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SEMI-ANALYTICAL SHAPE FUNCTIONS IN THE FINITE ELEMENT ANALYSIS OF RECTANGULAR PLATES

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This paper presents a method for the dynamic analysis of a thin, elastic, isotropic rectangular plate. The method is a hybrid of finite element theory and classical thin plate theory. The displacement functions are derived from Sanders' thin-shell equations, and are expanded in power series. Expressions for mass and stiffness are determined by precise analytical integration. The free vibrations of rectangular plates, with various boundary conditions, are studied by following this method. The results obtained reveal that the frequencies calculated in this way are in good agreement with those obtained by others.

1. INTRODUCTION

Rectangular plates are perhaps the most widely used structural elements. They are used in such fields as civil and naval engineering, and in aeronautical and space technology. A knowledge of the free vibration characteristics of rectangular plates enables engineers to design better and lighter structures. For this reason, the behaviour of rectangular plates has been the subject of on-going research for more than a hundred years.

The first mathematical model of the behaviour of the plate membrane was formulated by Euler in the 18th century. More than 50 years later, Lagrange developed the first correct differential equation for the free vibration of plates. Some time later, Navier (1785–1836) produced a method of calculating the mode shape and the frequencies for certain boundary-value problems. He used the trigonometric series introduced by Fourier to express the deflection of the plate.

Kirchoff (1824–1887) is considered the founder of modern plate theory which, by analyzing plates with substantial deflection, takes into account both the bending and the stretching of the plates. He concluded that the non-linear effects should not be ignored when dealing with large deflections and that the natural frequencies and mode shapes can be determined by the virtual work method. Love [1] applied Kirchoff's work to thick plates.

More recently, Leissa [2] summarized the work of several researchers in a book containing more than 500 references. The needs of the modern aircraft industry have led to advances in the study of rectangular plates. In 1956, Turner *et al.* [3] introduced the finite element method, which permits the complex-plate problem to be formulated, and, with the advent of high-speed computers, a variety of numerical methods using matrix algebra have been developed. Zienkiewicz [4] contributed to the formulation of different kinds of finite elements. Bogner *et al.* [5, 6] worked on an element using bi-cubic interpolation functions to simulate the displacement of the plate.

In order to predict both low and high frequencies with precision, we used the finite element method with many elements and developed a hybrid finite element method which is derived from Sanders' classical shell theory [7]. Various elements have been developed for close and open cylindrical [8–15], conical [16] and spherical [17] shells in vacuum or containing a fluid at rest or in motion. Whilst several well-known finite element codes such as NASTRAN, ABAQUS or ANSYS can solve the free vibrations of a rectangular plate in vacuum relatively easily, none can correctly predict the natural frequencies of a plate submerged in fluid. We needed a general, thin, rectangular plate element that could later be used for fluid–structure interaction analysis. The formulation of an analytical solution for a general rectangular plate element is quite complex: we had to find displacement functions compatible with both the plate equations of motion and the solution of a plate in contact with fluid. We decided, therefore, to expand the homogeneous solution of the bi-harmonic equation of the plate into a power series and, in so doing, we obtained a semi-analytical solution in the form of a polynomial which can be used in classical finite element theory.

In this article, we discuss the development of this element and its relative accuracy in comparison with other methods. We first determined the fundamental equations of the plate and, secondly, derived the displacement functions of plate theory and expanded them in power series. With these displacement functions, we were able to determine the mass and stiffness matrices required by the finite element method and, therefore, the free vibration characteristics of the plate.

2. FUNDAMENTAL EQUATIONS FOR A THIN RECTANGULAR PLATE

2.1. EQUILIBRIUM EQUATIONS

To establish the equilibrium equations of the plate, we use Sanders' equations [7] for cylindrical shells and assume the radius of the shell to be infinite. There three equations take into account both membrane effects and bending effects. Sanders based his equations on Love's first approximation [1] for thin shell, and showed that all strains vanish for any rigid-body motion. For the finite element method this theory satisfies the convergence criteria for small rigid-body motions.

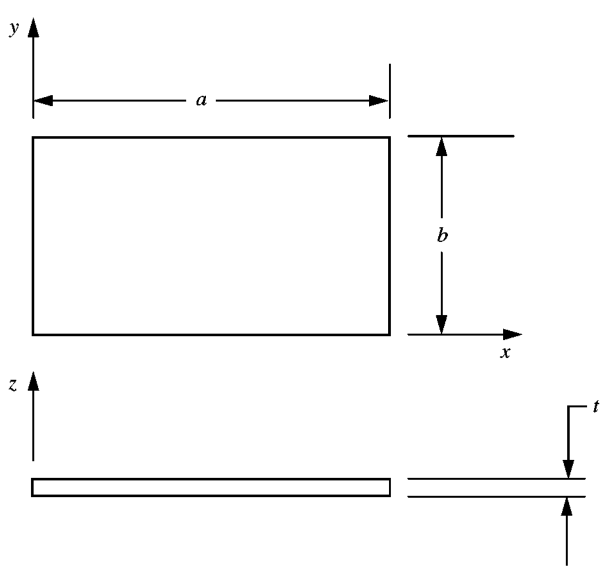

Figure 1. Geometry of rectangular plate's mean surface.

The geometry of the mean surface and the co-ordinate system used for this analysis are shown in Figure 1.

The equilibrium equations for a rectangular plate, following Sanders' theory, are written as

$$\frac{\partial \bar{N}_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0, \quad \frac{\partial N_{xx}}{\partial x} + \frac{\partial \bar{N}_{xy}}{\partial y} = 0, \quad \frac{\partial^2 M_{xx}}{\partial x^2} + 2\,\frac{\partial^2 \bar{M}_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} = 0, \tag{1}$$

where N_{xx}, N_{xy}, N_{yy}, M_{yy}, M_{xx} and M_{xy} are the force resultants (force per unit length of the middle surface) and x and y are the co-ordinates of the plate. The unit vectors corresponding to the stresses defined in equation (1) are indicated in Figure 2.

