (9d) becomes also redundant ($u_1 = 0$, $M^a = M^c$) and, as a result, the number (three) of the remaining boundary conditions matches the order (six) of the set of the two remaining simultaneous differential equations (13a and b).

3. HOMOGENEOUS AND SYMMETRICALLY LAMINATED BEAMS

Next assume that the elastic beam considered is made up of an arbitrary but odd number of specially orthotropic layers, which are symmetrically arranged with respect to the beam middle axis (a homogeneous, linearly elastic beam is a particular case ($N = 1$) of such a symmetric material lay up). Since the flexural dynamic behaviour of such a beam is uncoupled from its corresponding in-plane behaviour, only an odd choice of the shape function $\phi(z)$ can from now on be associated with the governing equations of the G3DOFBT. This makes it clear that the Timoshenko shape function (A.1) as well as the Bickford shape function [13], namely,

$$\phi(z) = z\left(1 - \frac{4z^2}{3h^2}\right), \tag{14}$$

are both acceptable with regard to the dynamic analysis of homogeneous and symmetrically laminated elastic beams.

It can then easily be verified that equations (8) and (12a) yield

$$\rho_1 = \rho_{01} = B^c_{11} = B^a_{11} = 0. \tag{15}$$

As a result, equation (13a), which is mainly related to the axial dynamic behaviour of the beam, uncouples from the flexural equations (13b, c). These are then obtained in the following form:

$$\begin{aligned} D^c_{11} w_{,xxxx} - D^a_{11} u_{1,xxx} &= -(\rho_0 w - \rho_2 w_{,xx} + \rho_{11} u_{1,x})_{,tt} \\ D^a_{11} w_{,xxx} - D^{aa}_{11} u_{1,xx} + A^a_{55} u_1 &= (\rho_{11} w_{,x} - \rho_{02} u_1)_{,tt}. \end{aligned} \tag{16}$$

It can similarly be verified that the first of the boundary conditions (9) uncouples from the remaining three boundary conditions, which can produce all the variational consistent sets of end boundary conditions that are associated with the beam flexural motion. It can further be verified (by setting $v = w_{,x}$) that, in accordance with the number (three) of the remaining end boundary conditions, equation (16) forms a sixth order set of simultaneous differential equations, with respect to the spatial co-ordinate.

In the case of the Euler–Bernoulli theory ($\phi = u_1 = 0$), the last of equations (16) as well as the last of the boundary conditions (9) are again dropped. This then leaves the fourth order differential equation (16a) as the only equation of motion, the solution of which, in conjunction with the remaining variationally consistent boundary conditions (9b, c), will determine the single unknown function, w. The corresponding, widely known equations of the Euler–Bernoulli theory are then obtained by further dropping the rotatory inertia terms $\rho_2 w_{,xx}$ and $\rho_2 w_{,x}$ from the differential equation (16a) and the boundary condition (9c) respectively. As already mentioned, however, the appearance of these terms is dictated by energy considerations inasmuch as is due to the variationally consistent manner in which the beam governing equations were derived (see also reference [19]).

Solutions of equation (16) that represent free harmonic vibrations with vibration frequency ω will be sought in section 6 by means of the following non-dimensional parameters:

$$w/L = \bar{w}(\xi)\cos(\omega t), \qquad u_1 = \bar{u}(\xi)\cos(\omega t) \quad (0 \leqslant \xi = x/L \leqslant 1). \tag{17}$$

Introduction of these parameters into equation (16) yield the following set of simultaneous ordinary differential equations:

$$\begin{aligned} \bar{w}^{(\mathrm{IV})} - \frac{D^a_{11}}{D^c_{11}}\bar{u}''' &= \bar{\omega}^2\left(\bar{w} - \frac{\rho_2}{\rho_0 L^2}\bar{w}'' + \frac{\rho_{11}}{\rho_0 L^2}\bar{u}'\right), \\ \frac{D^a_{11}}{D^c_{11}}\bar{w}''' - \frac{D^{aa}_{11}}{D^c_{11}}\bar{u}'' + \frac{A^a_{55}L^2}{D^c_{11}}\bar{u} &= -\bar{\omega}^2\left(\frac{\rho_{11}}{\rho_0 L^2}\bar{w}' - \frac{\rho_{02}}{\rho_0 L^2}\bar{u}\right), \end{aligned} \tag{18}$$

where a prime denotes ordinary differentiation with respect to ξ and

$$\bar{\omega}^2 = \omega^2 \frac{\rho_0 L^4}{D^c_{11}}. \tag{19}$$

Equation (18) can be uncoupled to give

$$\left[\left(\frac{D^a_{11}}{D^c_{11}}\right)^2 - \frac{D^{aa}_{11}}{D^c_{11}}\right] v^{(\mathrm{VI})} + \left[\left(2\frac{D^a_{11}}{D^c_{11}}\frac{\rho_{11}}{\rho_0 L^2} - \frac{D^{aa}_{11}}{D^c_{11}}\frac{\rho_2}{\rho_0 L^2} - \frac{\rho_{02}}{\rho_0 L^2}\right)\bar{\omega}^2 + \frac{A^a_{55}L^2}{D^c_{11}}\right] v^{(\mathrm{IV})}$$

$$+ \left[\left(\frac{A^a_{55}L^2}{D^c_{11}}\frac{\rho_2}{\rho_0 L^2} + \frac{D^{aa}_{11}}{D^c_{11}}\right)\bar{\omega}^2 + \left(\left(\frac{\rho_{11}}{\rho_0 L^2}\right)^2 - \frac{\rho_2\rho_{02}}{\rho_0^2 L^4}\right)\bar{\omega}^4\right] v''$$

$$+ \left[\frac{\rho_{02}}{\rho_0 L^2}\bar{\omega}^4 - \frac{A^a_{55}L^2}{D^c_{11}}\bar{\omega}^2\right] v = 0, \tag{20}$$

where v stands for both $\bar{w}$ and $\bar{u}$.

4. ORTHOGONALITY CONDITIONS FOR HOMOGENEOUS AND SYMMETRICALLY LAMINATED BEAMS

In order to obtain the orthogonality conditions associated with the flexural normal modes of a homogeneous or a symmetrically laminated beam, denote with a subscript "n" all quantities that are associated with the nth vibration mode, and insert expression (17) into equation (7b, c). These equations then yield the following relationships associated with the nth vibration mode.

$$(M^c_n)'' = -\omega_n^2 L(\rho_0 L^2 \bar{w}_n - \rho_2 \bar{w}''_n + \rho_{11}\bar{u}'_n), \tag{21}$$

$$(M^a_n)' - LQ^a_n = \omega_n^2 L(\rho_{11}\bar{w}'_n - \rho_{02}\bar{u}_n). \tag{22}$$

Multiplying equation (21) by $\bar{w}_m$, where now a subscript "m" denotes quantities associated with the mth vibration mode, and then integrating by parts gives

$$(M^c_n)'\bar{w}_m|_0^1 - (M^c_n)\bar{w}'_m|_0^1 + \int_0^1 M^c_n \bar{w}''_m \,\mathrm{d}\xi$$

$$= -\omega_n^2 L\left[\int_0^1 \rho_0 L^2 \bar{w}_n\bar{w}_m \,\mathrm{d}\xi - \rho_2\bar{w}'_n\bar{w}_m|_0^1 + \rho_2\int_0^1 \bar{w}^1_n\bar{w}'_m \,\mathrm{d}\xi\right.$$

$$\left. + \rho_{11}\bar{u}_n\bar{w}_m|_0^1 - \int_0^1 \rho_{11}\bar{u}_n\bar{w}'_m \,\mathrm{d}\xi\right],$$

which, after the use of the boundary conditions (9), is simplified as follows:

$$\int_0^1 M^c_n\bar{w}''_m \,\mathrm{d}\xi = \omega_n^2 L\left[-L^2\int_0^1 \rho_0\bar{w}_n\bar{w}_m \,\mathrm{d}\xi - \int_0^1 \rho_2\bar{w}'_n\bar{w}'_m \,\mathrm{d}\xi + \int_0^1 \rho_{11}\bar{u}_n\bar{w}'_m \,\mathrm{d}\xi\right]. \tag{23}$$

Similarly, multiplying equation (22) by $\bar{u}_m$ and then integrating by parts, gives

$$-\int_0^1 M_n^a \bar{u}_m' \, \mathrm{d}\xi - L\int_0^1 Q_n^a \bar{u}_m \, \mathrm{d}\xi = \omega_n^2 L\left[\int_0^1 \rho_{11} \bar{w}_n' \bar{u}_m \, \mathrm{d}\xi - \int_0^1 \rho_{02} \bar{u}_n \bar{u}_m \, \mathrm{d}\xi\right]. \tag{24}$$

Next adding equations (23) and (24), gives

$$\omega_n^2 L\left[\int_0^1 (-\rho_0 L^2 \bar{w}_n \bar{w}_m - \rho_2 \bar{w}_n' \bar{w}_m' + \rho_{11}(\bar{u}_n \bar{w}_m' + \bar{w}_n' \bar{u}_m) - \rho_{02} \bar{u}_n \bar{u}_m) \, \mathrm{d}\xi\right]$$

$$= \int_0^1 (M_m^c \bar{w}_n'' - M_m^a \bar{u}_n' - L Q_m^a \bar{u}_n) \, \mathrm{d}\xi. \tag{25}$$

An interchange of the subscripts m and n yields a similar equation that, subtracted from equation (25), yields

$$\int_0^1 \left[\bar{w}_m \bar{w}_n + \frac{\rho_{02}}{\rho_0 L^2} \bar{u}_m \bar{u}_n + \frac{\rho_2}{\rho_0 L^2} \bar{w}_m' \bar{w}_n' - \frac{\rho_{11}}{\rho_0 L^2} (\bar{u}_m \bar{w}_n' + \bar{w}_m' \bar{u}_n)\right] \mathrm{d}\xi = 0, \quad \omega_n \neq \omega_m. \tag{26}$$

It is noted that, this orthogonality condition has finally been derived by virtue of the definitions of M^c, M^a and Q^a given by expressions (11) and (4), whereas the two flexural modes considered were assumed to vibrate with different natural frequencies.

It should be noted that, since this orthogonality condition does not depend on the particular choice of the odd shape function involved, it is also independent of the particular beam theory employed. In more detail, a shape function of form (14) yields the orthogonality condition of the Bickford beam theory which is next used as the pilot beam theory of the present investigation. Moreover, a shape function of the form (A.1) yields an orthogonality condition for the Timoshenko beam theory that, with the use of the transformation $\psi = u_1 - w_{,x}$ and equation (A.4), can easily be shown to be identical to the corresponding condition obtained earlier in reference [8, 21, 22]. In the case of the Euler–Bernoulli theory, however ($\bar{u} = 0$), equation (26) yields a variationally consistent orthogonality condition which is slightly different from the well-known conventional condition [1, 19], in the sense that it also includes the third term that appears within the integral. The appearance of this extra term is evidently due to the contribution of the aforementioned boundary inertia term [see equation (9b)] as well as to its counterpart that appears in the differential equation of motion.

