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RESPONSE OF A DUFFING OSCILLATOR TO COMBINED DETERMINISTIC HARMONIC AND RANDOM EXCITATION

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The response of Duffing oscillator to combined deterministic harmonic and random excitation is investigated. The method of harmonic balance and the method of stochastic averaging are used to determine the response of the system. Theoretical analyses and numerical simulations show that when the intensity of the random excitation increases, the non-trivial steady state solution may change from a limit cycle to a diffused limit cycle. Under some conditions, the system may have two steady state solutions and jumps may exist.

1. INTRODUCTION

The study of the response of non-linear systems to narrow-band random excitation is of considerable importance. In the theory of non-linear random vibration, most results obtained so far are attributed to the response of non-linear oscillators to wideband random excitation. In comparison, results on the effect of narrowband excitation on non-linear oscillators are quite limited. Furthermore, some results in this area are disputable.

A typical example in this area is the response of a Duffing oscillator with a hardening spring to narrowband random excitation. It is well known from the theory of non-linear oscillation that if an oscillator with hardening non-linear stiffness is subjected to sinusoidal excitation, the response may exhibit the phenomenon of sharp jumps in amplitude [1]. The jump phenomenon may also occur if an oscillator with hardening non-linear stiffness is subjected to narrowband random excitation. This phenomenon was first observed experimentally and studied theoretically by Lyon *et al.* [2]. Later, it was further examined by a number of authors [3–10]. The main conclusion drawn from these studies is that under certain value areas of the parameters of the oscillator and excitation, the variance of the stationary displacement response of the Duffing oscillator to narrowband excitation is

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triple valued: among them only two are stable and realizable and the jump is the switch between these two stable branches.

However, using digital simulation method, Zhu *et al.* [11] pointed out that the jump of the Duffing oscillator under narrowband random excitation is essentially a transition of the response, from one more probable motion to another *vice versa*. In the case when jumps occur, all the statistics, including the variance of the displacement, of the stationary response are unique and independent of initial conditions. So, until now, the conclusion that there are multiple-valued stationary displacement responses of the Duffing oscillator under narrowband random excitation is disputable.

In this paper, the response of the Duffing oscillator to a combined deterministic harmonic and random excitation is investigated. The method of harmonic balance and the method of stochastic averaging are used to determine the response of the system. Theoretical analyses and numerical simulations show that when the intensity of the random excitation increases, the steady state solution may change form a limit cycle to a diffused limit cycle. Under some conditions the system may have two steady state solutions and jumps may exist.

2. FORMULATION OF PROBLEM

Consider the Duffing oscillation under a combined deterministic and random excitation

$$\ddot{u} + \varepsilon^2 \beta \dot{u} + \omega_0^2 u + \varepsilon^2 u^3 = h \cos \Omega t + \varepsilon \xi(t), \tag{1}$$

where dots indicate differentiation with respect to the time t, $\varepsilon \ll 1$ is a small parameter, β and ω_0 are stiffness coefficient and natural frequency, respectively, h, $\Omega > 0$ are constants and represent the amplitude and frequency of the deterministic harmonic excitations, $\xi(t)$ is a zero mean wideband random process with power spectrum $S(\omega)$. However, $h \cos \Omega t + \varepsilon \xi(t)$ can also be taken as a narrowband random process. We first determine the response of system (1) when $\xi(t) = 0$. In this case, equation (1) can be written as

$$\ddot{u} + \varepsilon^2 \beta \dot{u} + \omega_0^2 u + \varepsilon^2 u^3 = h \cos \Omega t. \tag{2}$$

Periodic solution of equation (2) can be solved by the method of harmonic balance [1]. Let the first order periodic solution of equation (2) be

$$u_0 = a_0 \cos(\Omega t + \gamma). \tag{3}$$

Substituting equation (3) into equation (2), neglecting the high order harmonic term $\cos[3(\Omega t + \gamma)]$, and equating coefficients of $\cos \Omega t$ and $\sin \Omega t$, one obtains

$$\begin{aligned} &(-a_0 \Omega^2 + a_0 \omega_0^2 + \tfrac{3}{4} \varepsilon^2 a_0^3) \cos \gamma - \varepsilon^2 \beta \Omega a_0 \sin \gamma = h, \\ &(-a_0 \Omega^2 + a_0 \omega_0^2 + \tfrac{3}{4} \varepsilon^2 a_0^3) \sin \gamma + \varepsilon^2 \beta \Omega a_0 \cos \gamma = 0. \end{aligned} \tag{4}$$