2.2. KINEMATICS EQUATIONS

The relation between the strain (ε) and the displacement for a rectangular plate is given as

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ 2\varepsilon_{xy} \\ \kappa_x \\ \kappa_y \\ \bar{\kappa}_{xy} \end{pmatrix} = \left\{ \begin{array}{c} \dfrac{\partial U}{\partial x} \\ \dfrac{\partial V}{\partial y} \\ \dfrac{\partial V}{\partial x} + \dfrac{\partial U}{\partial y} \\ -\dfrac{\partial^2 W}{\partial x^2} \\ -\dfrac{\partial^2 W}{\partial y^2} \\ -\dfrac{\partial^2 W}{\partial x \partial y} \end{array} \right\}, \tag{2}$$

where U and V are the in-plane displacements and W the deflection of the plate.

For an anisotropic and elastic material the relationship between stress and strain is

$$\{\sigma\} = [P]\{\varepsilon\}, \tag{3}$$

where $[P]$ is a 6×6 symmetric elasticity matrix. In the case of an isotropic material there is no coupling between membrane and bending effects, and the only non-vanishing terms are

$$P_{11} = P_{22} = D, \quad P_{44} = P_{55} = K, \quad P_{12} = P_{21} = vD, \quad P_{45} = P_{54} = vK,$$

$$P_{33} = \frac{(1-v)}{2} D, \quad P_{66} = \frac{(1-v)}{2} K. \tag{4}$$

where D and K are, respectively, the membrane and bending stiffness defined as

$$D = \frac{Et}{1-v^2}, \quad K = \frac{Et^3}{12(1-v^2)},$$

E being Young's modulus, v Poisson's ratio and t the plate thickness.

Next, we substitute equations (2) and (3) into the equilibrium equations to obtain the three equations of motion in terms of the in-plane and normal displacements of the plate's

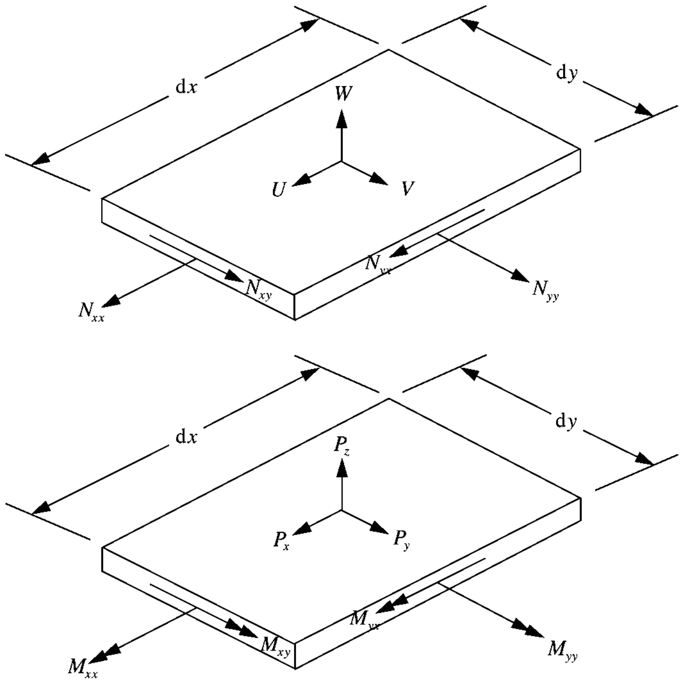

Figure 2. Differential element for a rectangular plate.

means surface (U, V, W):

$$P_{22}\frac{\partial^2 V}{\partial y^2} + P_{21}\frac{\partial^2 U}{\partial x \partial y} + P_{33}\left(\frac{\partial^2 U}{\partial x \partial y} + \frac{\partial^2 V}{\partial x^2}\right) = 0,$$

$$P_{11}\frac{\partial^2 U}{\partial x^2} + P_{12}\frac{\partial^2 V}{\partial x \partial y} + P_{33}\left(\frac{\partial^2 V}{\partial x \partial y} + \frac{\partial^2 U}{\partial y^2}\right) = 0, \tag{5}$$

$$P_{44}\frac{\partial^4 W}{\partial x^4} + \frac{\partial^4 W}{\partial x^2 \partial y^2}(P_{45} + P_{54} + 2P_{66}) + P_{55}\frac{\partial^4 W}{\partial y^4} = 0.$$

The first two equations (5) describe the membrane behaviour and the last equation defines the bending of a rectangular plate. By solving these equations, it is possible to find the displacement function in terms of the nodal displacements.

3. DISPLACEMENT FUNCTIONS

3.1. SOLUTION OF THE DIFFERENTIAL EQUATIONS

In this instance, we are dealing with the case of isotropic material. As can be seen in equation (5), the equations of motion are decoupled. It is possible, therefore, to consider the membrane and bending equations to be two different problems, each with its own solution.

The solution for the membrane differential equations in equation (5) is based on Szilard [18]. We assume the solution to be a bi-linear polynomial expressing the nodal displacements in U and V are, respectively, the in-plane displacement in the x and y directions as can be shown on an element in Figure 3.

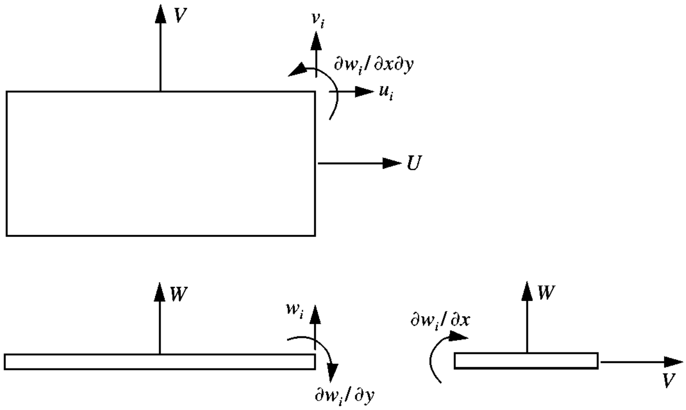

Figure 3. Displacements and degrees of freedom of a rectangular plate.

