Journal of Sound and Vibration (2001) **242**(3), 542–552
doi:10.1006/jsvi.2000.3370, available online at http://www.idealibrary.com on IDEAL®

ON THE EXISTENCE OF WEAK SOLUTIONS IN THE STUDY OF ANISOTROPIC PLATES

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(*Received* 22 *November* 1999, *and in final form* 24 *August* 2000)

1. INTRODUCTION

Differential equations and boundary conditions which describe physical phenomena are often obtained from physical principles by means of the variational calculus techniques. The necessary conditions for the existence of extremes of a functional lead to the Euler differential equation which involves unnecessary derivatives of higher order than the order of the derivatives included in the functional. Since this functional describes a certain type of energy, it is more natural, from a physical point of view, to look for a weak solution of the problem under consideration than to find its classical solution which does not exist for many common industrial problems.

A weak solution of a boundary value or eigenvalue problem may be obtained, under rather natural assumptions for the parameters of the problem, by variational methods. Let G be a domain in R^2 with a piecewise smooth boundary $\Gamma = \partial G$ and the operator

$$Au = \sum_{|\alpha|,|\beta| \leqslant 2} (-1)^{|\alpha|} D^{\alpha}(a_{\alpha\beta}(x) D^{\beta} u(x)) = \sum_{|\alpha|,|\beta| \leqslant 2} (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2}} \left(a_{\alpha\beta}(x) \frac{\partial^{|\beta|} u(x)}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}} \right), \qquad (1)$$

where $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$ are multi-index vectors whose co-ordinates are non-negative integers and $|\alpha|$, $|\beta|$ are the sums $|\alpha| = \alpha_1 + \alpha_2$, $|\beta| = \beta_1 + \beta_2$.

Let $a_{\alpha\beta}(x) \in C^{(|\alpha|)}(G), u(x) \in C^{(4)}(G)$. The summation in equation (1) is carried over all the vectors α and β for which $|\alpha|$, $|\beta| \leqslant 2$.

The equations which govern the statical and dynamical behaviour of isotropic, orthotropic and anisotropic plates with complicating effects are associated with operator A, as very particular cases.

Thus, if we define

$$a_{\alpha\beta} = \begin{cases} D_{11} & \text{when } \ \alpha = (2,0), \beta = (2,0), \\ 2(D_{12} + 2D_{66}) & \text{when } \ \alpha = (2,0), \beta = (0,2), \\ 4D_{16} & \text{when } \ \alpha = (2,0), \beta = (1,1), \\ D_{22} & \text{when } \ \alpha = (0,2), \beta = (0,2), \\ 4D_{26} & \text{when } \ \alpha = (0,2), \beta = (1,1). \end{cases}$$

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and $a_{\alpha\beta} = 0$ in the remaining cases, we obtain

$$Au = D_{11}\frac{\partial^4 u}{\partial x_1^4} + 4D_{16}\frac{\partial^4 u}{\partial x_1^3\partial x_2} + 2(D_{12} + 2D_{66})\frac{\partial^4 u}{\partial x_1^2\partial x_2^2} + 4D_{26}\frac{\partial^4 u}{\partial x_1\partial x_2^3} + D_{22}\frac{\partial^4 u}{\partial x_2^4}, \quad (2)$$

where u denotes the deflection of the mid-surface of the plate and D_{kl} the rigidities of the anisotropic plate. The notation follows the one used in reference [1]. Similarly, if we put

$$a_{\alpha\beta} = \begin{cases} D_{x_1} & \text{when } \alpha = (2,0), \beta = (2,0), \\ 2(\mu_{x_1}D_{x_2} + 2D_{x_1x_2}) & \text{when } \alpha = (2,0), \beta = (0,2), \\ D_{x_2} & \text{when } \alpha = (0,2), \beta = (0,2), \\ 0 & \text{in the other cases,} \end{cases}$$

and

$$a_{\alpha\beta} = \begin{cases} D & \text{when } \alpha = (2,0), \beta = (2,0), \\ 2D & \text{when } \alpha = (2,0), \beta = (0,2), \\ D & \text{when } \alpha = (0,2), \beta = (0,2), \\ 0 & \text{in the other cases,} \end{cases}$$