Equations (4) give the following frequency response formula of system (2):

$$(-a_0 \Omega^2 + a_0 \omega_0^2 + \tfrac{3}{4} \varepsilon^2 a_0^3)^2 + \varepsilon^4 \beta^2 \Omega^2 a_0^2 = h^2. \tag{5}$$

Applying the Floquet theory [1], one obtains the following necessary and sufficient condition of the stability of the periodic solution of equation (2):

$$(\omega_0^2 - \Omega^2 + \tfrac{3}{4}\varepsilon^2 a_0^2)(\omega_0^2 - \Omega^2 + \tfrac{9}{4}\varepsilon^2 a_0^2) + \varepsilon^4\beta^2\Omega^2 < 0. \tag{6}$$

Condition (6) shows that not all the branches given by equation (5) are stable. If there are three branches, among them only the largest and smallest one are stable and realizable and the jump is the switch between these two stable branches.

Next, we determine the effect of the noise, i.e., $\varepsilon\xi(t) \neq 0$, on the deterministic steady state motion. To this end, we let

$$u = u_0 + Y = a_0\cos(\Omega t + \gamma) + Y, \tag{7}$$

where Y is a perturbation term. Substituting equation (7) into equation (1) and neglecting the non-linear terms, one obtains the following linearization equation:

$$\ddot{Y} + \varepsilon^2[\beta\dot{Y} + 3a_0^2\cos^2(\Omega t + \gamma)Y] + \omega_0^2 Y = \varepsilon\xi(t). \tag{8}$$

Equation (8) can be solved by the method of stochastic averaging [12]. By introducing the following transformation:

$$Y(t) = A(t)\cos\Phi,\ \dot{Y}(t) = -A(t)\,\omega_0\sin\Phi,\quad \Phi = \omega_0 t + \Theta, \tag{9}$$

equation (8) can be written as the following standard form:

$$\begin{aligned}\dot{A} &= \frac{\varepsilon^2}{\omega_0}[-\beta A\omega_0\sin\Phi + 3a_0^2\cos^2(\Omega t + \gamma)A\cos\Phi]\sin\Phi - \frac{\varepsilon}{\omega_0}\sin\Phi\xi(t),\\ \dot{\Theta} &= \frac{\varepsilon^2}{A\omega_0}[-\beta A\omega_0\sin\Phi + 3a_0^2\cos^2(\Omega t + \gamma)A\cos\Phi]\cos\Phi - \frac{\varepsilon}{A\omega_0}\cos\Phi\xi(t).\end{aligned} \tag{10}$$

Taking stochastic averaging on equations (10), one obtains the following Ito equations:

$$\begin{aligned}\mathrm{d}A &= \left[-\frac{\beta A}{2}\varepsilon^2 + \frac{\pi S(\omega_0)}{2A\omega_0^2}\varepsilon^2\right]\mathrm{d}t + \frac{\sqrt{\pi S(\omega_0)}}{\omega_0}\varepsilon\,\mathrm{d}W_1(t),\\ \mathrm{d}\Theta &= \frac{3a_0^2}{4\omega_0}\varepsilon^2\,\mathrm{d}t + \frac{\sqrt{\pi S(\omega_0)}}{A\omega_0}\varepsilon\,\mathrm{d}W_2(t),\end{aligned} \tag{11}$$

where $W_1(t)$ and $W_2(t)$ are independent standard Wiener processes.

It is clear that $A(t)$ is a Markov process, its steady state probability density function $p(a)$ is governed by the following FPK equation:

$$\frac{\mathrm{d}}{\mathrm{d}a}\left[\left(\frac{\beta a}{2}\varepsilon^2 - \frac{\pi S(\omega_0)}{2a\omega_0^2}\varepsilon^2\right)p\right] + \frac{\pi S(\omega_0)\varepsilon^2}{2\omega_0^2}\frac{\mathrm{d}^2 p}{\mathrm{d}a^2} = 0. \tag{12}$$

The solution of equation (12) is

$$p(a) = \frac{a}{\sigma^2}\exp\left(-\frac{a^2}{2\sigma^2}\right),\quad \sigma^2 = \frac{\pi S(\omega_0)}{\beta\omega_0^2}. \tag{13}$$

The first and second moments of $A(t)$ are

$$EA(t) = \int_0^{+\infty} ap(a)\,\mathrm{d}a = \sqrt{\frac{\pi}{2}}\,\sigma, \quad EA^2(t) = \int_0^{+\infty} a^2 p(a)\,\mathrm{d}a = 2\sigma^2. \tag{14}$$