The polynomial expression will be

$$U(x, y) = C_1 + C_2\frac{x}{a} + C_3\frac{y}{b} + C_4\frac{xy}{ab},$$

$$V(x, y) = C_5 + C_6\frac{x}{a} + C_7\frac{y}{b} + C_8\frac{xy}{ab}, \tag{6}$$

where x and y are the element co-ordinate system and a and b are the length and width of the plate corresponding to the x and y co-ordinates. These assumed displacement functions contain the same number of unknown parameters C_i as the number of nodal displacements ($2\times 4 = 8$). The solution is rather crude but converges monotonically almost to the "exact" value for the problem of finding the maximum deflection [18].

In the case of bending, the bi-harmonic equation has a general solution of the following form:

$$W(x, y) = \sum_{i=1}^{n} C_i\,\mathrm{e}^{(x/a)+(y/b)}, \tag{7}$$

where W is the normal deflection of the plate element shown in Figure 3. Since it is a complex matter to find the characteristic equation, we expand the solution in a Taylor series. The number of terms in the series remains to be determined. Furthermore, the number of degrees of freedom describing the motion of the plate in its normal direction is governed by the number of terms in the series. Therefore, we add as many terms as the hermitian bi-cubic polynomial used by Bogner [6]. The expanded polynomial, which approximates the normal deflection of the element, is

$$W_p(x, y) = \frac{A_1}{36}\frac{x^3y^3}{ab} + \frac{A_2}{12}\frac{x^3y^2}{ab} + \frac{A_3}{6}\frac{x^3y}{ab} + \frac{A_4}{6}\frac{x^3}{a} + \frac{A_5}{12}\frac{x^2y^3}{ab} + \frac{A_6}{4}\frac{x^2y^2}{ab}$$
$$+ \frac{A_7}{2}\frac{x^2y}{ab} + \frac{A_8}{2}\frac{x^2}{a} + \frac{A_9}{6}\frac{xy^3}{ab} + \frac{A_{10}}{2}\frac{xy^2}{ab} + A_{11}\frac{xy}{ab} + A_{12}\frac{x}{a} + \frac{A_{13}}{6}\frac{y^3}{b}$$
$$+ \frac{A_{14}}{2}\frac{y^2}{b} + \frac{A_{15}}{6}\frac{y}{b} + A_{16}. \tag{8}$$

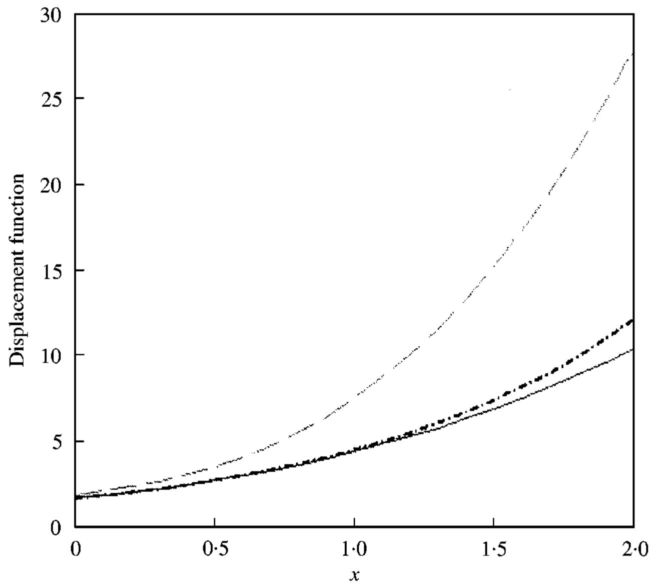

Figure 4. Comparison of displacement functions; equation (8), $W_p(x, y)$: (——), Bogner [8]; $W_b(x, y)$: (— —); and the exact solution, equation (7), $W(x, y)$: (– · · –).

Equation (8) gives 16 unknown parameters A_i corresponding, again, to the number of degrees of freedom per element for bending. Figure 3 shows the nodal degrees of freedom $(W, \partial W/\partial x, \partial W/\partial y, \partial^2 W/\partial x \partial y)$ which relate to the bending motion. Instead of a rotational degree of freedom about the z-axis, we have the second derivative of W, which gives the twisting strain, and ensures a continuity of slope between the elements. This gives conforming and compatible elements in bending.

Furthermore, when using a power series to express the displacement function, we approach the "exact" solution of the bending equation more closely than Bogner *et al.* [5, 6] did with the bi-cubic polynomial. The two displacement functions are compared to the exact equation (7) in Figure 4.

3.2. DISPLACEMENT FUNCTIONS FOR A FINITE ELEMENT

We can write the displacement, U, V and W in matrix form,

$$\begin{Bmatrix} U \\ V \\ W \end{Bmatrix} = [R]\,\{C\}, \tag{9}$$

where $[R]$ is a 3×24 matrix in which the components are the x and y terms of equations (6) and (8) without the unknown constants. The vector $\{C\}$ is given by

$$\{C\} = \{C_1, \ldots, C_{24}\}^t. \tag{10}$$

To determine these constants, we need to define 24 boundary conditions for the finite element. These 24 boundary conditions will be the 24-degrees-of-freedom per element, which means 6-degrees-of-freedom per node as follows:

$$\{\{\delta_1\}, \{\delta_2\}, \{\delta_3\}, \{\delta_4\}\}^{\mathrm{T}} = \{u_1, v_1, w_1, w_{1,x}, w_{1,y}, w_{1,xy}, u_2, v_2, w_2, w_{2,x}, w_{2,y}, w_{2,xy}, u_3, v_3, w_3, w_{3,x}, w_{3,y}, w_{3,xy}, u_4, v_4, w_4, w_{4,x}, w_{4,y}, w_{4,xy}, \quad (11)$$

where the δ_i's are the generalized nodal displacement and $w_{i,x}$ is the derivative of w_i with respect to x and so on. Then, we have to define a transformation matrix $[A]$ to relate the displacement functions $\{C_i\}$ and the nodal displacements $\{\delta_i\}$:

$$\{\delta_i\} = [A]\{C_i\}. \quad (12)$$

$[A]$ is a 24×24 matrix listed in Appendix A. The terms of matrix $[A]$ are obtained from matrix $[R]$ by going from node 1 to 4 and setting the value of x to 0 or a, and of y to 0 or b. By multiplying equation (12) by $[A]^{-1}$ and substituting into equation (9) we obtain

$$\begin{Bmatrix} U \\ V \\ W \end{Bmatrix} = [R] \quad [A]^{-1} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix} = [N] \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix}, \quad (13)$$

where $[N]$ is the displacement function matrix for a finite element of the rectangular plate.