5. PURELY SHEAR VIBRATION OF SIMPLY SUPPORTED BEAMS

Downs [9] considered the free vibration of a homogeneous isotropic Timoshenko beam having both of its ends simply supported and showed that the earlier relevant studies [8, 21] failed to trace a certain, purely shear, vibration mode where the beam vibrates without transverse deflection ($w = 0$). The existence of this vibration mode is an important discovery in the study of the Timoshenko beam theory. This is due to the fact that associated to it, purely shear vibration frequency is precisely at the point of separation between a lower and an upper frequency regime, the upper regime revealing the existence of a second spectrum of free vibration frequencies. The fact that the upper frequency regime of the Timoshenko beam theory yields a second spectrum of frequencies was first claimed by Trail-Nash and Kollar [23], though Anderson [24] also found a second set of Timoshenko beam frequencies. Since then (1953), however, the second-frequency spectrum of the Timoshenko beam theory became an issue of serious controversy and debate.

In brief, some investigators [25–27] departed from the solution of the Timoshenko beam equations, which clearly yields the so-called second-frequency spectrum at least for simply supported beams, and revealed the physical and/or mathematical mechanisms responsible for its formation beyond the value of the purely shear vibration frequency. On the contrary, noting that the frequencies of this second spectrum are not always close enough to corresponding exact elasticity predictions, Stephen [28] ignored the results of the mathematical analysis and suggested that the observed controversy is academic unless the second spectrum of frequencies has some physical significance. Hence, he concluded that essentially this should be judged only with reference to the extent that the values of these frequencies agree with the corresponding exact three-dimensional elasticity results. As will become evident in the following, the present authors are in favour of the line followed by Stephen [28]. It will also be shown that the purely shear vibration mode in SS beams is predictable by means of any shear deformable beam theory. It will be finally shown that a second-frequency spectrum, of the nature discovered in the case of the Timoshenko beam theory, is also present in the case of the Bickford beam theory.

Under these considerations, and for the general case of the homogeneous orthotropic or even the symmetrically laminated beam considered in the present formulation, it is next assumed that $w = 0$. It is then an easy matter to show that the following displacement fields,

$$u_1 = c\cos(\omega t), \quad w = 0, \tag{27}$$

which represent a purely shear vibration mode, satisfies equation (16) identically. Here, c is a constant vibration amplitude whereas,

$$\omega^2 = \frac{A_{55}^a}{\rho_{02}} \quad \text{or} \quad \bar{\omega}^2 = \frac{\rho_0 L^4 A_{55}^a}{\rho_{02} D_{11}^c} = \eta^2. \tag{28}$$

With any odd function of z being an acceptable shape function choice, it becomes obvious that the results presented in this section are considerably more general than those presented in reference [9], in the sense that the purely shear mode in SS beams is predictable by means of any shear deformable beam theory. Hence, the analytical part of reference [9] is evidently a particular case of the present formulation.

As an application of this statement, consider the particular case of homogeneous orthotropic beam ($N = 1$) which will now become the main concern of the present investigation. Upon inserting the shape function (A.1) into equations (8b) and (12g), and accounting appropriately for the contribution of the well-known transverse shear correction factor, K, equation (28) yields the purely shear vibration frequency parameter of a homogeneous orthotropic Timoshenko beam in the following form:

$$\eta_T = 12\left(\frac{L}{h}\right)^2 \sqrt{K\frac{Q_{55}}{Q_{11}}}. \tag{29}$$

It then becomes an easy matter to show that, in the more special case of a homogeneous isotropic beam, equation (29) yields precisely the purely shear vibration frequency predicted in reference [9].

The accurate determination of the values of shear correction factors that appear in uniform shear deformable beam, plate and shell theories has been a major concern for several investigators. Some of the relevant methods applied in connection with complicated material arrangements led to the evaluation of these factors by means of rather complicated mathematical formulae (e.g., references [29–31]). Early investigations, which dealt with vibrations of structural components made of homogeneous isotropic material (e.g., references [32–34]), determined the simple value $K = \pi^2/12$ for the shear correction factors involved by matching the approximate pure shear vibration frequencies predicted with their corresponding exact elasticity predictions. It is further noted that this value of the shear correction factor ($\pi^2/12 \cong 0{\cdot}822467$) is not far from its static counterpart [32, 35], namely $K = 5/6 \cong 0{\cdot}833333$.

Similarly upon inserting the shape function (14) into equations (8b) and (12g), equation (28) yields the purely shear vibration frequency parameter of a homogeneous orthotropic Bickford beam in the following form:

$$\eta_B = 12\left(\frac{L}{h}\right)^2 \sqrt{\frac{14}{17}\frac{Q_{55}}{Q_{11}}}. \tag{30}$$

This result clearly shows that the Bickford theory, which will be next employed as the pilot beam theory of the present investigation, not only predicts the purely shear vibration mode, but, most importantly, it employs correction factor values in an intrinsic manner. As far as homogeneous isotropic and orthotropic beams are concerned, the present value of the shear correction factor employed intrinsically, namely $K = 14/17 \cong 0{\cdot}823529$, is not far from the one used in reference [9] ($K = 0{\cdot}85$). Moreover, it is remarkably close to (it is in fact between) and aforementioned most commonly used relevant values, namely $\pi^2/12 \cong 0{\cdot}822467$ and $5/6 \cong 0{\cdot}833333$, and could therefore be successfully used as an appropriate value for the shear correction factor of the Timoshenko beam theory. However as far as other vibration modes are concerned, there is little information available in the literature on which the above values of K can assist the Timoshenko beam theory to improve its frequency predictions. It is therefore suggested that a proper assessment of the frequency predictions of either the Timoshenko theory (for different values of K) or the Bickford beam theory should be based on comparisons with corresponding results based on appropriate exact elasticity solutions, such as the solution presented in reference [36] for homogeneous isotropic beams.

6. FREQUENCY EQUATIONS AND CHARACTERISTIC FUNCTIONS FOR HOMOGENEOUS BICKFORD BEAMS

Equations (8) and (12) suggest that the following relationships hold in the particular case of a homogeneous beam ($N = 1$) with constant material density:

$$\frac{D_{11}^a}{D_{11}^c} = \frac{\rho_{11}}{\rho_2} = \frac{\int_{-h/2}^{h/2} z\phi(z)\,\mathrm{d}z}{\int_{-h/2}^{h/2} z^2\,\mathrm{d}z}, \quad \frac{D_{11}^{aa}}{D_{11}^c} = \frac{\rho_{02}}{\rho_2} = \frac{\int_{-h/2}^{h/2} [\phi(z)]^2\,\mathrm{d}z}{\int_{-h/2}^{h/2} z^2\,\mathrm{d}z}. \tag{31}$$

With the use of these relationships, equation (20) is further simplified as follows:

$$v^{(\mathrm{VI})} + (2\Omega^2 + A_1)v^{(\mathrm{IV})} + [\Omega^4 + (A_1 + A_2)\Omega^2]v'' + (A_2\Omega^4 - kA_1\Omega^2)v = 0, \tag{32}$$

where

$$A_1 = \frac{A_{55}L^2}{AD_{11}^c}, \quad A_2 = \frac{\rho_0\rho_{02}L^2}{A\rho_2^2}, \quad A = \frac{\rho_{11}^2 - \rho_2\rho_{02}}{\rho_2^2}, \quad k = \frac{\rho_0 L^2}{\rho_2}, \quad \Omega^2 = \frac{\rho_2\bar{\omega}^2}{\rho_0 L^2}. \tag{33}$$

Either of the choices $\phi(z) = 0$ and z yields $A = 0$, which makes both the parameters A_1 and A_2 singular. It is therefore of importance to note that the effect of these choices, $\phi(z) = 0$ and z, on equation (32) becomes clear only when the whole equation is multiplied through by A. It is then obvious that, when either the Euler–Bernoulli or the Timoshenko beam theory is encountered, A is nullified, thus converting the, otherwise sixth order, ordinary differential equation (32) into a corresponding fourth order equation.

As has already been mentioned, however, the main implications of these choices of $\phi(z)$ that correspond to the Euler–Bernoulli theory and the Timoshenko beam theory, respectively, have already been considered elsewhere [1, 8, 11, 21] and are therefore outside the purposes of this paper. The present study is primarily interested on non-linear choices of $\phi(z)$, namely for choices of the shape function that make $A \neq 0$. It can be shown in this respect that, for any odd (non-linear) polynomial form of $\phi(z)$, A takes a negative value and, therefore, A_1 and A_2 are also negative. Under these considerations, it is further shown in Appendix B that the squares of all the three double roots of the algebraic auxiliary equation of the differential equation (32) are real. These can be expressed in the following form:

$$\lambda_i^2 = \mu_i + v \quad (i = 1, 2, 3), \tag{34}$$

where μ_i $(i = 1, 2, 3)$ are the real functions of Ω^2 given by equation (B.8) and

$$v = -\tfrac{1}{3}(2\Omega^2 + A_1). \tag{35}$$

Further developments are possible only when a specific non-linear form is decided for $\phi(z)$. Hence, the Bickford shape function [equation (14)] is introduced at this point and is employed as the pilot shape function throughout the remaining of the present study. As can then be verified with the use of relationship (31), all the quantities defined in equation (33) are independent of the constant material density, ρ. Hence, in this particular case, the numerical value of the coefficient of the highest (sixth) order derivative that appears in the governing differential equation of the homogeneous Bickford beam theory is found to be $A = -0{\cdot}007619$. As was expected, A takes now a non-zero value. Its particularly small magnitude, however, reveals that, if necessary, A can be employed as a perturbation parameter in the case of a possible asymptotic analysis, in which the differential equation of a homogeneous Timoshenko beam would be regarded as the basic approximation of the corresponding Bickford's equation.

As far as the Bickford beam theory is concerned, it is observed that

$$\lambda_1^2 < 0, \quad \lambda_2^2 \begin{cases} > 0 & \text{if } 0 \leqslant \bar{\omega} < \eta_B, \\ < 0 & \text{if } \bar{\omega} > \eta_B, \end{cases} \quad \lambda_3^2 > 0, \tag{36}$$

where η_B is the pure shear vibration frequency parameter defined by equation (30). Therefore, the general solution of the differential equation (32) yields: (1) for $0 \leqslant \bar{\omega} < \eta_B$,

$$\begin{aligned} \bar{w} &= B_1 \cos(\lambda_1 \xi) + B_2 \sin(\lambda_1 \xi) + B_3 \cosh(\lambda_2 \xi) + B_4 \sinh(\lambda_2 \xi) \\ &\quad + B_5 \cosh(\lambda_3 \xi) + B_6 \sinh(\lambda_3 \xi), \\ \bar{u} &= R_1 B_2 \cos(\lambda_1 \xi) - R_1 B_1 \sin(\lambda_1 \xi) + R_2 B_4 \cosh(\lambda_2 \xi) + R_2 B_3 \sinh(\lambda_2 \xi) \\ &\quad + R_3 B_6 \cosh(\lambda_3 \xi) + R_3 B_5 \sinh(\lambda_3 \xi), \end{aligned} \tag{37}$$

and (2) for $\bar{\omega} > \eta_B$,

$$\begin{aligned}\bar{w} &= C_1\cos(\lambda_1\xi) + C_2\sin(\lambda_1\xi) + C_3\cos(\lambda_4\xi) + C_4\sin(\lambda_4\xi) \\ &\quad + C_5\cosh(\lambda_3\xi) + C_6\sinh(\lambda_3\xi), \\ \bar{u} &= R_1C_2\cos(\lambda_1\xi) - R_1C_1\sin(\lambda_1\xi) + R_4C_4\cos(\lambda_4\xi) - R_4C_3\sin(\lambda_4\xi) \\ &\quad + R_3C_6\cosh(\lambda_3\xi) + R_3C_5\sinh(\lambda_3\xi),\end{aligned} \tag{38}$$

where $\lambda_4^2 = -\lambda_2^2$. Here, B_i and C_i $(i = 1, 2, \ldots, 6)$ are arbitrary constants of integration whereas

$$R_i = \begin{cases} \dfrac{5(h^2\lambda_i^2\bar{\omega}^2 - 12\lambda_i^4 + 12\bar{\omega}^2)}{4\lambda_i(h^2\bar{\omega}^2 - 12\lambda_i^2)} & \text{if } i = 1,\ 4, \\ \dfrac{5(h^2\lambda_i^2\bar{\omega}^2 + 12\lambda_i^4 - 12\bar{\omega}^2)}{4\lambda_i(h^2\bar{\omega}^2 + 12\lambda_i^2)} & \text{if } i = 2,\ 3. \end{cases} \tag{39}$$

In a similar fashion to the Timoshenko beam model [8], the split of all the possible values of $\bar{\omega}$ into two regimes will yield, after the application of appropriate boundary conditions in equations (37) and (38), two different forms of the frequency equation as well as two different forms of corresponding normal modes.