Combining equations (7), (9) and (14), one obtains

$$Eu^2 = E[a_0 \cos(\Omega t + \gamma) + A(t)\cos\Phi]^2 = \frac{a_0^2}{2} + \sigma^2 = \frac{a_0^2}{2} + \frac{\pi S(\omega_0)}{\beta\omega_0^2}. \tag{15}$$

In some parameter areas, the amplitude a_0 of periodic solution given by equation (5) has three branches, among them only the largest and smallest one are stable and realizable. However, if $S(\omega_0)$ is small enough, the noise $\varepsilon\xi(t)$ will not change the stability of stable branches, hence, there will be three stationary displacement variances given by equation (15) and among them only the largest and smallest ones are stable and realizable.

3. NUMERICAL SIMULATION

In this paper, the power spectrum of $\xi(t)$ is taken as

$$S(\omega) = \begin{cases} S_0, & 0 < \omega \leqslant 2\omega_0, \\ 0, & \omega > 2\omega_0. \end{cases} \tag{16}$$

For numerical simulation it is more convenient to use the pseudorandom signal given by [12]

$$\xi(t) = \sqrt{\frac{4\omega_0 S_0}{N}} \sum_{k=1}^{N} \cos\left[\frac{\omega_0}{N}(2k-1)t + \varphi_k\right], \tag{17}$$

where φ_k's are independent and uniformly distributed in $(0,2\pi]$.

In the numerical simulation, the parameters in system (1) and (17) are chosen as follows:

$$\beta = \omega_0 = h = 1, \quad \varepsilon^2 = 0{\cdot}1, \quad N = 1000.$$

The governing equation (1) is numerically integrated by the fourth order Runge–Kutta algorithm, and the numerical results are shown in Figures 1 to 4. When $\xi(t) = 0$, the

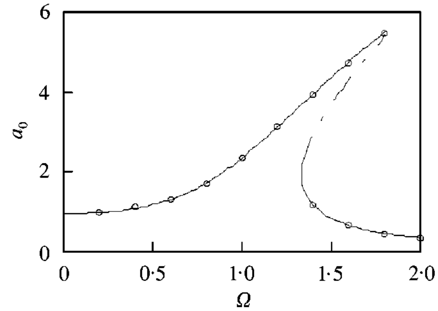

Figure 1. Frequency response of system (1): ——, stable solution; – – –, unstable solution; ∘∘∘ numerical solution.

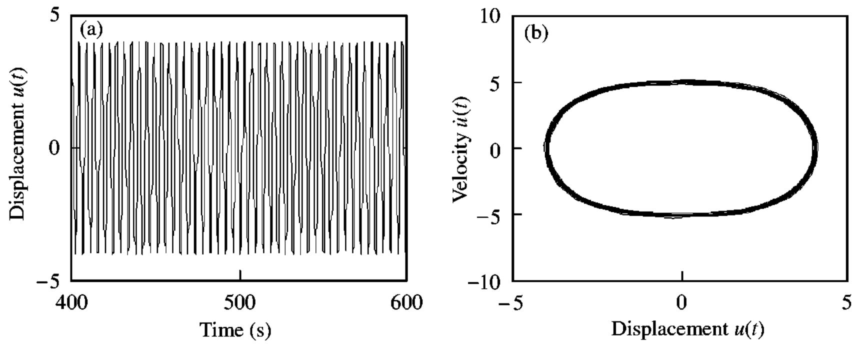

Figure 2. Numerical results of equation (1): $u(0) = -4{\cdot}0$, $\dot{u}(0) = -5{\cdot}5$: (a) time history of $u(t)$; (b) phase plot.

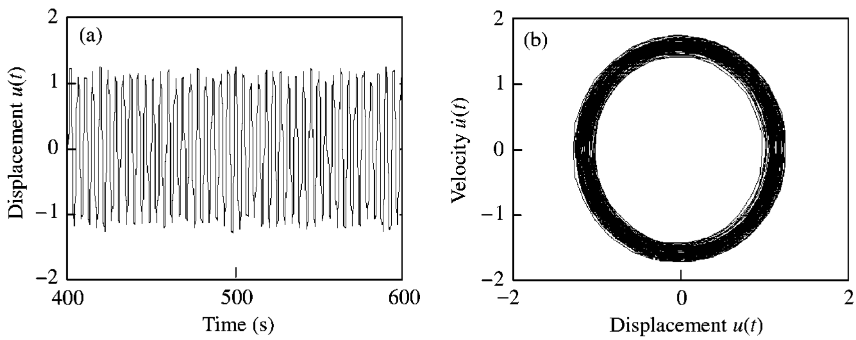

Figure 3. Numerical results of equation (1): $u(0) = -1{\cdot}0$, $\dot{u}(0) = -1{\cdot}5$: (a) time history of $u(t)$; (b) phase plot.

variations of the steady state response with Ω are shown in Figure 1, for comparison the theoretical results given by equation (5) are also shown in Figure 1.