4. STRESS AND STRAIN VECTORS

The strain vector can be found by using equations (2) and (13):

$$\{\varepsilon\} = [D] \; [R] \; [A^{-1}] \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix} = [B] \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix}, \quad (14)$$

where $[D]$ is a matrix containing the derivative operators from equation (2). After defining the strain vector, we can use it and refer to equation (3) for the stress vector:

$$\{\sigma\} = [P] \; \{\varepsilon\} = [P] \; [B] \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix}. \quad (15)$$

5. MASS AND STIFFNESS MATRICES FOR ONE FINITE ELEMENT

Using the finite element theory [4], the mass and stiffness matrices may be expressed as

$$[m] = \rho t \int_0^a \int_0^b [N]^{\mathrm{T}} \; [N] \, \mathrm{d}A, \quad [k] = \int_0^a \int_0^b [B]^{\mathrm{T}} [P] \; [B] \, \mathrm{d}A, \quad (16, 17)$$

where $\mathrm{d}A = \mathrm{d}y \, \mathrm{d}x$. The matrices $[N]$, $[P]$ and $[B]$ are given in equations (13), (14) and (3). Integrating equations (16) and (17) over x and y, we obtain

$$[m] = [A^{-1}][S][A^{-1}], \quad [k] = [A^{-1}][G][A^{-1}], \quad (18, 19)$$

TABLE 1

Constants and exponents for symmetric submatrices

i	j	A_{ij}	B_{ij}	$E_{ij}^{(1)}$	$E_{ij}^{(2)}$	$E_{ij}^{(3)}$	$E_{ij}^{(4)}$	$E_{ij}^{(5)}$	$E_{ij}^{(6)}$	$E_{ij}^{(7)}$	$E_{ij}^{(8)}$	L_{ij}	M_{ij}	i	j	B_{ij}	$E_{ij}^{(5)}$	$E_{ij}^{(6)}$	$E_{ij}^{(7)}$	$E_{ij}^{(8)}$	L_{ij}	M_{ij}
1	1	1/9	1/63504	1/3	1/3	1/4	1/4	1/756	1/400	1/450	1/756	2	2	5	15	1/180	0	0	0	0		
1	2	1/6	1/18144	1/2	0	0	1/2	1/504	1/160	1/180	1/216	2	2	5	16	1/144	0	0	0	0		
1	3	1/6	1/7560	0	1/2	1/2	0	0	1/120	1/8	1/90	0	0	6	6	1/400	1/20	1/9	1/18	1/20	2	2
1	4	1/4	1/6048	0	0	0	0	0	0	1/60	1/72		2	6	7	1/160	0	1/6	1/12	1/8		2
1	5		1/18144					1/216	1/160	1/180	1/504	2	2	6	8	1/120	0	0	1/6	1/6		2
1	6		1/5184					1/144	1/64	1/96	1/144	2	2	6	9	1/576	1/16	1/16	1/16	0	2	
1	7		1/2160					0	1/48	1/72	1/60	0	2	6	10	1/160	1/8	1/6	1/12	0	2	
1	8		1/1728					0	0	1/48	1/48		2	6	11	1/64	0	1/4	0	0		
1	9		1/7560					1/90	1/120	1/90	0	2		6	12	1/48	0	0	0	0		
1	10		1/2160					1/60	1/48	1/72	0	2		6	13	1/432	1/12	0	1/8	0	2	
1	11		1/900					0	1/36	0	0			6	14	1/120	1/6	0	1/6	0	2	
1	12		1/720					0	0	0	0			6	15	1/48	0	0	0	0		
1	13		1/6048					1/72	0	1/60	0	2		6	16	1/36	0	0	0	0		
1	14		1/1728					1/48	0	1/48	0	2		7	7	1/60	0	1/3	0	1/3		2
1	15		1/720					0	0	0	0			7	8	1/40	0	0	0	1/2		2
1	16		1/576					0	0	0	0			7	9	1/240	0	1/12	1/6	0		
2	2	1/3	1/5040	1	0	0	0	1/252	1/60	1/90	1/60	2	2	7	10	1/64	0	1/4	1/4	0		
2	3	1/4	1/2016	0	0	1	0	0	1/40	1/60	1/20	0	2	7	11	1/24	0	1/2	0	0		
2	4	1/2	1/1512	0	0	0	0	0	0	1/30	1/18	0	2	7	12	1/16	0	0	0	0		
2	5		1/5184					1/144	1/96	10/576	1/144	2	2	7	13	1/180	0	0	1/3	0		
2	6		1/1440					1/72	1/24	1/36	1/40	2	2	7	14	1/48	0	0	1/2	0		
2	7		1/576					0	1/16	1/48	1/16		2	7	15	1/18	0	0	0	0		
2	8		1/432					0	0	1/24	1/12		2	7	16	1/12	0	0	0	0		
2	9		1/2160					1/60	1/48	1/24	0	2		8	8	1/20		0	0	1		2
2	10		1/600					1/30	1/18	1/18	0	2		8	9	1/192	0	0	1/4	0		
2	11		1/240					0	1/12	0	0			8	10	1/48	0	0	1/2	0		
2	12		1/180					0	0	0	0			8	11	1/16	0	0	0	0		
2	13		1/1728					1/48	0	1/16	0	2		8	12	1/8	0	0	0	0		
2	14		1/480					1/24	0	1/12	0	2		8	13	1/144	0	0	1/2	0		
2	15		1/192					0	0	0	0			8	14	1/36	0	0	1	0		
2	16		1/144					0	0	0	0			8	15	1/12	0	0	0	0		
3	3	1/3	1/756	0	1	0	0	0	1/20	0	1/9		2	8	16	1/6	0	0	0	0		