With the application of appropriate boundary conditions, equations (37) or (38) yield six homogeneous algebraic equations for the six unknown constants B_i or C_i respectively. For a non-trivial solution, the determinant of the coefficients of those equations should be equal to zero. In each case, this condition will lead to the frequency equation, the roots of which provide the natural vibration frequencies sought. The corresponding normal modes are then obtained as the characteristic functions of this generalized eigenvalue problem. The frequency equations thus obtained are presented for 10 different sets of end boundary conditions in Appendix C. These sets are formed in accordance with equation (9), by considering all the possible combinations of the following boundary conditions applied on either $\xi = 0$ or 1:

1. Clamped end (C):

$$\bar{w} = \bar{w}_{,\xi} = \bar{u} = 0.$$

2. Simply supported (S):

$$\bar{w} = \bar{w}_{,\xi\xi} = \bar{u}_{,\xi} = 0.$$

3. Free (F):

$$\bar{w}_{,\xi\xi\xi} - \frac{D_{11}^a}{D_{11}^c}\,\bar{u}_{,\xi\xi} + \bar{\omega}^2\left(\frac{\rho_2}{\rho_0 L^2}\,\bar{w}_{,\xi} - \frac{\rho_{11}}{\rho_0 L^2}\,\bar{u}\right) = \bar{w}_{,\xi\xi} = \bar{u}_{,\xi} = 0. \tag{40}$$

4. Guided (G):

$$\bar{w}_{,\xi\xi\xi} - \frac{D_{11}^a}{D_{11}^c}\,\bar{u}_{,\xi\xi} + \bar{\omega}^2\left(\frac{\rho_2}{\rho_0 L^2}\,\bar{w}_{,\xi} - \frac{\rho_{11}}{\rho_0 L^2}\,\bar{u}\right) = \bar{w}_{,\xi} = \bar{u} = 0.$$

The characteristic functions of the Bickford beam theory were obtained with the help of the computer package "REDUCE" [37] and, for the purposes of this investigation, they are presented in Appendix D for the 10 different sets of end boundary conditions already mentioned.

7. NUMERICAL EXAMPLES

As far as the Euler–Bernoulli theory is concerned, the auxiliary equation that corresponds to equation (B.1) produces essentially a single root (λ, say). For any given set of end boundary conditions, the values of λ that nullify the corresponding frequency equation are always positive invariant quantities of both the thickness and the material properties of the beam [1]. Hence, with $\bar{\omega}$ being now defined according to

$$\bar{\omega} = (\lambda L)^2, \tag{41}$$

the corresponding non-dimensional Euler–Bernoulli theory frequency parameters, which are associated with purely flexural motion only, occur as invariant quantities of both the thickness and the material properties of the beam. Such predictions, which ignore completely the effects of the transverse shear motion, do not fit with the dynamic analysis of moderately thick or even thin but highly reinforced beams, the high reinforcement of which essentially increases their effective thickness.

It is of particular interest to notice that, for thin beams having no reinforcement, both the values of λ_1 and λ_2 predicted by means of equation (B.1) are almost identical and equal to the single value of λL predicted through the Euler–Bernoulli theory for the same set of end boundary conditions. On the other hand, the value of λ_3 is always several orders of magnitude higher than λ_1 and λ_2, regardless of the values of the beam thickness and stiffness. Hence, following the suggestion of one of the referees of this paper, it can be shown that in the limiting case in which

$$h \to 0, \quad \lambda_1 = \lambda_1 = \lambda, \quad \lambda_3 \to \infty, \tag{42}$$

the frequency equations (C.1)–(C.10) as well as the characteristic functions (D.1)–(D.10) are reduced to their corresponding Euler–Bernoulli counterparts. However, upon increasing the thickness or the Q_{11}/Q_{55} ratio, the values of λ_1 and λ_2 separate, with the latter becoming much smaller than the former though always remaining positive. This trend is partially illustrated in Tables 1–3 that deal with SS and CC beams only. Detailed numerical results for other sets of end boundary conditions may be given in a future publication.

As far as SS Euler–Bernoulli beams are concerned, the values of λ that yield the corresponding values of $\bar{\omega}$, in accordance to equation (41), are integer multiples of π [1]. In this respect, it is of further interest to note that, although the values of λ_2 decrease substantially with increasing h/L or Q_{11}/Q_{55}, the values of λ_1 remain essentially constant and approximately equal to their corresponding λL values. Hence, upon replacing λL with λ_1, observe further that the frequency equation (C.5) or (C.15) becomes essentially identical to the corresponding frequency equation of the Euler–Bernoulli beam theory [1]. These observations lead to the conclusion that it is the replacement of the simple relation (41) with the complicated formulas (34), (35) and (B.8) that, in the case of Bickford's theory, makes the values of $\bar{\omega}$ vary with changing h/L or Q_{11}/Q_{55}.

In order to illustrate this, consider any of the first six natural frequency parameters of a SS Bickford beam, which are tabulated in Table 1 for several realistic values of h/L or Q_{11}/Q_{55}. The bottom row of this table shows also the values of the corresponding Euler–Bernoulli beam theory frequency parameters (EBBT), with an accuracy of eight significant figures. As already mentioned, the exact values of these frequency parameters are

TABLE 1

The first six frequency parameters $\bar{\omega}$ of a SS beam for several values of the stiffness and aspect ratios

h/L	Q_{11}/Q_{55}	I	II	III	IV	V	VI
0·15	80	5·9551039	14·351772	23·352359	33·423026	44·882196	54·111963†
	50	6·8028376	17·253428	28·140848	39·813891	52·611402	66·771933
	10	8·8750537	28·495280	51·171799	74·535354	89·222774	122·33138
	2	9·5766252	35·442025	72·024133	114·82887	161·14376	209·46979
0·1	80	7·3833396	19·609941	32·291487	45·569857	59·786487	75·201808
	50	8·0688830	23·022671	38·820213	55·000417	71·832131	89·581255
	10	9·3844302	33·167490	64·114381	97·846339	132·56864	167·70455
	2	9·7352061	37·480638	79·744556	132·65278	192·97629	258·36495
0·05	80	9·0147529	29·533359	53·595935	78·439765	103·59854	129·16595
	50	9·3050676	32·275532	61·225539	92·090683	123·53341	155·28085
	10	9·7405518	37·537721	79·875483	132·66996	192·39296	256·45752
	2	9·8353358	38·940825	86·189627	149·92255	228·19082	318·97822
0·02	80	9·7159012	37·182452	78·341141	128·68654	184·58349	243·59587
	50	9·7720906	37·986320	81·778474	137·51619	201·72207	271·57741
	10	9·8485738	39·145455	87·169609	152·79871	234·61075	330·99783
	2	9·8640905	39·390479	88·383617	156·52445	243·38015	348·41687
0·01	80	9·8304757	38·863605	85·803934	148·72981	225·36882	313·36456
	50	9·8449424	39·088362	86·888767	151·94528	232·62669	327·11390
	10	9·8643328	39·394295	88·402443	156·58182	243·51379	348·67844
	2	9·8682248	39·456362	88·714933	157·56191	245·88340	353·53447
EBBT		9·8696044	39·478418	88·826440	157·91367	246·74011	355·30576

†The value of the frequency parameter η_B that corresponds to purely shear vibration [see equation (30)].

$\bar{\omega} = (\lambda L)^2 = (m\pi)^2$ ($m = 1, 2, \dots$). The present analysis has shown that, within an accuracy of 12 significant figures, the corresponding values of λ_1 remain always approximately equal to $m\pi$, regardless of the value of either h/L or Q_{11}/Q_{55}. Contrary to this, the frequency values obtained on the basis of Bickford's theory vary substantially with changing values of h/L and Q_{11}/Q_{55}.

In more detail, Table 1 shows that, for the thinnest ($h/L = 0{\cdot}01$) and less reinforced, essentially isotropic beam ($Q_{11}/Q_{55} = 2$), there is a remarkably good agreement between the corresponding frequency parameters based on the Euler–Bernoulli theory and the Bickford theory. This, however, does not proceed beyond the third significant figure for the fundamental, the second and the fourth frequency parameters, or beyond the second significant figure for the three remaining frequencies. As was expected on the other hand, all the frequency parameters of Bickford's theory decrease continuously with increasing the beam reinforcement or the beam thickness. Hence, the fundamental frequency of Bickford's beam becomes as low as about 60% of its Euler–Bernoulli theory value for a moderately thick ($h/L = 0{\cdot}15$) and highly reinforced ($Q_{11}/Q_{55} = 80$) beam. On the other hand, the variation of h/L or Q_{11}/Q_{55} influences the higher vibration frequencies to a much higher degree. Hence, for the same moderately thick and highly reinforced beam, the second frequency becomes as low as 36% whereas the fifth and sixth frequencies are as low as 18 and 16% of the corresponding Euler–Bernoulli theory counterparts respectively. Finally, it should be noted that the frequency value of the purely shear mode, which is essentially infinite according to the Euler–Bernoulli theory, also decreases dramatically with increasing h/L or Q_{11}/Q_{55}. Hence, as shown in Table 1, this purely shear vibration

TABLE 2

The value of λ_1 that corresponds to first six frequencies of a CC beam for several values of the stiffness and aspect ratios. The results indicated with a star fall into the upper frequency regime

h/L	Q_{11}/Q_{55}	I	II	III	IV	V	VI
0·15	80	3·8200459	6·9189158	10·259381	13·551761	16·818827	20·050483*
	50	3·8960789	6·8523273	10·136869	13·410880	16·681140	19·920744*
	10	4·3240151	7·0730380	10·049429	13·134828	16·296565	19·479200
	2	4·6276938	7·5909561	10·567298	13·546566	16·546442	19·568261
0·1	80	3·9697725	6·8411576	10·067921	13·318835	16·580017	19·821254
	50	4·0940784	6·8775855	10·010724	13·213942	16·451760	19·683741
	10	4·4994417	7·3324954	10·248850	13·240030	16·305864	19·420559
	2	4·6806846	7·7186163	10·758629	13·786632	16·813843	19·846247
0·05	80	4·3407371	7·0958971	10·060717	13·140809	16·293822	19·476424
	50	4·4491727	7·2484039	10·169643	13·187831	16·285169	19·427121
	10	4·6597584	7·6665352	10·670798	13·667116	16·673453	19·699127
	2	4·7169360	7·8155599	10·924717	14·024620	17·118101	20·206773
0·02	80	4·6393068	7·6173267	10·594874	13·573495	16·574789	19·606654
	50	4·6713426	7·6952861	10·715217	13·722929	16·734291	19·759118
	10	4·7179228	7·8187341	10·929984	14·031888	17·126822	20·216189
	2	4·7278943	7·8469129	10·983436	14·117236	17·249313	20·379713
0·01	80	4·7055086	7·7846155	10·867409	13·936062	16·996076	20·051739
	50	4·7145630	7·8094484	10·912646	14·004771	17·088889	20·167103
	10	4·7269650	7·8443386	10·978437	14·109058	17·237236	20·363061
	2	4·7295012	7·8516163	10·992518	14·132076	17·271186	20·409819
EBBT		4·7300407	7·8532046	10·995608	14·137165	17·278760	20·420352

parameter, η_B, has become sixth in the order of the frequency parameters of the moderately thick ($h/L = 0{\cdot}15$) and highly reinforced ($Q_{11}/Q_{55} = 80$) SS beam.

