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LETTERS TO THE EDITOR

THE MODES OF NON-HOMOGENEOUS DAMPED BEAMS

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This short note is concerned with computing the eigenvalues and eigenfunctions of a continuous beam model with damping, using the separation of variables approach. The beam considered has different stiffness, damping and mass properties in each of two parts. Pinned boundary conditions are assumed at each end, although other boundary conditions may be applied at the ends quite simply. Although applications are not considered in detail, one possible example is a thin beam partly submerged in a fluid. The fluid would add considerable damping and mass to the beam structure, and possibly some stiffness. Yang and Zhang [1] calculated these added mass and damping coefficients for parallel flat plates.

1. THE MODEL AND BOUNDARY CONDITIONS

Suppose an Euler–Bernoulli beam consists of two parts, one of length L_1 with stiffness k_1, damping coefficient c_1, and mass per unit length m_1, and the second of length L_2, with parameters k_2, c_2 and m_2. These parameters are assumed to be constant along each beam segment, and contain contributions from the beam and any surrounding fluid. Figure 1 shows the beam on pinned–pinned supports diagrammatically. The equations of free motion for the beams is a simple extension of the undamped equations [2], and are

$$k_i \frac{\partial^4 w_i}{\partial x^4} + m_i \frac{\partial^2 w_i}{\partial t^2} + c_i \frac{\partial w_i}{\partial t} = 0 \quad \text{for } i = 1, 2, \tag{1}$$

where $w_i(x, t)$ is the transverse displacement in the ith portion and x is the axial position along the beam. The boundary conditions for this system depend on conditions at the ends of the beam and also conditions to join the two beams together. In the following we concentrate on pinned connections at the beam ends, and so these boundary conditions are

$$w_1(0) = 0, \quad \frac{\partial^2 w_1}{\partial x^2}(0) = 0, \tag{2, 3}$$

$$w_2(L) = 0, \quad \frac{\partial^2 w_2}{\partial x^2}(L) = 0, \tag{4, 5}$$

where $L = L_1 + L_2$ is the total length of the beam. At the interface, one must have that the displacement, slope, moment and shear force are all continuous, so that

$$w_1(L_1) = w_2(L_1), \quad \frac{\partial w_1}{\partial x}(L_1) = \frac{\partial w_2}{\partial x}(L_1), \tag{6, 7}$$

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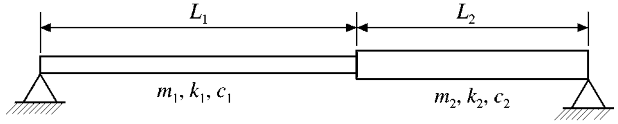

Figure 1. The pinned–pinned beam layout.

$$k_1 \frac{\partial^2 w_1}{\partial x^2}(L_1) = k_2 \frac{\partial^2 w_2}{\partial x^2}(L_1), \quad k_1 \frac{\partial^3 w_1}{\partial x^3}(L_1) = k_2 \frac{\partial^3 w_2}{\partial x^3}(L_1) \tag{8, 9}$$

The standard approach to solving the equations of motion is by separating the variables, and the same procedure may be applied here. Thus, assuming that

$$w_i(x, t) = X_i(x) T_i(t) \tag{10}$$

and substituting into equation (1) gives

$$\frac{k_i}{m_i} \frac{X_i^{(\mathrm{IV})}}{X_i} = -\frac{\ddot{T}_i + (c_i/m_i)\dot{T}_i}{T_i} = \kappa_i, \tag{11}$$

where the κ_i are constants to be determined, the overdots represent differentiation with respect to time, and the superscript (IV) represents the fourth derivative with respect to x. To satisfy the boundary conditions (6)–(9), these time functions must be equal, so that $T_1(t) = T_2(t) = T(t)$. Thus, the differential equation for X_i may be written as

$$X_i^{(\mathrm{IV})} - \frac{m_i}{k_i} \kappa_i X_i = 0. \tag{12}$$

The κ_i are determined from the time function from equation (11), as shown in the next two sections.