3	4	1/2	1/504	0	0	0	0	0	0	0	1/6		2	9	9	1/756	1/9	1/20	0	0	2
3	5		1/2160					0	1/48	1/24	1/60		2	9	10	1/216	1/6	1/8	0	0	2
3	6		1/576					0	1/16	1/16	1/16		2	9	11	1/90	0	1/6	0	0	
3	7		1/216					0	1/8	0	1/6		2	9	12	1/72	0	0	0	0	
3	8		1/144					0	0	0	1/4		2	9	13	1/504	1/6	0	0	0	2
3	9		1/900					0	1/36	1/9	0			9	14	1/144	1/4	0	0	0	2
3	10		1/240					0	1/12	1/6	0			9	15	1/60	0	0	0	0	
3	11		1/90					0	1/6	0	0			9	16	1/48	0	0	0	0	
3	12		1/60					0	0	0	0			10	10	1/60	1/3	1/3	0	0	2
3	13		1/720					0	0	1/6	0			10	11	1/24	0	1/2	0	0	2
3	14		1/192					0	0	1/4	0			10	12	1/18	0	0	0	0	
3	15		1/72					0	0	0	0			10	13	1/144	1/4	0	0	0	2
3	16		1/48					0	0	0	0			10	14	1/40	1/2	0	0	0	2
4	4	1	1/252	0	0	0	0	0	0	0	1/3		2	10	15	1/16	0	0	0	0	
4	5		1/1728					0	0	1/16	1/48		2	10	16	1/12	0	0	0	0	
4	6		1/432					0	0	1/8	1/12		2	11	11	1/9	0	0	0	0	
4	7		1/144					0	0	0	1/4		2	11	12	1/6	0	0	0	0	
4	8		1/72					0	0	0	1/2		2	11	13	1/60	0	0	0	0	
4	9		1/720					0	0	1/6	0			11	14	1/16	0	0	0	0	
4	10		1/180					0	0	1/3	0			11	15	1/6	0	0	0	0	
4	11		1/60					0	0	0	0			11	16	1/4	0	0	0	0	
4	12		1/30					0	0	0	0			12	12	1/3	0	0	0	0	
4	13		1/576					0	0	1/4	0			12	13	1/48	0	0	0	0	
4	14		1/144					0	0	1/2	0			12	14	1/12	0	0	0	0	
4	15		1/48					0	0	0	0			12	15	1/4	0	0	0	0	
4	16		1/24					0	0	0	0			12	16	1/2	0	0	0	0	
5	5		1/5040					1/60	1/60	1/252	1/90	2	2	13	13	1/252	1/3	0	0	0	2
5	6		1/1440					1/40	1/24	1/36	1/72	2	2	13	14	1/72	1/2	0	0	0	2
5	7		1/600					0	1/18	1/18	1/30		2	13	15	1/30	0	0	0	0	
5	8		1/480					0	0	1/12	1/24		2	13	16	1/24		0	0	0	
5	9		1/2016					1/20	1/40	1/60	0	2		14	14	1/20	1	0	0	0	2
5	10		1/576					1/16	1/16	1/48	0	2		14	15	1/8	0	0	0	0	
5	11		1/240					0	1/12	0	0			14	16	1/6	0	0	0	0	
5	12		1/192					0	0	0	0			15	15	1/3	0	0	0	0	
5	13		1/1512					1/18	0	1/30	0	2		15	16	1/2	0	0	0	0	
5	14		1/432					1/12	0	1/24	0	2		16	16	1	0	0	0	0	

where $[S]$ and $[G]$ are 24×24 real symmetrical matrices. For the mass matrix, $[S]$ is partitioned as

$$[S]=\begin{bmatrix} S^{(u)}_{4\times 4} & \vdots & 0 & \vdots & 0 \\ \cdots & \vdots & \cdots & \vdots & \cdots \\ 0 & \vdots & S^{(v)}_{4\times 4} & \vdots & 0 \\ \cdots & \vdots & \cdots & \vdots & \cdots \\ 0 & \vdots & 0 & \vdots & S^{(w)}_{16\times 16} \end{bmatrix}. \tag{20}$$

The elements of the symmetrical submatrices are given by

$$S^{(u)}_{ij}=S^{(v)}_{ij}=A_{ij}\,a\,b,\quad S^{(w)}_{ij}=B_{ij}\,a\,b, \tag{21}$$

where the constants A_{ij} and B_{ij} are found in Table 1.

In the case of the stiffness matrix, $[G]$ is partitioned as

$$[G]=\begin{bmatrix} G^{(u)}_{4\times 4} & \vdots & G^{(uv)}_{4\times 4} & \vdots & 0 \\ \cdots & \vdots & \cdots & \vdots & \cdots \\ G^{(vu)}_{4\times 4} & \vdots & G^{(v)}_{4\times 4} & \vdots & 0 \\ \cdots & \vdots & \cdots & \vdots & \cdots \\ 0 & \vdots & 0 & \vdots & G^{(w)}_{16\times 16} \end{bmatrix}. \tag{22}$$

We can see from this matrix that a rectangular plate element has a coupling between u and v in the membrane part and no coupling between membrane and bending for an isotropic material. The elements of the submatrices are given by equations (23), where the constants $E^{(k)}_{ij}$, $k=1,2,\dots,8$, and the exponents L_{ij}, M_{ij} are given in Table 1:

$$G^{(u)}_{ij}=\frac{D}{ab}\left(E^{(1)}_{ij}b^2+(1-v)a^2E^{(2)}_{ij}\right),\quad G^{(vu)}_{ij}=D\left(E^{(3)}_{ij}v+(1-v)E^{(4)}_{ij}\right),$$

$$G^{(v)}_{ij}=\frac{D}{ab}\left(E^{(2)}_{ij}b^2+(1-v)a^2E^{(1)}_{ij}\right),\quad G^{(uv)}_{ij}=D\left(E^{(4)}_{ij}v+(1-v)E^{(3)}_{ij}\right),$$

$$G^{(w)}_{ij}=\frac{K}{ab}\left(E^{(5)}_{ij}\left(\frac{a}{b}\right)^{L_{ij}}+E^{(6)}_{ij}\,\frac{(1-v)}{2}+E^{(7)}_{ij}v+E^{(8)}_{ij}\left(\frac{b}{a}\right)^{M_{ij}}\right). \tag{23}$$

6. CALCULATION AND DISCUSSION

6.1. FREE VIBRATIONS

The complete plate is subdivided into finite elements, each of which is a smaller rectangular plate. The positions of the nodal points are chosen in such a way that the local co-ordinate system of the element is parallel to the global co-ordinate system of the plate.