In a similar fashion, Tables 2 and 3 show the variation of λ_1 and $\bar{\omega}$, respectively, that correspond to the first six frequency parameters of a CC beam obtained on the basis of Bickford's theory, Further, these tables also make clear that not only $\bar{\omega}$, but also λ_1, decreases continuously in this case with increasing the beam thickness or the material reinforcement. It is denoted, however, that although there is a relatively moderate variation of λ_1 (Table 2), the natural frequencies of the CC Bickford beam (Table 3) can be influenced by the variation of h/L and Q_{11}/Q_{55} to a much higher degree than their corresponding SS counterparts (Table 1). Hence, the fundamental frequency of the CC Bickford beam becomes as low as about 34% of its Euler–Bernoulli theory value in the case of the afore-mentioned moderately thick ($h/L = 0{\cdot}15$) and highly reinforced ($Q_{11}/Q_{55} = 80$) beam. Moreover, the second frequency of that CC Bickford beam becomes as low as 26% whereas its fifth and sixth frequencies are as low as about 16·5 and 15% of the corresponding Euler–Bernoulli theory counterparts respectively.

It should be noted, however that, unlike the previous SS beam case, the frequency equation (C.1) or (C.11) of the CC Bickford's beam is dependent on the value of all three λ_i parameters ($i = 1, 2, 3$) and is therefore remarkably different than the corresponding Euler–Bernoulli theory frequency equation [1]. It should be also noted that apart from the values of λ_1 and $\bar{\omega}$ denoted with a star in Tables 2 and 3, respectively, all the remaining results fall into the lower frequency regime and were therefore obtained on the basis of the first set of frequency equations (C.1). The fact that the results denoted with a star fall into the upper frequency regime, and were therefore obtained on the basis of the second set of

TABLE 3

The first six frequency parameters $\bar{\omega}$ of a CC beam for several values of the stiffness and aspect ratios. The results indicated with a star fall into the upper frequency regime

h/L	Q_{11}/Q_{55}	I	II	III	IV	V	VI
0·15	80	7·7391457	16·107434	25·907782	36·856287	49·307952	63·380430*
	50	9·2641424	19·186510	30·703829	43·132122	56·843191	71·943529*
	10	15·561863	34·064502	55·787245	78·797028	102·70263	127·22911
	2	20·119312	49·670003	87·047526	128·99118	173·88775	220·71378
0·1	80	10·510574	21·835026	34·948967	48·878431	63·935992	80·244267
	50	12·365071	25·986978	41·802283	58·407047	75·942755	94·476308
	10	18·358770	42·994119	72·799084	105·23971	139·23082	174·12323
	2	21·266688	55·231139	101·11545	155·35093	215·51253	279·89198
0·05	80	16·050902	35·600165	58·593067	83·016716	108·33006	134·33304
	50	17·705133	40·799042	68·449337	98·281758	129·34406	161·15307
	10	21·104405	54·633693	99·779763	152·95998	211·71441	274·28739
	2	22·076772	59·813922	114·76807	184·56815	267·51158	361·58431
0·02	80	20·812201	53·259645	96·239280	146·14139	200·63178	258·10507
	50	21·353409	55·954363	103·43674	160·33461	224·03667	292·54658
	10	22·152300	60·340253	116·49895	189·04411	276·43820	377·05007
	2	22·324740	61·360631	119·82187	197·09331	292·67882	405·97743
0·01	80	21·946360	59·169823	112·84718	180·59676	260·20664	349·56554
	50	22·102627	60·064306	115·64253	187·05357	272·57004	370·39921
	10	22·317209	61·329682	119·74766	196·95673	292·46439	405·67397
	2	22·361098	61·594129	120·62929	199·15357	297·04350	414·13005
EBBT		22·373285	61·672823	120·90339	199·85945	298·55554	416·99079

frequency equations (C.1), did not appear to influence the afore-mentioned discussion in any way at least as far as CC beams are concerned. An issue that needs further clarification is the fact that the second set of SS frequency equations (C.15) introduces a second spectrum of vibration frequencies. As already mentioned in Section 5, in accordance with the line suggested in reference [28], the usefulness of the second-frequency branch of the Bickford beam theory should be judged only with reference to the extent that the values of these frequencies agree with the corresponding exact three-dimensional elasticity predictions (see, for instance, reference [36]).

8. CONCLUSIONS

This paper is considered as a step towards the generalization of a certain but basic class of refined theories that describe the plane motion of shear deformable elastic beams as well as the construction and the documentation of their frequency equations, characteristic functions and relevant orthogonality conditions, at least as far as plane flexural motion is concerned. Such new sets of characteristic functions are consistent with corresponding higher order plate or shell theories and could therefore play an important role in the dynamic and static analysis of corresponding structural components.

In more detail, appropriate combinations of characteristic functions in two dimensions form a natural two-dimensional functional basis, to which the unknown solution of a relevant plate or shell boundary value problem can be expanded upon. Such a solution can then be approached very accurately by means of an error-minimization procedure (e.g., Ritz method or Galerkin approach). As far as classical plate and shell models are concerned, such error minimization approaches have already been applied successfully (see, for instance, references [2–5]) and, due to the orthogonality conditions that the characteristic functions of the Euler–Bernoulli theory obey, their convergence was found to be reasonably fast. In dealing with dynamic analyses of highly reinforced plate and shell components, the accurate modelling of which requires the consideration of transverse shear deformation effects, solution expansions in terms of the Euler–Bernoulli theory characteristic functions appear to be inadequate. It appears therefore that, corresponding expansions in terms of the characteristic functions of appropriate shear deformable beam theories, like the ones considered in the investigation, might be found to be reasonable alternatives of the orthogonal polynomial expansions that are already in use [38–41].

Thus, the proposed developments were initially based on a generalization of the relevant, refined laminated beam theories (G3DOFBT) whereas purely flexural plane vibrations were considered by confining G3DOFBT to the motion of symmetric laminates. With the particular case of a homogeneous orthotropic beam being only an application of this formulation, the remainder of the paper presented a continuation of the relevant studies that dealt with the construction of the frequency equations, the characteristic functions and the orthogonality conditions of the Euler–Bernoulli [1] and Timoshenko [8, 9, 11, 21] beams subjected to different sets of end boundary conditions. Further developments were, however, possible only after the specification of a particular refined beam theory.

Under these considerations, the homogeneous orthotropic version of the Bickford's theory [13] was chosen to be the pilot beam theory of the study, though the derivations presented could be developed in a similar fashion for other shear deformable theories of this nature (e.g., references 14, 15, 16]). The main analytical complication that arises in refined shear deformable beam model is due to the corresponding auxiliary algebraic equation (B.1) which, unlike its fourth-degree counterpart arising in either the Euler–Bernoulli theory or the Timoshenko beam theory, is of the sixth degree. As a result, the roots of this auxiliary equation are eventually obtained by means of three characteristic values, λ_i ($i = 1, 2, 3$), whereas the corresponding equation of the Timoshenko beam needs the determination of two such characteristic values only and that of the even simpler Euler–Bernoulli theory just one.

It is of particular interest to notice that, as the present generalized formulation revealed, the existence of purely shear vibration modes in SS beams (that is, flexural modes without transverse deflection [9]), are predicted by means of any shear deformable beam theory. Moreover, regardless of the shear deformable theory considered, the corresponding purely shear vibration frequency, η, splits the whole range of possible natural vibration frequencies into a "lower" and a "upper" frequencies regime, each one of which has different sets of frequency and normal mode equations. Hence, the corresponding observations detailed in reference [8], for the Timoshenko beam, and in this investigation for the Bickford beam, become particular cases of this more general statement. Moreover, comparisons of the approximate pure shear vibration frequencies predicted with their corresponding exact elasticity predictions revealed that, unlike the Timoshenko beam theory, the Bickford beam theory is essentially capable of employing values of an appropriate transverse shear correction factor in an intrinsic manner.

As far as the homogeneous orthotropic version of the Bickford beam theory is concerned, the identification of the border, η_B, between its "lower" and "upper" regime of natural frequency predictions, allowed the analytical construction of both the frequency and the corresponding normal mode equations for 10 different sets of end boundary conditions. It should be noted, however, that the vast majority of the preliminary numerical results presented and discussed in this paper dealt with natural frequencies that lie into the "lower" frequency regime of SS and CC beams. These were referred to as the first six frequencies of SS and CC Bickford beams and, for several realistic values of h/L or Q_{11}/Q_{55}, were compared with corresponding Euler–Bernoulli theory results that, being well documented in the literature [1], are known invariant quantities of the beam geometrical and material properties. The very few times that the comparisons carried out required the use of the second set of frequency equations referred to the sixth frequency of particularly thick and highly reinforced CC Bickford beams.

It was also noted that the frequency value of the purely shear mode, η_B, which is essentially infinite according to the Euler–Bernoulli theory, is decreasing so dramatically with increasing h/L or Q_{11}/Q_{55} that has become sixth in the order of the frequency parameters of a moderately thick ($h/L = 0{\cdot}15$) and highly reinforced ($Q_{11}/Q_{55} = 80$) SS beam. However, this and other advanced relevant problems (as the fact, for instance, that the second set of SS frequency equations (C.15) appears to introduce another branch of vibration frequencies), are issues that need further clarification. Such problems can be resolved by comparing with corresponding results based on an appropriate plane stress elasticity solution [36] and they evidently include further numerical studies. They were therefore considered to be beyond the scope of this investigation and are left for future research investigations.