2. OVERDAMPED MODES

Overdamped modes have a time function given by

$$T(t) = \mathrm{e}^{-\lambda t} \tag{13}$$

for some positive λ to be determined. Equation (13) may also be multiplied by any constant. Substituting into equation (11) gives,

$$\kappa_i = (c_i/m_i)\lambda - \lambda^2. \tag{14}$$

The κ_i given by equation (14) is real, but may be negative or positive and the form of the solution will change depending on this sign. Suppose both κ_i are positive, then the standard solutions to equation (12), incorporating the boundary conditions (2)–(5) are

$$X_1(x) = A_1 \sin \mu_1 x + B_1 \sinh \mu_1 x \tag{15}$$

and

$$X_2(x) = A_2 \sin \mu_2 (L - x) + B_2 \sinh \mu_2 (L - x) \tag{16}$$

for some constants A_i and B_i, and where

$$\mu_i^4 = \frac{m_i}{k_i}\,\kappa_i. \tag{17}$$

Now, following the standard route, the boundary conditions (6)–(9) are applied to obtain four equations in the unknown parameters. These are

$$\begin{bmatrix} \sin\mu_1 L_1 & \sinh\mu_1 L_1 & -\sin\mu_2 L_2 & -\sinh\mu_2 L_2 \\ \mu_1\cos\mu_1 L_1 & \mu_1\cosh\mu_1 L_1 & \mu_2\cos\mu_2 L_2 & \mu_2\cosh\mu_2 L_2 \\ -k_1\mu_1^2\sin\mu_1 L_1 & k_1\mu_1^2\sinh\mu_1 L_1 & k_2\mu_2^2\sin\mu_2 L_2 & -k_2\mu_2^2\sinh\mu_2 L_2 \\ -k_1\mu_1^3\cos\mu_1 L_1 & k_1\mu_1^3\cosh\mu_1 L_1 & -k_2\mu_2^3\cos\mu_2 L_2 & k_2\mu_2^3\cosh\mu_2 L_2 \end{bmatrix} \begin{Bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{Bmatrix} = 0. \tag{18}$$

Thus, for a non-trivial solution the matrix should be singular and so

$$f(\mu_1, \mu_2) = \det \begin{bmatrix} \sin\mu_1 L_1 & \sinh\mu_1 L_1 & -\sin\mu_2 L_2 & -\sinh\mu_2 L_2 \\ \mu_1\cos\mu_1 L_1 & \mu_1\cosh\mu_1 L_1 & \mu_2\cos\mu_2 L_2 & \mu_2\cosh\mu_2 L_2 \\ -k_1\mu_1^2\sin\mu_1 L_1 & k_1\mu_1^2\sinh\mu_1 L_1 & k_2\mu_2^2\sin\mu_2 L_2 & -k_2\mu_2^2\sinh\mu_2 L_2 \\ -k_1\mu_1^3\cos\mu_1 L_1 & k_1\mu_1^3\cosh\mu_1 L_1 & -k_2\mu_2^3\cos\mu_2 L_2 & k_2\mu_2^3\cosh\mu_2 L_2 \end{bmatrix} = 0. \tag{19}$$

Using equations (14) and (17), μ_1 and μ_2 may be written in terms of λ, and equation (19) becomes

$$f_\lambda(\lambda) = f(\mu_1(\lambda), \mu_2(\lambda)) = 0. \tag{20}$$

This gives an equation to determine the solutions for λ. These solutions are then used to compute μ_1 and μ_2, and thus the matrix in equation (19). This matrix is singular, and for distinct eigenvalues has a null space of dimension 1. The null space may be calculated using the singular value decomposition, and the resulting vector gives the parameters A_i and B_i, and thus the eigenfunction from equations (15) and (16).

One also needs to consider negative κ_i in the solutions of equation (12). In this case, the general solution, incorporating the boundary conditions (2)–(5), is

$$X_1(x) = A_1\cosh\eta_1 x\sin\eta_1 x + B_1\sinh\eta_1 x\cos\eta_1 x \tag{21}$$

and

$$X_2(x) = A_2\cosh\eta_2(L-x)\sin\eta_2(L-x) + B_2\sinh\eta_2(L-x)\cos\eta_2(L-x), \tag{22}$$

where now

$$\eta_i^4 = -\frac{m_i}{4k_i}\,\kappa_i. \tag{23}$$

Applying the same procedure as before, using the boundary conditions (6)–(9), gives equations for the unknown parameters, and yields the following equation instead of equation (18):