Once the stiffness and the mass matrices have been obtained it is possible to construct the global matrices for the complete plate using the finite element assembly technique. If N is the number of nodes then $[M]$ and $[K]$ are two matrices of order $6N$. In the case of free vibration, the equations of motion are

$$[M]\{\ddot{\delta}_T\}+[K]\{\delta_T\}=\{0\}, \tag{24}$$

where $\{\delta\}_T$ is the vector for global displacement of the whole shell:

$$\{\delta_T\} = \{\{\delta_1\}, \{\delta_2\}, \dots, \{\delta_N\}\}^t, \tag{25}$$

N being the number of nodes. We specify

$$\{\delta_T\} = \{\delta_{0,T}\} \sin(\omega t + \phi), \tag{26}$$

where ϕ is the natural angular frequency and P is the phase angle.

By introducing equations (24) and (25), we obtain the typical eigenvalue and eigenvector problem:

$$\det[[K] - \omega^2 [M]] = 0. \tag{27}$$

We have proven in earlier sections that the scattering equations are decoupled from the bending equations. For this reason, the solution of equation (27) gives us both the bending and in-plane modes. The shape of the eigenvector for each mode will permit us to differentiate the bending modes from the in-plane modes.

6.2. CONVERGENCE

The accuracy of the finite element method depends on the number of elements used to discretize the physical problem. A preliminary set of calculations was undertaken to determine the requisite number of finite elements for an accurate determination of the natural frequencies. Calculations were made with a rectangular steel plate having the following properties: $a = 609{\cdot}6$ mm, $b = 304{\cdot}8$ mm, $t = 2{\cdot}54$ mm, $E = 196 \times 10^9$ N/m^2, $\nu = 0{\cdot}3$, $\rho = 7{\cdot}86$ kg/m^3 and with the number of elements $N = 1, 2, 4, 8, 16, 32, 64$. The boundary conditions were the result of the plate being simply supported on all edges. The results for the first six natural frequencies are given in Figure 5. We conclude that the convergence of the system is fast. About eight elements are needed for convergence. For convergence at higher frequencies, a larger number of elements must be used. The reason for this is simple: since we are using polynomials to represent the mode shapes, we need more degrees of freedom, and hence a greater number of elements, to have a satisfactory representation of the higher mode shapes.

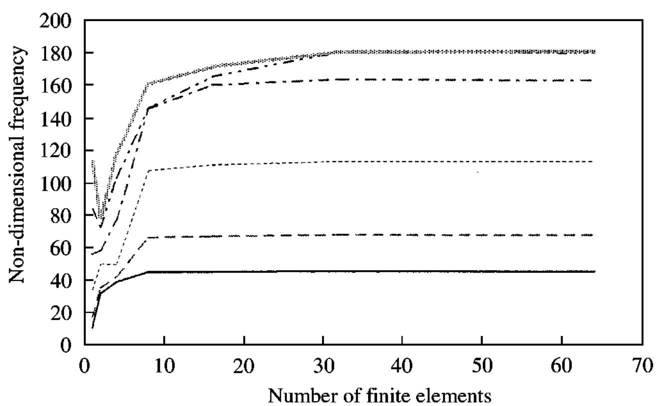

Figure 5. Non-dimensional natural frequency, $\Omega = \omega\, a^2 (\rho t/K)^{1/2}$, of a simply supported rectangular plate as a function of the number of finite elements for the first six modes; m, n: (—), 1,1; (- - -), 2,1; (· · · ·), 3,1; (– · –), 1,2; (– · · –), 2,2; (· · · · · ·), 4,1.

6.3. CALCULATIONS FOR RECTANGULAR PLATES

The eigenvalues of a uniform rectangular plate with different boundary conditions may be calculated in a simpler way. In fact, Leissa [2] gives a good summary and all the tables needed to solve the kind of problems discussed here. Our main aim is to test the validity of the mass and stiffness matrices as developed in this paper.

We first determine the natural frequencies of the rectangular plate and compare that calculation to its exact solution. This comparison enables us to give the relative accuracy of the method for 64 elements. Figure 6 gives the first six mode shapes and natural frequencies computed. By looking at the deformed shape, we can tell the number of axial modes in both directions where m is the longest side (x-axis) and n is the shortest side (y-axis). The error between the finite element model and the exact solution is given in Table 2.

Table 2 shows fairly good results for the finite element method as compared with the exact solution. The error varies from 3 to 15% depending on the mode. As can be seen, the

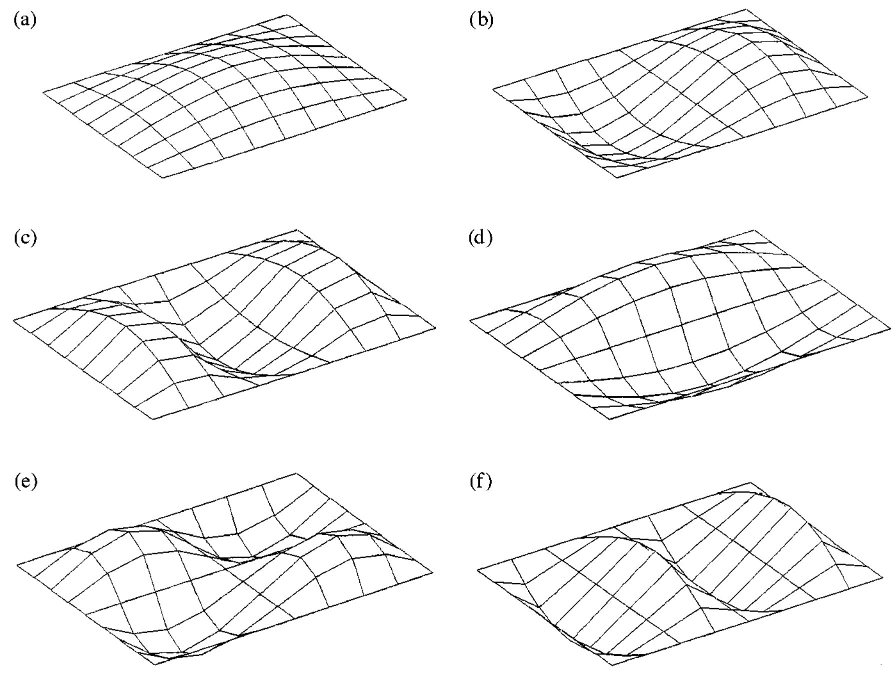

Figure 6. Computed mode shapes of the simply supported plate: (a) $f_{1,1} = 76{\cdot}16$ *Hz*, (*b*) $f_{2,1} = 114{\cdot}69$ Hz, (c) $f_{3,1} = 191{\cdot}27$ Hz, (d) $f_{1,2} = 275{\cdot}44$ Hz, (e) $f_{2,2} = 303{\cdot}94$ Hz, (f) $f_{4,1} = 305{\cdot}19$ Hz.