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APPENDIX A

With the introduction of the Timoshenko shape function, namely

$$\phi(z) = z, \tag{A.1}$$

the transformation $\psi = u_1 - w_{,x}$ brings the in-plane displacement approximation (1a) into the familiar relevant form,

$$U(x, y, z, t) = u(x, t) + z\psi(x, t), \tag{A.2}$$

whereas,

$$N^c = N = \int_{-h/2}^{h/2} \sigma_x \,\mathrm{d}z, \quad M^c = M^a = M = \int_{-h/2}^{h/2} \sigma_x z \,\mathrm{d}z, \quad Q^a = Q = \int_{-h/2}^{h/2} \tau_{xz} \,\mathrm{d}z, \tag{A.3}$$

and

$$\rho_1 = \rho_{01} = \int_{-h/2}^{h/2} \rho z \,\mathrm{d}z, \quad \rho_2 = \rho_{11} = \rho_{02} = \int_{-h/2}^{h/2} \rho z^2 \,\mathrm{d}z. \tag{A.4}$$

As a result of these simplifications, the equations of motion (7) of the Timoshenko-type laminated beam theory are obtained in the following conventional form:

$$\begin{aligned} N_{,x} &= (\rho_0 u + \rho_1 \psi)_{,tt}, \\ M_{,xx} &= (\rho_0 w + \rho_2 \psi_{,x} + \rho_1 u_{,x})_{,tt}, \\ M_{,x} - Q &= (\rho_1 u + \rho_2 \psi)_{,tt}, \end{aligned} \tag{A.5}$$

Moreover, with the use of equation (A.5c), the essential part of the boundary condition (9b) is transformed into the following:

$$Q \text{ prescribed}, \tag{A.6}$$

which agrees with previously published literature [42, 43].

APPENDIX B

The auxiliary equation of the differential equation (32) is as follows:

$$\lambda^6 + (2\Omega^2 + A_1)\lambda^4 + (\Omega^2 + A_1 + A_2)\Omega^2\lambda^2 + (A_2\Omega^2 - kA_1)\Omega^2 = 0. \tag{B.1}$$

With the help of the following transformation.

$$\lambda^2 = \mu - \tfrac{1}{3}(2\Omega^2 + A_1), \tag{B.2}$$

equation (B.1) yields

$$\mu^3 + \tfrac{1}{3}[-(A_1 + \Omega^2)^2 + (A_1 + 3A_2)\Omega^2]\mu + \tfrac{2}{27}(A_1 + \Omega^2)^3 - \tfrac{1}{27}[3A_1(A_1 + 3A_2 + 9k + 3\Omega^2) + (4\Omega^2 - 9A_2)\Omega^2]\Omega^2 = 0. \tag{B.3}$$

Since $A_1 < 0$ and $A_2 < 0$, the coefficient of μ in equation (B.3) is clearly negative. Since it can be further shown that the coefficient of μ^0 is also negative, equation (B.3) can be reduced to the following simplified form:

$$\mu^3 - \left(\frac{s^2 + r^2}{4}\right)^{1/3}\mu + \frac{r}{3\sqrt{3}} = 0, \tag{B.4}$$

where

$$r = 3\sqrt{3}\,[\tfrac{2}{27}(A_1 + 2\Omega^2)^3 - \tfrac{1}{3}(A_1 + 2\Omega^2)(A_1 + A_2 + \Omega^2)\Omega^2 + (A_2\Omega^2 - kA_1)\Omega^2],$$
$$s^2 = 4[\tfrac{1}{3}(2\Omega^2 + A_1)^2 - (A_1 + A_2 + \Omega^2)]^3 - r^2. \tag{B.5}$$

According to the theory of cubic algebraic equations [44], the value of the quantity

$$\theta = -\frac{4r^2}{27(r^2 + s^2)}, \tag{B.6}$$

dictates the form of the roots of equation (B.4). Since in the present case it is clearly

$$-\tfrac{4}{27} < \theta < 0, \tag{B.7}$$

equation (B.4) has three real roots [44] which, after some algebraic manipulations, can be expressed as follows:

$$\mu_1 = -\left[\frac{16(r^2 - s^2)}{27}\right]^{1/6}\cos\left[\frac{1}{3}\tan^{-1}\left(\frac{s}{r}\right) + \frac{\pi}{3}\right],$$
$$\mu_2 = -\left[\frac{16(r^2 + s^2)}{27}\right]^{1/6}\cos\left[\frac{1}{3}\tan^{-1}\left(\frac{s}{r}\right) - \frac{\pi}{3}\right], \tag{B.8}$$
$$\mu_3 = \left[\frac{16(r^2 + s^2)}{27}\right]^{1/6}\cos\left[\frac{1}{3}\tan^{-1}\left(\frac{s}{r}\right)\right].$$

Hence, equation (B.2) returns real values to the squares of all the three double roots of the algebraic equation (B.1), which are given in accordance with equations (34) and (35).

APPENDIX C

The frequency equations obtained for $0 \leqslant \bar{\omega} < \eta_B$, are as follows:

(a) Clamped–clamped beam (CC):

$$P_2P_3\sin\lambda_1(1 - \cosh\lambda_2\cosh\lambda_3) + P_1P_3\sinh\lambda_2(1 - \cos\lambda_1\cosh\lambda_3) - P_1P_2\sinh\lambda_3(1 - \cos\lambda_1\cosh\lambda_2) + \tfrac{1}{2}(-P_1^2 + P_2^2 + P_3^2)\sin\lambda_1\sinh\lambda_2\sinh\lambda_3 = 0 \tag{C.1}$$

(b) Clamped–simply supported beam (CS):

$$P_1 \tan\lambda_1 - P_2 \tanh\lambda_2 + P_3 \tanh\lambda_3 = 0. \tag{C.2}$$

(c) Clamped–free beam (CF):

$$\frac{P_2P_3}{\lambda_2\lambda_3}\cos\lambda_1\left(1+\frac{\lambda_2^2+\lambda_3^2}{2\lambda_2\lambda_3}\sinh\lambda_2\sinh\lambda_3\right)+\frac{P_1P_3}{\lambda_1\lambda_3}\cosh\lambda_2\left(1-\frac{\lambda_1^2-\lambda_3^2}{2\lambda_1\lambda_3}\sin\lambda_1\sinh\lambda_3\right)$$
$$-\frac{P_1P_2}{\lambda_1\lambda_2}\cosh\lambda_3\left(1-\frac{\lambda_1^2-\lambda_2^2}{2\lambda_1\lambda_2}\sin\lambda_1\sinh\lambda_2\right)-\frac{1}{2}\left(\frac{P_1^2}{\lambda_1^2}+\frac{P_2^2}{\lambda_2^2}+\frac{P_3^2}{\lambda_3^2}\right)\cos\lambda_1\cosh\lambda_2\cosh\lambda_3=0. \tag{C.3}$$

(d) Clamped–guided beam (CG):

$$P_1 \cot\lambda_1 + P_2 \coth\lambda_2 - P_3 \coth\lambda_3 = 0. \tag{C.4}$$

(e) Simply supported–simply supported beam (SS):

$$\sin\lambda_1 = 0. \tag{C.5}$$

(f) Simply supported–free beam (SF):

$$\frac{P_1}{\lambda_1^2}\cot\lambda_1 - \frac{P_2}{\lambda_2^2}\coth\lambda_2 + \frac{P_3}{\lambda_3^2}\coth\lambda_3 = 0. \tag{C.6}$$

(g) Simply supported–guided beam (SG):

$$\cos\lambda_1 = 0. \tag{C.7}$$

(h) Free–free beam (FF):

$$\frac{P_2P_3}{\lambda_2^2\lambda_3^2}\sin\lambda_1(1-\cosh\lambda_2\cosh\lambda_3)-\frac{P_1P_3}{\lambda_1^2\lambda_3^2}\sinh\lambda_2(1-\cos\lambda_1\cosh\lambda_3)$$
$$+\frac{P_1P_2}{\lambda_1^2\lambda_2^2}\sinh\lambda_3(1-\cos\lambda_1\cosh\lambda_2)+\frac{1}{2}\left(-\frac{P_1^2}{\lambda_1^4}+\frac{P_2^2}{\lambda_2^4}+\frac{P_3^2}{\lambda_3^4}\right)\sin\lambda_1\sinh\lambda_2\sinh\lambda_3=0. \tag{C.8}$$

(i) Free–guided beam (FG):

$$\frac{P_1}{\lambda_1^2}\tan\lambda_1 + \frac{P_2}{\lambda_2^2}\tanh\lambda_2 - \frac{P_3}{\lambda_3^2}\tanh\lambda_3 = 0. \tag{C.9}$$

(j) Guided–guided beam (GG):

$$\sin\lambda_1 = 0, \tag{C.10}$$

where

$$P_1 = \lambda_1(\lambda_2^2-\lambda_3^2)(h^2\bar{\omega}^2-12\lambda_1^2)(h^2\bar{\omega}^2+12\lambda_2^2+12\lambda_3^2),$$
$$P_2 = \lambda_2(\lambda_1^2+\lambda_3^2)(h^2\omega^2+12\lambda_2^2)(h^2\bar{\omega}^2-12\lambda_1^2+12\lambda_3^2),$$
$$P_3 = \lambda_3(\lambda_1^2+\lambda_2^2)(h^2\bar{\omega}^2+12\lambda_3^2)(h^2\bar{\omega}^2-12\lambda_1^2+12\lambda_2^2).$$

Similarly, the frequency equations obtained for $\bar{\omega} > \eta_B$, are as follows.

(a) Clamped–clamped beam (CC):

$$S_2S_3\sin\lambda_1(1-\cos\lambda_4\cosh\lambda_3) - S_1S_3\sin\lambda_4(1-\cos\lambda_1\cosh\lambda_3) + S_1S_2\sinh\lambda_3(1-\cos\lambda_1\cos\lambda_4) - \tfrac{1}{2}(S_1^2+S_2^2-S_3^2)\sin\lambda_1\sin\lambda_4\sinh\lambda_3 = 0. \tag{C.11}$$

(b) Clamped–simply supported beam (CS):

$$S_1\tan\lambda_1 - S_2\tan\lambda_4 - S_3\tanh\lambda_3 = 0. \tag{C.12}$$

(c) Clamped–free beam (CF):

$$-\frac{S_2S_3}{\lambda_4\lambda_3}\cos\lambda_1\left(1-\frac{\lambda_4^2-\lambda_3^2}{2\lambda_4\lambda_3}\sinh\lambda_4\sinh\lambda_3\right) + \frac{S_1S_3}{\lambda_1\lambda_3}\cos\lambda_4\left(1-\frac{\lambda_1^2-\lambda_3^2}{2\lambda_1\lambda_3}\sin\lambda_1\sinh\lambda_3\right) - \frac{S_1S_2}{\lambda_1\lambda_4}\cosh\lambda_3\left(1-\frac{\lambda_1^2+\lambda_4^2}{2\lambda_1\lambda_4}\sin\lambda_1\sin\lambda_4\right) + \frac{1}{2}\left(\frac{S_1^2}{\lambda_1^2}+\frac{S_2^2}{\lambda_4^2}+\frac{S_3^2}{\lambda_3^2}\right)\cos\lambda_1\cos\lambda_4\cosh\lambda_3 = 0. \tag{C.13}$$