Next, we determine the effect of the noise term $\xi(t)$ on the primary response. When $\Omega = 1{\cdot}4$, $S(\omega_0) = 0{\cdot}0025$, for different initial values, the numerical results of equation (1) are shown in Figures 2 and 3.

Figures 2 and 3 show that when $\xi(t)$ is small enough, in some parameter area of Ω, for different initial values the stationary displacement variances of the response of system (1) may be different. The random noise $\xi(t)$ will change the steady state response of system (1) from a limit cycle to a diffused limit cycle. Further numerical simulation shows that when the intensity of the random excitation increases, the width of the diffused limit cycle will increase.

When $S(\omega_0) = 0{\cdot}0025$, the variations of the steady state response with Ω are shown in Figure 4, for comparison the theoretical results given by equation (15) are also shown in Figure 4.

Further numerical simulation shows that when $\xi(t)$ is small enough, the multi-valuedness is responsible for a jump phenomenon. From equations (5), (6) and Figure 4, it can be shown that $\Omega_1 = 1{\cdot}34$, $\Omega_2 = 1{\cdot}80$ are bifurcation points of the steady state response of system (1). When $\Omega < \Omega_1$ or $\Omega > \Omega_2$, the steady state response of system (1) is unique; when $\Omega_1 < \Omega < \Omega_2$, the response has three branches among them only the highest and lowest ones are stable. Jump may occur at the points $\Omega = \Omega_1, \Omega_2$.

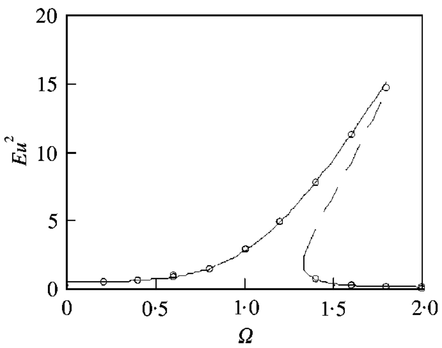

Figure 4. Frequency response of system (1): ——, stable solution; – – –, unstable solution; ∘∘∘ numerical solution.

4. CONCLUSION AND DISCUSSION

For the first time, the method of harmonic balance is used to analyze the response of a non-linear system under deterministic and random excitation. The method of stochastic averaging is used to analyze the effect of the random noise on the response of the Duffing oscillator. So far, exact solutions of non-linear system under random excitation are only available for a very limited number of problems. Thus, approximate methods have been developed and used to treat many of these problems. These include the method of equivalent or stochastic linearization, perturbation methods, stochastic averaging and series expansions, etc. The approximate methods in the determining the system can be extended to random system. For example, in recent year, Rajan and Davies [7], Nayfeh and Serhan [13] have extended the method of multiple scales to the analysis of non-linear systems under random external excitations, and the authors [14] extended this method to the analysis of non-linear systems under random parameter excitation. In this paper, for the first time we extend the method of harmonic balance to the analysis of the response of Duffing oscillator under combined deterministic harmonic and random excitation.

Theoretical analyses and numerical simulations show that when $\xi(t)$ is small enough, in some parameter area of Ω, for different initial values the stationary displacement variances of the response of system (1) may be different. The random noise $\xi(t)$ will change the steady state response of system (1) from a limit cycle to a diffused limit cycle. When the intensity of the random excitation increases, the width of the diffused limit cycle will increase. When $\xi(t)$ become large enough, the lowest branches of the stationary displacement response will lose their stability and the stationary response will be unique. These conclusions are in accordance with the physical instinct. When $\xi(t)$ is small enough, the deterministic harmonic term $h \cos \Omega t$ will play a decisive role in the response of system (1), so phenomenon of multiple-valued steady state response and jump can be observed in some parameter areas.

There are many models to describe the narrowband random excitation. The character of the narrowband random excitation in different models may be different, while some of them may be similar to the deterministic harmonic excitation. So it is possible that the steady state response of a Duffing oscillator under such kind of narrowband random will be multiple valued.

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