TABLE 2

Relative error between the exact solution of a simply supported rectangular plate and a 64 finite element model

m, n	1, 1	2, 1	3, 1	1, 2	2, 2	4, 1
Exact solution; f (Hz)	83·50	133·61	217·12	283·9	334·0	334·0
Our method; f (Hz)	76·16	114·69	191·27	275·44	303·94	305·19
% error	8·79	14·16	11·91	2·98	9·00	8·63

TABLE 3

Comparison of a cantilever rectangular steel plate natural frequencies

		Frequency in Hz, for values of m of		
n	Type	1	2	3
1	Theoretical [19]	69·5	436	1220
	NASTRAN	67·6	416·4	1151
	Experimental [20]	64	405	1120
	Our method	67·58	420·62	1170·41
3	Theoretical [19]	1610	2260	
	NASTRAN	1474	1996	
	Experimental [20]	1606	ND	
	Our method	1532.91	1798.72	

relative error does not increase as we increase the order of the mode. This could be explained by the fact that we use a high order polynomial which approximates some mode shapes more exactly than others. The error will now rely more on the ability of the shape function to represent each mode shape than the number of elements or the order of the mode.

We used a second set of calculations in order to compare our method with experimental values and other numerical methods. The calculations were carried out using two different boundary conditions: clamped on the shortest side and simply supported on two opposite edges. As there is no exact solution to the problem of the cantilevered plate, we verified the performance of our method against that of other numerical solutions. Results were compared to the solution computed using a quadrilateral element with linear shape functions of NASTRAN. A solution was also obtained by Martin [18], who used a variational procedure similar to the Rayleigh–Ritz method for a cantilevered rectangular steel plate of dimensions: $a = 130{\cdot}0$ mm, $b = 70{\cdot}1$ mm and $t = 1{\cdot}35$ mm. Our results are shown in Table 3 and compared to those of Martin [19], NASTRAN and to the experimental data of Grinsted [20]. Figure 7 shows the associated eigenvectors.

We calculated the natural frequencies of the cantilevered plate using an 8×8 element model. As can be seen, the results obtained by this method are in satisfactory agreement with those obtained using the other theory and with the experimental results. The natural frequencies and mode shapes of the rectangular steel plate simply supported on the two opposite shortest sides were also calculated. Since there is an analytical and an “exact” solution to the problem, the analysis increases our confidence in the calculation of the symmetrical model. To do this, we analyzed a steel plate which has the following dimensions: $a = 609{\cdot}6$ mm, $b = 304{\cdot}8$ mm and $t = 2{\cdot}54$ mm. The results obtained by our method were calculated using an 8×8 element model and are compared to the analytical solution in Table 4. As can be seen, all the modes are computed with relatively good accuracy.

7. CONCLUSION

The objective of this paper was to present a new method for deriving the displacement functions of a thin rectangular plate and, subsequently, to use these displacement functions

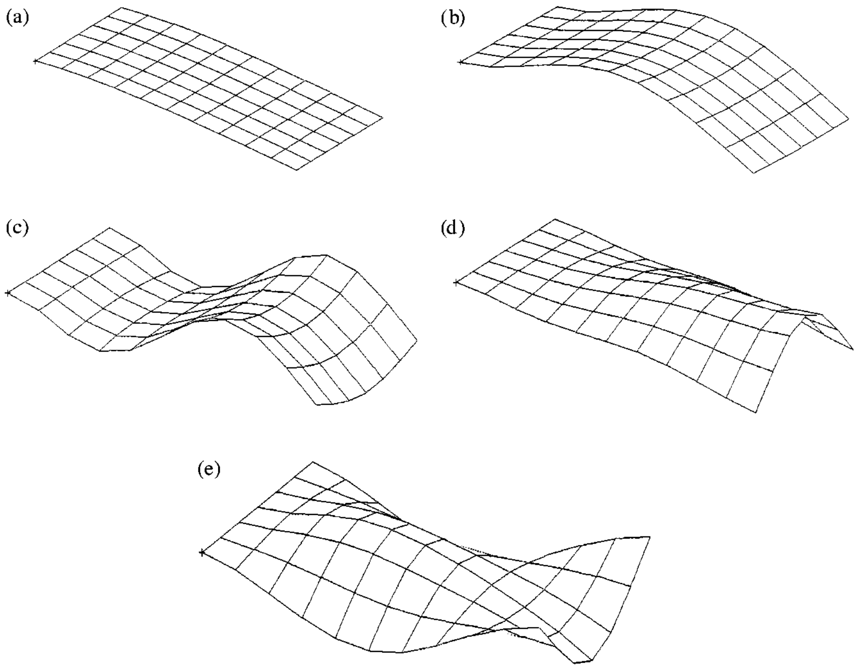

Figure 7. Computed mode shapes of the cantilever plate: (a) $m = 1$, $n = 1$; (b) $m = 2$, $n = 1$; (c) $m = 3$, $n = 1$; (d) $m = 1$, $n = 3$; (e) $m = 2$, $n = 3$.

TABLE 4

Natural frequencies (In Hz) for a rectangular steel plate simply supported on opposite edges calculated numerically and analytically

m, n	1, 1	2, 1	3, 1	1, 3
Analytical solution	16·7	66·8	150·3	178·5
Our method	15·98	64·37	145·86	161·30
% error	4·31	3·64	2·95	9·636

in dynamic analysis and fluid–structure interaction. The mass and stiffness matrices of a 24-degrees-of-freedom rectangular element were developed.