(d) Clamped–guided beam (CG):

$$S_1\cot\lambda_1 - S_2\cot\lambda_4 + S_3\coth\lambda_3 = 0. \tag{C.14}$$

(e) Simply supported–simply supported (SS):

$$\sin\lambda_1\sin\lambda_4 = 0. \tag{C.15}$$

(f) Simply supported–free beam (SF):

$$\frac{S_1}{\lambda_1^2}\cot\lambda_1 - \frac{S_2}{\lambda_4^2}\cot\lambda_4 - \frac{S_3}{\lambda_3^2}\coth\lambda_3 = 0. \tag{C.16}$$

(g) Simply supported–guided beam (SG):

$$\cos\lambda_1\cos\lambda_4 = 0. \tag{C.17}$$

(h) Free–free beam (FF):

$$\frac{S_2S_3}{\lambda_4^2\lambda_3^2}\sin\lambda_1(1-\cosh\lambda_4\cosh\lambda_3) - \frac{S_1S_3}{\lambda_1^2\lambda_3^2}\sin\lambda_4(1-\cos\lambda_1\cosh\lambda_3) - \frac{S_1S_2}{\lambda_1^2\lambda_4^2}\sinh\lambda_3(1-\cos\lambda_1\cos\lambda_4) + \frac{1}{2}\left(\frac{S_1^2}{\lambda_1^4}+\frac{S_2^2}{\lambda_4^4}-\frac{S_3^2}{\lambda_3^4}\right)\sin\lambda_1\sin\lambda_4\sinh\lambda_3 = 0. \tag{C.18}$$

(i) Free–guided beam (FG):

$$\frac{S_1}{\lambda_1^2}\tan\lambda_1 - \frac{S_2}{\lambda_4^2}\tanh\lambda_4 + \frac{S_3}{\lambda_3^2}\tanh\lambda_3 = 0. \tag{C.19}$$

(j) Guided–guided beam (GG):

$$\sin\lambda_1\sin\lambda_4 = 0, \tag{C.20}$$

where

$$S_1 = \lambda_1(\lambda_4^2 + \lambda_3^2)(h^2\bar{\omega}^2 - 12\lambda_1^2)(h^2\bar{\omega}^2 - 12\lambda_4^2 + 12\lambda_3^2),$$

$$S_2 = \lambda_4(\lambda_1^2 + \lambda_3^2)(h^2\bar{\omega}^2 - 12\lambda_4^2)(h^2\bar{\omega}^2 - 12\lambda_1^2 + 12\lambda_3^2),$$

$$S_3 = \lambda_3(\lambda_1^2 - \lambda_4^2)(h^2\bar{\omega}^2 + 12\lambda_3^2)(h^2\bar{\omega}^2 - 12\lambda_1^2 - 12\lambda_4^2).$$

APPENDIX D

In order to provide expressions for the characteristic functions that correspond to the lower frequency regime, $0 \leqslant \bar{\omega} < \eta_B$, it is convenient to define the following auxiliary functions:

$$\Gamma(c_1, c_2, c_3, c_4, c_5, c_6) = c_1 \sin\lambda_1 \cosh\lambda_2 + c_2 \cos\lambda_1 \sinh\lambda_2 + c_3 \sin\lambda_1 \cosh\lambda_3 + c_4 \cos\lambda_1 \sinh\lambda_3 + c_5 \sinh\lambda_2 \cosh\lambda_3 + c_6 \cosh\lambda_2 \sinh\lambda_3$$

and

$$\Delta(c_1, c_2, c_3, c_4, c_5, c_6, c_7) = c_1 \sin\lambda_1 \sinh\lambda_2 + c_2 \cos\lambda_1 \cosh\lambda_2 + c_3 \sin\lambda_1 \sinh\lambda_3 + c_4 \cos\lambda_1 \cosh\lambda_3 + c_5 \sinh\lambda_2 \sinh\lambda_2 \sinh\lambda_3 + c_6 \cosh\lambda_2 \cosh\lambda_3 + c_7.$$

After these definitions, these characteristic functions are given as follows:

(a) Clamped–clamped beam (CC):

$$\begin{aligned} \bar{w} &= B[\Delta_1 \cos\lambda_1\xi + \Gamma_1 \sin\lambda_1\xi + \Delta_2 \cosh\lambda_2\xi + \Gamma_2 \sinh\lambda_2\xi \\ &\quad + \Delta_3 \cosh\lambda_3\xi + \Gamma_3 \sinh\lambda_3\xi], \\ \bar{u} &= B[R_1\Gamma_1 \cos\lambda_1\xi - R_1\Delta_1 \sin\lambda_1\xi + R_2\Gamma_2 \cosh\lambda_2\xi + R_2\Delta_2 \sinh\lambda_2\xi \\ &\quad + R_3\Gamma_3 \cosh\lambda_3\xi + R_3\Delta_3 \sinh\lambda_3\xi], \end{aligned} \tag{D.1}$$

where

$$\Gamma_1 = \Gamma(\lambda_1, \lambda_2, -\lambda_1, -\lambda_3, -\lambda_2, \lambda_3)P_1,$$

$$\Gamma_2 = -\Gamma(\lambda_1, \lambda_2, -\lambda_1, -\lambda_3, -\lambda_2, \lambda_3)P_2,$$

$$\Gamma_3 = \Gamma(\lambda_1, \lambda_2, -\lambda_1, -\lambda_3, -\lambda_2, \lambda_3)P_3,$$

$$\Delta_1 = \Delta(-\lambda_2, \lambda_1, \lambda_3, -\lambda_1, 0, 0, 0)P_1 + \Delta(0, 0, 0, 0, -\lambda_3, \lambda_2, -\lambda_2)P_2 + \Delta(0, 0, 0, 0, -\lambda_2, \lambda_3, -\lambda_3)P_3,$$

$$\Delta_2 = \Delta(0, 0, -\lambda_3, \lambda_1, 0, 0, -\lambda_1)P_1 + \Delta(\lambda_1, \lambda_2, 0, 0, \lambda_3, -\lambda_2, 0)P_2 + \Delta(0, 0, -\lambda_1, -\lambda_3, 0, 0, \lambda_3)P_3,$$

$$\Delta_3 = \Delta(\lambda_2, -\lambda_1, 0, 0, 0, 0, \lambda_1)P_1 + \Delta(-\lambda_1, -\lambda_2, 0, 0, 0, 0, \lambda_2)P_2 + \Delta(0, 0, \lambda_1, \lambda_3, \lambda_2, -\lambda_3, 0)P_3.$$

(b) Clamped–simply supported beam (CS):

$$\bar{w} = B\left[\frac{P_1}{\cos\lambda_1}\sin\lambda_1(\xi-1) - \frac{P_2}{\cosh\lambda_2}\sinh\lambda_2(\xi-1) + \frac{P_3}{\cosh\lambda_3}\sinh\lambda_3(\xi-1)\right],$$
$$\bar{u} = B\left[\frac{R_1P_1}{\cos\lambda_1}\cos\lambda_1(\xi-1) - \frac{R_2P_2}{\cosh\lambda_2}\cosh\lambda_2(\xi-1) + \frac{R_3P_3}{\cosh\lambda_3}\cosh\lambda_3(\xi-1)\right]. \tag{D.2}$$

(c) Clamped–free beam (CF):

$$\begin{aligned}\bar{w} &= B[\Delta_4\cos\lambda_1\xi + \Gamma_4\sin\lambda_1\xi + \Delta_5\cosh\lambda_2\xi + \Gamma_5\sinh\lambda_2\xi \\ &\quad + \Delta_6\cosh\lambda_3\xi + \Gamma_6\sinh\lambda_3\xi],\\ \bar{u} &= B[R_1\Gamma_4\cos\lambda_1\xi - R_1\Delta_4\sin\lambda_1\xi + R_2\Gamma_5\cosh\lambda_2\xi + R_2\Delta_5\sinh\lambda_2\xi \\ &\quad + R_3\Gamma_6\cosh\lambda_3\xi + R_3\Delta_6\sinh\lambda_3\xi],\end{aligned} \tag{D.3}$$

where

$$\begin{aligned}\Gamma_4 &= \Gamma(\lambda_2^3\lambda_3, \lambda_1^3\lambda_3, -\lambda_2\lambda_3^3, -\lambda_1^3\lambda_2, \lambda_1\lambda_3^3, -\lambda_1\lambda_2^3)P_1,\\ \Gamma_5 &= -\Gamma(\lambda_2^3\lambda_3, \lambda_1^3\lambda_3, -\lambda_2\lambda_3^3, -\lambda_1^3\lambda_2, \lambda_1\lambda_3^3, -\lambda_1\lambda_2^3)P_2,\\ \Gamma_6 &= \Gamma(\lambda_2^3\lambda_3, \lambda_1^3\lambda_3, -\lambda_2\lambda_3^3, -\lambda_1^3\lambda_2, \lambda_1\lambda_3^3, -\lambda_1\lambda_2^3)P_3,\\ \Delta_4 &= \Delta(-\lambda_1^3\lambda_3, \lambda_2^3\lambda_3, \lambda_1^3\lambda_2, -\lambda_2\lambda_3^3, 0, 0, 0)P_1\\ &\quad + \Delta(0,0,0,0,\lambda_1\lambda_2^3, -\lambda_1\lambda_3^3, \lambda_1\lambda_2^2\lambda_3)P_2\\ &\quad + \Delta(0,0,0,0,\lambda_1\lambda_3^3, -\lambda_1\lambda_2^3, \lambda_1\lambda_2\lambda_3^2)P_3,\\ \Delta_5 &= \Delta(0,0,-\lambda_1^3\lambda_2, \lambda_2\lambda_3^3, 0, 0, \lambda_1^2\lambda_2\lambda_3)P_1\\ &\quad + \Delta(\lambda_2^3\lambda_3, \lambda_1^3\lambda_3, 0, 0, -\lambda_1\lambda_2^3, \lambda_1\lambda_3^3, 0)P_2\\ &\quad + \Delta(0,0,-\lambda_2\lambda_3^3, -\lambda_1^3\lambda_2, 0, 0, -\lambda_1\lambda_2\lambda_3^2)P_3,\\ \Delta_6 &= \Delta(\lambda_1^3\lambda_3, -\lambda_2^3\lambda_3, 0,0,0,0,-\lambda_1^2\lambda_2\lambda_3)P_1\\ &\quad + \Delta(-\lambda_2^3\lambda_3, -\lambda_1^3\lambda_3, 0,0,0,0,-\lambda_1\lambda_2^2\lambda_3)P_2\\ &\quad + \Delta(0,0,\lambda_2\lambda_3^3, \lambda_1^3\lambda_2, -\lambda_1\lambda_3^3, \lambda_1\lambda_2^3, 0)P_3.\end{aligned}$$

(d) Clamped–guided beam (CG):

$$\bar{w} = B\left[\frac{P_1}{\sin\lambda_1}\cos\lambda_1(\xi-1) + \frac{P_2}{\sinh\lambda_2}\cosh\lambda_2(\xi-1) - \frac{P_3}{\sinh\lambda_3}\cosh\lambda_3(\xi-1)\right],$$
$$\bar{u} = B\left[\frac{R_1P_1}{\sin\lambda_1}\sin\lambda_1(\xi-1) - \frac{R_2P_2}{\sinh\lambda_2}\sinh\lambda_2(\xi-1) + \frac{R_3P_3}{\sinh\lambda_3}\sinh\lambda_3(\xi-1)\right]. \tag{D.4}$$