$$\det\begin{bmatrix} ch_1 s_1 & sh_1 c_1 & -ch_2 s_2 & -sh_2 c_2 \\ \eta_1(ch_1 c_1 + sh_1 s_1) & \eta_1(ch_1 c_1 - sh_1 s_1) & \eta_2(ch_2 c_2 + sh_2 s_2) & \eta_2(ch_2 c_2 - sh_2 s_2) \\ 2k_1\eta_1^2 sh_1 c_1 & -2k_1\eta_1^2 ch_1 s_1 & -2k_2\eta_2^2 sh_2 c_2 & 2k_2\eta_2^2 ch_2 s_2 \\ 2k_1\eta_1^3(ch_1 c_1 - sh_1 s_1) & -2k_1\eta_1^3(ch_1 c_1 + sh_1 s_1) & 2k_2\eta_2^3(ch_2 c_2 - sh_2 s_2) & -2k_2\eta_2^3(ch_2 c_2 + sh_2 s_2) \end{bmatrix} \tag{24}$$

$$= f(\eta_1, \eta_2) = f(\eta_1(\lambda), \eta_2(\lambda)) = 0,$$

where

$$s_i = \sin \eta_i L_i, \qquad c_i = \cos \eta_i L_i, \qquad sh_i = \sinh \eta_i L_i, \quad ch_i = \cosh \eta_i L_i. \tag{25}$$

Of course, this implies that both the κ parameters are negative. If one is positive and the other is negative, then the obvious combination of the spatial response should be substituted into the expression for the matrix and thus f.

3. UNDERDAMPED MODES

In some respects, the process to compute the underdamped eigenvalues is very similar to that for the overdamped eigenvalues. The time function is now

$$T(t) = e^{\lambda t}. \tag{26}$$

Although this looks very similar to equation (13), there are three crucial differences. λ is a complex eigenvalue, as opposed to real. Thus, the negative sign used in equation (13) to consider positive real λ is not required. Finally, the eigenfunction will be complex, and both the eigenvalue and eigenfunction occur as complex conjugate pairs, so that their sum produces a real response. From equation (11), the κ_i are complex and given by

$$\kappa_i = -(c_i/m_i)\,\lambda - \lambda^2. \tag{27}$$

The spatial solutions, from equation (12) and incorporating boundary conditions (2)–(5), are now

$$X_1(x) = A_1 \sinh v_{11}x + B_1 \sinh v_{12}x \tag{28}$$

and

$$X_2(x) = A_2 \sinh v_{21}(L - x) + B_2 \sinh v_{22}(L - x), \tag{29}$$

where $\pm v_{i1}$ and $\pm v_{i2}$ are the four distinct complex solutions of

$$v_i^4 = \frac{m_i}{k_i}\,\kappa_i \tag{30}$$

TABLE 1

The physical parameters for the numerical example

	Beam 1	Beam 2	
		Case 1	Case 2
L_i	1 m	2 m	2 m
m_i	10 kg/m	20 kg/m	40 kg/m
c_i	0	100 N s/m^2	10 kN s/m^2
k_i	100 N/m^2	100 N/m^2	100 N/m^2

and A_i and B_i are now complex. Applying the boundary conditions (6)–(9) as before, gives

$$\begin{bmatrix} \sinh v_{11}L_1 & \sinh v_{12}L_1 & -\sinh v_{21}L_2 & -\sinh v_{22}L_2 \\ v_{11}\cosh v_{11}L_1 & v_{12}\cosh v_{12}L_1 & v_{21}\cosh v_{21}L_2 & v_{22}\cosh v_{22}L_2 \\ k_1v_{11}^2\sinh v_{11}L_1 & k_1v_{12}^2\sinh v_{12}L_1 & -k_2v_{21}^2\sinh v_{21}L_2 & -k_2v_{22}^2\sinh v_{22}L_2 \\ k_1v_{11}^3\cosh v_{11}L_1 & k_1v_{12}^3\cosh v_{12}L_1 & k_2v_{21}^3\cosh v_{21}L_2 & k_2v_{22}^3\cosh v_{22}L_2 \end{bmatrix} \begin{Bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{Bmatrix} = 0. \tag{31}$$

Thus, λ is determined to make the above matrix singular, and the null space gives the constants A_i and B_i, that are used to produce the eigenfunctions.