The convergence of the method was established and the natural frequencies were obtained for various boundary conditions and different modes. These results were compared with those of other authors and theories and satisfactory agreement was found. The advantage of this theory is that it is capable of modelling any non-uniform rectangular plates subjected to any boundary conditions without any new investigations. It is true that some results often give errors larger than 10%, but this is a numerical method which may execute real existing cases and its main advantage compared to analytical method is its flexibility. Our method has been tested with commercial FEM codes for the case of rectangular plate in vacuum.

This method combines the advantage of finite element analysis and the precision of a formulation which uses displacement functions derived from this plate theory.

A paper currently under preparation will deal with the dynamics of rectangular plates submerged in fluid. A more general quadrilateral element will be used and further investigation will be done on the displacement function in order to predict the natural frequencies of anti-symmetrical modes.

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APPENDIX A: MATRIX DEFINITION

The matrix $[A]$ is defined by

$$
\begin{bmatrix}
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{a}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{b}&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{a\cdot b}&0&0&0&0&0\\
0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&\frac{3}{a}&0&0&0&\frac{2}{a}&0&0&0&\frac{1}{a}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&\frac{1}{b}&0&0&0&\frac{1}{b}&0&0&0&\frac{1}{b}&0&0&0&\frac{1}{b}&0\\
0&0&0&0&0&0&0&0&0&0&\frac{3}{a\cdot b}&0&0&0&\frac{2}{a\cdot b}&0&0&0&\frac{1}{a\cdot b}&0&0&0&0&0\\
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&\frac{3}{a}&\frac{3}{a}&\frac{3}{a}&\frac{3}{a}&\frac{2}{a}&\frac{2}{a}&\frac{2}{a}&\frac{2}{a}&\frac{1}{a}&\frac{1}{a}&\frac{1}{a}&\frac{1}{a}&0&0&0&0\\
0&0&0&0&0&0&0&0&\frac{3}{b}&\frac{2}{b}&\frac{1}{b}&0&\frac{3}{b}&\frac{2}{b}&\frac{1}{b}&0&\frac{3}{b}&\frac{2}{b}&\frac{3}{b}&0&\frac{3}{b}&\frac{2}{b}&\frac{1}{b}&0\\
0&0&0&0&0&0&0&0&\frac{9}{a\cdot b}&\frac{6}{a\cdot b}&\frac{3}{a\cdot b}&0&\frac{6}{a\cdot b}&\frac{4}{a\cdot b}&\frac{2}{a\cdot b}&0&\frac{3}{a\cdot b}&\frac{2}{a\cdot b}&\frac{1}{a\cdot b}&0&0&0&0&0\\
0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{a}&\frac{1}{a}&\frac{1}{a}&\frac{1}{a}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{3}{b}&\frac{2}{b}&\frac{1}{b}&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{3}{a\cdot b}&\frac{2}{a\cdot b}&\frac{1}{a\cdot b}&0&0&0&0&0
\end{bmatrix}
$$

APPENDIX B: NOMENCLATURE

a	length of the rectangular plate
[A]	defined by equation (12)
A_i	defined by equation (8), $i = 0, 1, 2, 3$
b	width of the rectangular plate
[B]	defined by equation (14)
[C]	defined by equation (10)
C_i	defined by equation (6)
D	membrane stiffness, $Et/(1 - v^2)$
[D]	defined by equation (14)
E	Young's modulus
$E^{(k)}_{i,j}$	defined in equation (23), $i = 1, 2, \dots, 16; j = 1, 2, \dots, 16$
$f_{i,j}$	natural frequency (Hz)
[G]	defined by equation (22)
$G^{(u)}_{i,j}$, $G^{(v)}_{i,j}$, $G^{(w)}_{i,j}$	defined by equation (23), $i = 1, 2, \dots, 16; j = 1, 2, \dots, 16$
[k]	stiffness matrix of one element
K	bending stiffness, $Et^3/12(1 - v^2)$
[K]	stiffness matrix of the total plate
$L_{i,j}$	defined in equation (23), $i = 1, 2, \dots, 16; j = 1, 2, \dots, 16$
[m]	mass matrix of one element
m	axial mode number parallel to the x-axis
[M]	mass matrix of the total plate
$M_{i,j}$	defined in equation (23), $i = 1, 2, \dots, 16; j = 1, 2, \dots, 16$
$M_{x,x}$, $\bar{M}_{xy}$, $M_{y,y}$	bending moments of a rectangular plate
n	axial mode number parallel to the y-axis
N	number of finite elements
[N]	defined by equation (13)
$N_{x,x}$, $N_{x,y}$, N_y	stress components of a rectangular plate
[P]	elasticity matrix
$P_{i,j}$	terms of the elasticity matrix, $i = 1, 2, \dots, 6; j = 1, 2, \dots, 6$
[S]	defined by equation (20)
$S^{(u)}_{i,j}$, $S^{(v)}_{i,j}$, $S^{(w)}_{i,j}$	defined by equation (21), $i = 1, 2, \dots, 16; j = 1, 2, \dots, 16$
t	thickness of the rectangular plate
u_i, v_i, w_i	nodal displacements, $i = 1, \dots, 4$
U, V, W	in-plane and normal displacement of a rectangular plate
x, y	length and width co-ordinate of the plate
$w_{i,x}$, $w_{i,y}$, $w_{i,xy}$	nodal rotations and twisting, $i = 1, \dots, 4$
W_p	displacement function defined by equation (8)

Greek letters

δ_i	degree of freedom at node i, $i = 1, \dots, 4$
$\{\delta_i\}$	degrees of freedom at node i
$\{\delta_T\}$	degrees of freedom for the total plate
$\{\delta_{0,T}\}$	amplitude of the plate motion
$\{\varepsilon\}$	strain vector
ε_x, ε_y, ε_{xy}	deformations of the plate reference surface
κ_x, κ_y, κ_{xy}	changes in curvature and twisting of the plate reference surface
ρ	density of the plate material
$\{\sigma\}$	stress vector
v	Poisson's ratio
ω	angular natural frequency, rad/s^{-1}
Ω	non-dimensional frequency, $\omega a^2 \, (\rho t/K)^{1/2}$
Ψ	phase angle