(e) Simply supported–simply supported beam (SS):

$$\bar{w} = B\sin\lambda_1\xi, \qquad \bar{u} = BR_1\cos\lambda_1\xi. \tag{D.5}$$

(f) Simply supported–free beam (SF):

$$\bar{w} = B\left[\frac{P_1}{\lambda_1 \sin\lambda_1}\sin\lambda_1\xi + \frac{P_2}{\lambda_2 \sinh\lambda_2}\sinh\lambda_2\xi - \frac{P_3}{\lambda_3 \sinh\lambda_3}\sinh\lambda_3\xi\right],$$
$$\bar{u} = B\left[\frac{R_1P_1}{\lambda_1 \sin\lambda_1}\cos\lambda_1\xi + \frac{R_2P_2}{\lambda_2 \sinh\lambda_2}\cosh\lambda_2\xi - \frac{R_3P_3}{\lambda_3 \sinh\lambda_3}\cosh\lambda_3\xi\right], \tag{D.6}$$

(g) Simply supported–guided beam (SG):

$$\bar{w} = B\sin\lambda_1\xi, \qquad \bar{u} = BR_1\cos\lambda_1\xi. \tag{D.7}$$

(h) Free–free beam (FF):

$$\bar{w} = B[\Gamma_7\cos\lambda_1\xi + \Delta_7\sin\lambda_1\xi + \Gamma_8\cosh\lambda_2\xi + \Delta_8\sinh\lambda_2\xi + \Gamma_9\cosh\lambda_3\xi + \Delta_9\sinh\lambda_3\xi],$$
$$\bar{u} = B[R_1\Delta_7\cos\lambda_1\xi - R_1\Gamma_7\sin\lambda_1\xi + R_2\Delta_8\cosh\lambda_2\xi + R_2\Gamma_8\sinh\lambda_2\xi + R_3\Delta_9\cosh\lambda_3\xi + R_3\Gamma_9\sinh\lambda_3\xi], \tag{D.8}$$

where

$$\Gamma_7 = \Gamma(-\lambda_1^3, \lambda_2^3, \lambda_1^3, -\lambda_3^3, -\lambda_2^3, \lambda_3^3)\lambda_2\lambda_3P_1,$$
$$\Gamma_8 = \Gamma(-\lambda_1^3, \lambda_2^3, \lambda_1^3, -\lambda_3^3, -\lambda_2^3, \lambda_3^3)\lambda_1\lambda_3P_2,$$
$$\Gamma_9 = -\Gamma(-\lambda_1^3, \lambda_2^3, \lambda_1^3, -\lambda_3^3, -\lambda_2^3, \lambda_3^3)\lambda_1\lambda_2P_3,$$
$$\Delta_7 = \Delta(\lambda_2^4\lambda_3, \lambda_1^3\lambda_2\lambda_3, -\lambda_2\lambda_3^4, -\lambda_1^3\lambda_2\lambda_3, 0, 0, 0)P_1,$$
$$+\,\Delta\left(0, 0, 0, 0, -\frac{\lambda_1^2\lambda_3^4}{\lambda_2}, \lambda_1^2\lambda_2^2\lambda_3, -\lambda_1^2\lambda_2^2\lambda_3\right)P_2$$
$$+\,\Delta\left(0, 0, 0, 0, -\frac{\lambda_1^2\lambda_2^4}{\lambda_2}, \lambda_1^2\lambda_2\lambda_3^2, -\lambda_1^2\lambda_2\lambda_3^2\right)P_3,$$
$$\Delta_8 = \Delta\left(0, 0, -\frac{\lambda_2^2\lambda_3^4}{\lambda_1}, -\lambda_1^2\lambda_2^2\lambda_3, 0, 0, \lambda_1^2\lambda_2^2\lambda_3\right)P_1,$$
$$+\,\Delta(\lambda_1^4\lambda_3, -\lambda_1\lambda_2^3\lambda_3, 0, 0, -\lambda_1\lambda_3^4, \lambda_1\lambda_2^3\lambda_3, 0)P_2$$
$$+\,\Delta\left(0, 0, -\frac{\lambda_1^4\lambda_2^2}{\lambda_3}, \lambda_1\lambda_2^2\lambda_3^2, 0, 0, -\lambda_1\lambda_2^2\lambda_3^2\right)P_3,$$
$$\Delta_9 = \Delta\left(\frac{\lambda_2^4\lambda_3^2}{\lambda_1}, \lambda_1^2\lambda_2\lambda_3^2, 0, 0, 0, 0, -\lambda_1^2\lambda_2\lambda_3^2\right)P_1,$$
$$+\,\Delta\left(-\frac{\lambda_1^4\lambda_3^2}{\lambda_1}, \lambda_1\lambda_2^2\lambda_3^2, 0, 0, 0, 0, -\lambda_1\lambda_2^2\lambda_3^2\right)P_2$$
$$+\,\Delta(0, 0, \lambda_1^4\lambda_2, -\lambda_1\lambda_2\lambda_3^3, -\lambda_1\lambda_2^4, \lambda_1\lambda_2\lambda_3^3, 0\Big)P_3.$$

(i) Free–guided beam (FG):

$$\bar{w} = B\left[-\frac{P_1}{\lambda_1 \cos\lambda_1}\cos\lambda_1(\xi-1) - \frac{P_2}{\lambda_2 \cosh\lambda_2}\cosh\lambda_2(\xi-1) + \frac{P_3}{\lambda_3 \cosh\lambda_3}\cosh\lambda_3(\xi-1)\right],$$

$$\bar{u} = B\left[\frac{R_1 P_1}{\lambda_1 \cos\lambda_1}\sin\lambda_1(\xi-1) - \frac{R_2 P_2}{\lambda_2 \cosh\lambda_2}\sinh\lambda_2(\xi-1) + \frac{R_3 P_3}{\lambda_3 \sinh\lambda_3}\sinh\lambda_3(\xi-1)\right]. \tag{D.9}$$

(j) Guided–guided beam (GG):

$$\bar{w} = B\cos\lambda_1\xi, \qquad \bar{u} = -BR_1 \sin\lambda_1\xi. \tag{D.10}$$

Similarly, in order to provide expressions for the characteristic functions that correspond to the upper frequency regime, $\bar{\omega} > \eta_B$, it is convenient to define the following auxiliary functions:

$$Z(c_1, c_2, c_3, c_4, c_5, c_6) = c_1 \sin\lambda_1 \cos\lambda_4 + c_2 \cos\lambda_1 \sin\lambda_4 + c_3 \sin\lambda_1 \cosh\lambda_3 + c_4 \cos\lambda_1 \sinh\lambda_3 + c_5 \sin\lambda_4 \cosh\lambda_3 + c_6 \cos\lambda_4 \sinh\lambda_3$$

and

$$H(c_1, c_2, c_3, c_4, c_5, c_6, c_7) = c_1 \sin\lambda_1 \sin\lambda_4 + c_2 \cos\lambda_1 \cos\lambda_4 + c_3 \sin\lambda_1 \sinh\lambda_3 + c_4 \cos\lambda_1 \cosh\lambda_3 + c_5 \sin\lambda_4 \sinh\lambda_3 + c_6 \cos\lambda_4 \cosh\lambda_3 + c_7.$$

After these definitions, these characteristic functions are given as follows:

(a) Clamped–clamped beam (CC):

$$\begin{aligned}\bar{w} &= C[H_1 \cos\lambda_1\xi + Z_1 \sin\lambda_1\xi + H_2 \cos\lambda_4\xi + Z_2 \sin\lambda_4\xi + H_3 \cosh\lambda_3\xi + Z_3 \sinh\lambda_3\xi],\\ \bar{u} &= C[R_1 Z_1 \cos\lambda_1\xi - R_1 H_1 \sin\lambda_1\xi + R_4 Z_2 \cos\lambda_4\xi - R_4 H_2 \sin\lambda_4\xi + R_3 Z_3 \cosh\lambda_3\xi + R_3 H_3 \sinh\lambda_3\xi],\end{aligned} \tag{D.11}$$

where

$$Z_1 = -Z(-\lambda_1, \lambda_4, \lambda_1, \lambda_3, -\lambda_4, -\lambda_3)S_1,$$

$$Z_2 = Z(-\lambda_1, \lambda_4, \lambda_1, \lambda_3, -\lambda_4, -\lambda_3)S_2,$$

$$Z_3 = Z(-\lambda_1, \lambda_4, \lambda_1, \lambda_3, -\lambda_4, -\lambda_3)S_3,$$

$$H_1 = H(\lambda_4, \lambda_1, \lambda_3, -\lambda_1, 0, 0, 0)S_1 + H(0, 0, 0, 0, -\lambda_3, \lambda_4, -\lambda_4)S_2 + H(0, 0, 0, 0, -\lambda_4, -\lambda_3, \lambda_3)S_3,$$

$$H_2 = H(0, 0, -\lambda_3, \lambda_1, 0, 0, -\lambda_1)S_1 + H(\lambda_1, \lambda_4, 0, 0, \lambda_3, -\lambda_4, 0)S_2$$
$$+ H(0, 0, \lambda_1, \lambda_3, 0, 0, -\lambda_3)S_3,$$
$$H_3 = H(-\lambda_4, -\lambda_1, 0, 0, 0, 0, \lambda_1)S_1 + H(-\lambda_1, -\lambda_4, 0, 0, 0, 0, \lambda_4)S_2$$
$$+ H(0, 0, -\lambda_1, -\lambda_3, \lambda_4, \lambda_3, 0)S_3.$$

(b) Clamped–simply supported beam (CS):

$$\bar{w} = C\left[\frac{S_1}{\cos\lambda_1}\sin\lambda_1(\xi - 1) - \frac{S_2}{\cos\lambda_4}\sin\lambda_4(\xi - 1) - \frac{S_3}{\cosh\lambda_3}\sinh\lambda_3(\xi - 1)\right],$$
$$\bar{u} = C\left[\frac{R_1 S_1}{\cos\lambda_1}\cos\lambda_1(\xi - 1) - \frac{R_4 S_2}{\cos\lambda_4}\cos\lambda_4(\xi - 1) - \frac{R_3 S_3}{\cosh\lambda_3}\cosh\lambda_3(\xi - 1)\right]. \tag{D.12}$$