4. NUMERICAL EXAMPLE

The computation will be demonstrated using a beam with the parameters given in Table 1. The eigenvalues and eigenfunctions are computed using the procedure outlined above, and also using a finite element model for comparison. The finite element model has 30 elements of equal length, giving a total of 60 degrees of freedom. Two cases are considered: the first represents a lightly damped case, and the second the situation when a more viscous fluid surrounds beam 2. Table 2 shows the results for the lower modes for both cases. Case 2 has a large number of overdamped modes, whereas all the modes in case 1 are underdamped. Figure 2 shows the first four eigenfunctions for case 1. Figures 3 and 4 show the first six overdamped modes and the first four underdamped modes for case 2 respectively. Of particular interest in Figure 4 is the way that the majority of the displacement in the lower modes is local to the undamped part of the beam. As the damping in the second beam is increased this effect becomes more pronounced.

5. CONCLUSIONS

This paper has outlined a method to compute the eigenvalues and eigenfunctions of a continuous, damped beam, consisting of two parts. Both the overdamped and the underdamped eigenvalues and associated eigenfunctions have been computed for two different sets of parameters. For high damping the lower underdamped modes seem to be local to the undamped part of the beam. The procedure given assumed the boundary conditions to be pinned. In the general case the spatial solution would require four

TABLE 2

The lower eigenvalues for the numerical example

Case 1		Case 2	
Continuous model	Finite element model	Continuous model	Finite element model
$-2\cdot2552 \pm 1\cdot2711j$	$-2\cdot2552 \pm 1\cdot2711j$	$-0\cdot014824$	$-0\cdot014824$
$-1\cdot7936 \pm 10\cdot903j$	$-1\cdot7936 \pm 10\cdot903j$	$-0\cdot33642$	$-0\cdot33642$
$-1\cdot5741 \pm 24\cdot863j$	$-1\cdot5741 \pm 24\cdot863j$	$-2\cdot2122$	$-2\cdot2123$
$-1\cdot7876 \pm 43\cdot165j$	$-1\cdot7876 \pm 43\cdot166j$	$-8\cdot5798$	$-8\cdot5808$
$-1\cdot8781 \pm 68\cdot118j$	$-1\cdot8781 \pm 68\cdot121j$	$-26\cdot524$	$-26\cdot535$
$-1\cdot6984 \pm 99\cdot327j$	$-1\cdot6983 \pm 99\cdot339j$	$-5\cdot5929 \pm 35\cdot044j$	$-5\cdot5946 \pm 35\cdot044j$
$-1\cdot6775 \pm 133\cdot66j$	$-1\cdot6774 \pm 133\cdot69j$	$-90\cdot984$	$-91\cdot215$
$-1\cdot8549 \pm 174\cdot00j$	$-1\cdot8546 \pm 174\cdot06j$	$-18\cdot528 \pm 124\cdot77j$	$-18\cdot554 \pm 124\cdot78j$
		$-142\cdot99$	$-142\cdot73$
		$-114\cdot30 \pm 112\cdot95j$	$-114\cdot26 \pm 113\cdot15j$
		$-212\cdot31$	$-212\cdot29$
		$-112\cdot10 \pm 183\cdot55j$	$-112\cdot03 \pm 183\cdot90j$
		$-235\cdot08$	$-235\cdot08$
		$-245\cdot03$	$-245\cdot03$