(c) Clamped–free beam (CF):

$$\bar{w} = C[H_4\cos\lambda_1\xi + Z_4\sin\lambda_1\xi + H_4\cos\lambda_4\xi + Z_4\sin\lambda_4\xi$$
$$+ H_6\cosh\lambda_3\xi + Z_6\sinh\lambda_3\xi],$$
$$\bar{u} = C[R_1 Z_4\cos\lambda_1\xi - R_1 H_4\sin\lambda_1\xi + R_4 Z_5\cos\lambda_4\xi - R_4 H_5\sin\lambda_4\xi \tag{D.13}$$
$$+ R_3 Z_6\cosh\lambda_3\xi + R_3 H_6\sinh\lambda_3\xi],$$

where

$$Z_4 = Z(-\lambda_4^3\lambda_3, \lambda_1^3\lambda_3, -\lambda_4\lambda_3^3, -\lambda_1^3\lambda_4, \lambda_1\lambda_3^3, \lambda_1\lambda_4^3)S_1,$$
$$Z_5 = -Z(-\lambda_4^3\lambda_3, \lambda_1^3\lambda_3, -\lambda_4\lambda_3^3, -\lambda_1^3\lambda_4, \lambda_1\lambda_3^3, \lambda_1\lambda_4^3)S_2,$$
$$Z_6 = -Z(-\lambda_4^3\lambda_3, \lambda_1^3\lambda_3, -\lambda_4\lambda_3^3, -\lambda_1^3\lambda_4, \lambda_1\lambda_3^3, \lambda_1\lambda_4^3)S_3,$$
$$H_4 = H(-\lambda_1^3\lambda_3, -\lambda_4^3\lambda_3, \lambda_1^3\lambda_4, -\lambda_4\lambda_3^3, 0, 0, 0)S_1$$
$$+ H(0, 0, 0, 0, -\lambda_1\lambda_4^3, \lambda_1\lambda_3^3, \lambda_1\lambda_4^2\lambda_3)S_2$$
$$+ H(0, 0, 0, 0, -\lambda_1\lambda_3^3, -\lambda_1\lambda_4^3, -\lambda_1\lambda_4\lambda_3^2)S_3,$$
$$H_5 = H(0, 0, -\lambda_1^3\lambda_4, \lambda_4\lambda_3^3, 0, 0, \lambda_1^2\lambda_4\lambda_3)S_1$$
$$+ H(-\lambda_4^3\lambda_3, -\lambda_1^3\lambda_3, 0, 0, \lambda_1\lambda_4^3, -\lambda_1\lambda_3^3, 0)S_2$$
$$+ H(0, 0, \lambda_4\lambda_3^3, \lambda_1^3\lambda_4, 0, 0, \lambda_1\lambda_4\lambda_3^2)S_3,$$
$$H_6 = H(\lambda_1^3\lambda_3, \lambda_4^3\lambda_3, 0, 0, 0, 0, -\lambda_1^2\lambda_4\lambda_3)S_1$$
$$+ H(\lambda_4^3\lambda_3, \lambda_1^3\lambda_3, 0, 0, 0, 0, -\lambda_1\lambda_4^2\lambda_3)S_2$$
$$+ H(0, 0, -\lambda_4\lambda_3^3, -\lambda_1^3\lambda_4, \lambda_1\lambda_3^3, -\lambda_1\lambda_4^3, 0)S_3.$$

(d) Clamped–guided beam (CG):

$$\bar{w} = C\left[\frac{S_1}{\sin\lambda_1}\cos\lambda_1(\xi-1) - \frac{S_2}{\sin\lambda_4}\cos\lambda_4(\xi-1) + \frac{S_3}{\sinh\lambda_3}\cosh\lambda_3(\xi-1)\right],$$

$$\bar{u} = C\left[-\frac{R_1S_1}{\sin\lambda_1}\sin\lambda_1(\xi-1) + \frac{R_4S_2}{\sin\lambda_4}\sin\lambda_4(\xi-1) + \frac{R_3S_3}{\sinh\lambda_3}\sinh\lambda_3(\xi-1)\right]. \tag{D.14}$$

(e) Simply supported–simply supported beam (SS):

$$\bar{w} = C\sin\lambda_1\xi, \qquad \bar{u} = CR_1\cos\lambda_1\xi. \tag{D.15}$$

or

$$\bar{w} = C\sin\lambda_4\xi, \qquad \bar{u} = CR_4\cos\lambda_4\xi.$$

(f) Simply supported–free beam (SF):

$$\bar{w} = C\left[\frac{S_1}{\lambda_1\sin\lambda_1}\sin\lambda_1\xi - \frac{S_2}{\lambda_4\sin\lambda_4}\sin\lambda_4\xi + \frac{S_3}{\lambda_3\sinh\lambda_3}\sinh\lambda_3\xi\right],$$

$$\bar{u} = C\left[\frac{R_1S_1}{\lambda_1\sin\lambda_1}\cos\lambda_1\xi - \frac{R_4S_2}{\lambda_4\sin\lambda_4}\cos\lambda_4\xi + \frac{R_3S_3}{\lambda_3\sinh\lambda_3}\cosh\lambda_3\xi\right]. \tag{D.16}$$

(g) Simply supported–guided beam (SG):

$$\bar{w} = C\sin\lambda_1\xi, \qquad \bar{u} = CR_1\cos\lambda_1\xi.$$

or (D.17)

$$\bar{w} = C\sin\lambda_4\xi, \qquad \bar{u} = CR_4\cos\lambda_4\xi.$$

(h) Free–free beam (FF):

$$\bar{w} = C[Z_7\cos\lambda_1\xi + H_7\sin\lambda_1\xi + Z_8\cos\lambda_4\xi + H_8\sin\lambda_4\xi + Z_9\cosh\lambda_3\xi + H_9\sinh\lambda_3\xi],$$

$$\bar{u} = C[R_1H_7\cos\lambda_1\xi - R_1Z_7\sin\lambda_1\xi + R_4H_8\cos\lambda_4\xi - R_4Z_8\sin\lambda_4\xi + R_3H_9\cosh\lambda_3\xi + R_3Z_9\sinh\lambda_3\xi], \tag{D.18}$$

where

$$Z_7 = -\Gamma(-\lambda_1^3, \lambda_4^3, \lambda_1^3, -\lambda_3^3, -\lambda_4^3, \lambda_3^3)\lambda_4\lambda_3S_1,$$

$$Z_8 = \Gamma(-\lambda_1^3, \lambda_4^3, \lambda_1^3, -\lambda_3^3, -\lambda_4^3, \lambda_3^3)\lambda_1\lambda_3S_2,$$

$$Z_9 = -\Gamma(-\lambda_1^3, \lambda_4^3, \lambda_1^3, -\lambda_3^3, -\lambda_4^3, \lambda_3^3)\lambda_1\lambda_4S_3,$$

$$H_7 = -\Delta(\lambda_4^4\lambda_3, \lambda_1^3\lambda_4\lambda_3, -\lambda_4\lambda_3^4, -\lambda_1^3\lambda_4\lambda_3, 0, 0, 0)S_1, + \Delta\left(0, 0, 0, 0, -\frac{\lambda_1^2\lambda_3^4}{\lambda_4}, -\lambda_1^2\lambda_4^2\lambda_3, \lambda_1^2\lambda_4^2\lambda_3\right)S_2 + \Delta\left(0, 0, 0, 0, -\frac{\lambda_1^2\lambda_4^4}{\lambda_4}, \lambda_1^2\lambda_4\lambda_3^2, -\lambda_1^2\lambda_4\lambda_3^2\right)S_3,$$

$$H_8 = \Delta\left(0, 0, -\frac{\lambda_4^2\lambda_3^4}{\lambda_1}, -\lambda_1^2\lambda_4^2\lambda_3, 0, 0, \lambda_1^2\lambda_4^2\lambda_3\right)S_1,$$

$$+ \Delta(-\lambda_1^4\lambda_3, -\lambda_1\lambda_4^3\lambda_3, 0, 0, \lambda_1\lambda_3^4, \lambda_1\lambda_4^3\lambda_3, 0)S_2$$

$$+ \Delta\left(0, 0, \frac{\lambda_1^4\lambda_4^2}{\lambda_3}, -\lambda_1\lambda_4^2\lambda_3^3, 0, 0, \lambda_1\lambda_4^2\lambda_3^2\right)S_3,$$

$$H_9 = -\Delta\left(\frac{\lambda_4^4\lambda_3^2}{\lambda_1}, \lambda_1^2\lambda_4\lambda_3^2, 0, 0, 0, 0, -\lambda_1^2\lambda_4\lambda_3^2\right)S_1,$$

$$+ \Delta\left(-\frac{\lambda_1^4\lambda_3^2}{\lambda_4}, -\lambda_1\lambda_4^2\lambda_3^2, 0, 0, 0, 0, \lambda_1\lambda_4^2\lambda_3^2\right)S_2$$

$$+ \Delta(0, 0, \lambda_1^4, \lambda_4, -\lambda_1\lambda_4\lambda_3^3, -\lambda_1\lambda_4^4, \lambda_1\lambda_4\lambda_3^3, 0\Big)S_3.$$

(i) Free–guided beam (FG):

$$\bar{w} = C\left[\frac{S_1}{\lambda_1\cos\lambda_1}\cos\lambda_1(\xi-1) - \frac{S_2}{\lambda_4\cos\lambda_4}\cos\lambda_4(\xi-1) + \frac{S_3}{\lambda_3\cosh\lambda_3}\cosh\lambda_3(\xi-1)\right],$$

$$\bar{u} = C\left[-\frac{R_1S_1}{\lambda_1\cos\lambda_1}\sin\lambda_1(\xi-1) + \frac{R_4S_2}{\lambda_4\cosh\lambda_2}\sin\lambda_4(\xi-1) + \frac{R_3S_3}{\lambda_3\sinh\lambda_3}\sinh\lambda_3(\xi-1)\right]. \tag{D.19}$$

(j) Guided–guided beam (GG):

$$\bar{w} = C\cos\lambda_1\xi, \qquad \bar{u} = -CR_1\sin\lambda_1\xi$$

or (D.20)

$$\bar{w} = C\cos\lambda_4\xi, \qquad \bar{u} = -CR_4\sin\lambda_4\xi.$$

APPENDIX E: NOMENCLATURE

A_1, A_2, A, k, Ω	quantities defined by equation (33)
A_{11}^c, B_{11}^c, B_{11}^a, D_{11}^c, D_{11}^a, D_{11}^{aa}, A_{55}^a	rigidities defined by equation (12)
e_x^c, e_{xz}^a, k_x^c, k_x^a	strains and curvatures acting on the beam middle-axis
h	thickness of the beam
K	transverse shear correction factor
L	length of the beam
M^c, M^a	moment resultants
N^c, Q^a	force resultants
$Q_{11}^{(k)}$, $Q_{55}^{(k)}$	reduced elastic stiffnesses
R_i	quantities defined by equation (39)
U, W	displacement components
u, w, u_1	displacement functions
$\bar{u}$, $\bar{w}$	non-dimensional variables defined by equation (17)
x, y, z	Cartesian co-ordinate parameters

ε_x, γ_{xz}	strain components
η	purely shear frequency parameter
η_T	Timoshenko theory shear frequency parameter
η_B	Bickford theory shear frequency parameter
λ_1, λ_2, λ_3	characteristic roots defined by equations (B.2) and (B.8)
v	quantity defined by equation (35)
ξ	non-dimensional variable defined by equation (17)
ρ_0, ρ_1, ρ_2, ρ_{01}, ρ_{02}, ρ_{11}	inertial coefficients defined by equation (8)
σ_x, τ_{xz}	stress components
$\phi(z)$	shape function
ω	frequency
$\bar{\omega}$	non-dimensional frequency parameter
^{a}	denotes the additional quantities due to the choice $\phi'(0) = 1$
^{c}	denoted the quantities associated with E–B theory
^{k}	denotes the kth layer
$(\ )'$	denotes the ordinary differentiation
$(\ )_{,i}$	denotes partial differentiation