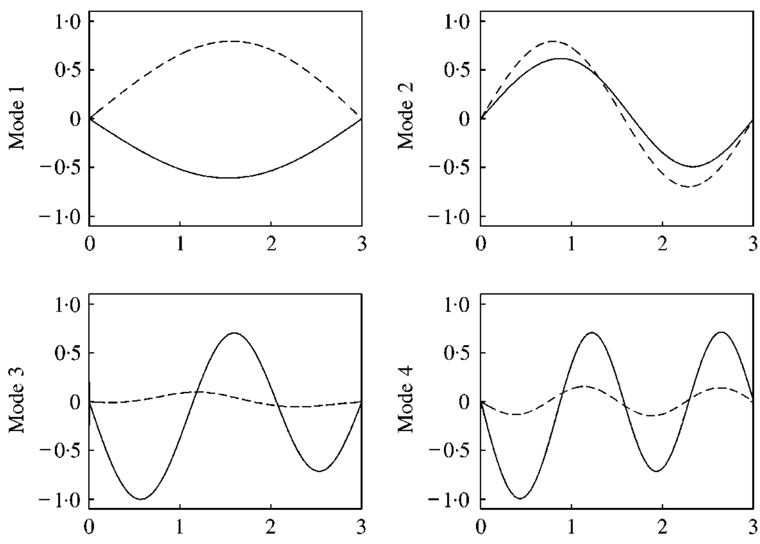

Figure 2. The first four underdamped modes for Case 1 (solid is real part, dashed is imaginary part).

unknown parameters for each beam section, and the boundary conditions equivalent to pinned–pinned could not be incorporated explicitly. The result would a search for a zero determinant of an 8×8 matrix rather than a 4×4.

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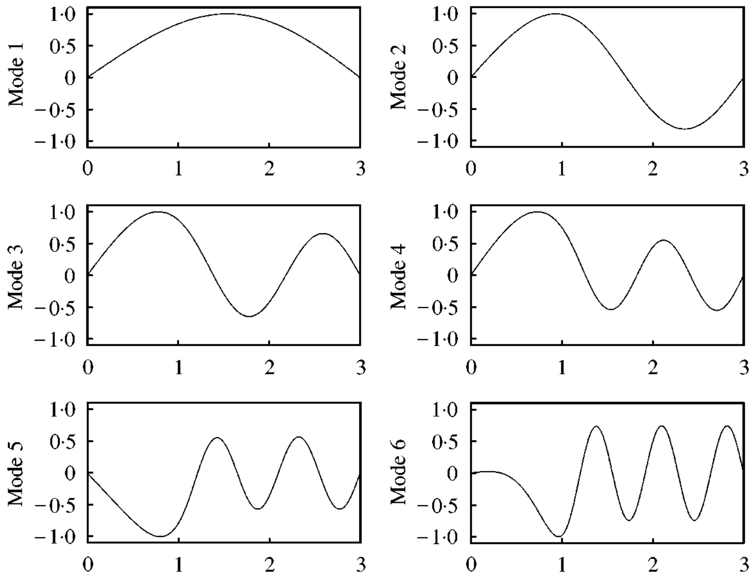

Figure 3. The first six overdamped modes for Case 2.

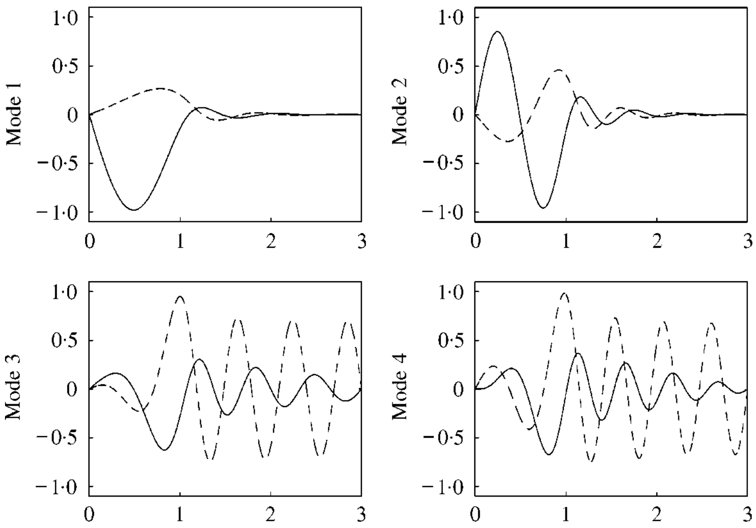

Figure 4. The first four underdamped modes for Case 2 (solid is real part, dashed is imaginary part).

REFERENCES

1. Y.-R. YANG and J.-Y. ZHANG 1997 *Journal of Sound and Vibration* **203**, 795–804. Frequency analysis of a parallel flat plate-type structure in still water, Part I: a multi-span beam.
2. D. J. INMAN 1994 *Engineering Vibration*. Englewood Cliffs, NJ; Prentice-Hall International Inc.