we obtain the operators $Au = D_{x_1}(\partial^4 u/\partial x_1^4) + 2(\mu_{x_1}D_{x_2} + 2D_{x_1x_2})(\partial^4 u/\partial x_1^2\partial x_2^2) + D_{x_2}(\partial^4 u/\partial x_2^4)$ and $Au = D(\partial^4 u/\partial x_1^4) + 2D(\partial^4 u/\partial x_1^2\partial x_2^2) + D(\partial^4 u/\partial x_2^4)$, which correspond, respectively, to the orthotropic and the isotropic plates [2–4]. Free transverse vibrations of a thin elastic plate are governed by the partial differential equation

$$A(u(x_1, x_2, t)) = -\rho h\frac{\partial^2 u(x_1, x_2, t)}{\partial t^2}, \quad (3)$$

where ρ denotes the density of the plate material and h the plate thickness.

In the case of normal modes of vibrations, one takes $u(x_1, x_2, t) = w(x_1, x_2)\cos\omega t$, consequently (3) is reduced to

$$A(w(x_1, x_2)) - \rho h\omega^2 w(x_1, x_2) = 0. \quad (4)$$

The statical behaviour of the plate when a load q is applied, is governed by

$$A(w(x_1, x_2)) = q(x_1, x_2). \quad (5)$$

Let G be $\{(x_1, x_2); 0 \leqslant x_1 \leqslant a; 0 \leqslant x_2 \leqslant b\}$ (see Figure 1). Both in the dynamical and statical cases, the boundary conditions which correspond to a rectangular anisotropic plate with edges elastically restrained against rotation are

$$w(0, x_2) = w(a, x_2) = w(x_1, 0) = w(x_1, b) = 0 \quad (6a)$$

$$r_1\frac{\partial w(0, x_2)}{\partial x_1} = \left(D_{11}\frac{\partial^2 w(0, x_2)}{\partial x_1^2} + D_{12}\frac{\partial^2 w(0, x_2)}{\partial x_2^2} + 2D_{16}\frac{\partial^2 w(0, x_2)}{\partial x_1\partial x_2}\right), \quad (6b)$$

$$r_2\frac{\partial w(a, x_2)}{\partial x_1} = -\left(D_{11}\frac{\partial^2 w(a, x_2)}{\partial x_1^2} + D_{12}\frac{\partial^2 w(a, x_2)}{\partial x_2^2} + 2D_{16}\frac{\partial^2 w(a, x_2)}{\partial x_1\partial x_2}\right), \quad (6c)$$

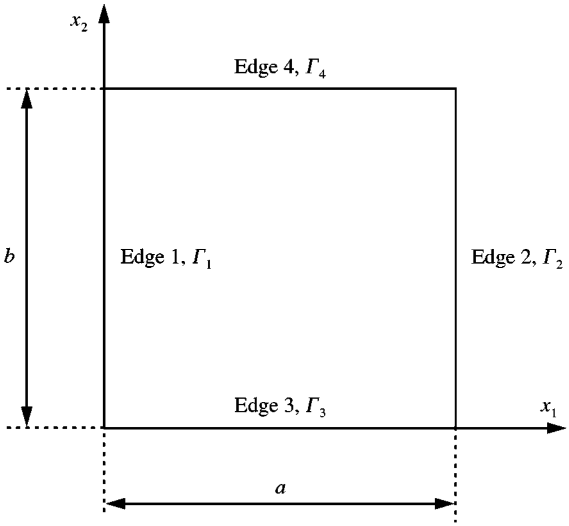

Figure 1. Rectangular plate under study (numbers at the edges are used as subscripts in defining edge restraint parameters).

$$r_3 \frac{\partial w(x_1, 0)}{\partial x_2} = \left(D_{12} \frac{\partial^2 w(x_1, 0)}{\partial x_1^2} + D_{22} \frac{\partial^2 w(x_1, 0)}{\partial x_2^2} + 2D_{26} \frac{\partial^2 w(x_1, 0)}{\partial x_1 \partial x_2} \right), \tag{6d}$$

$$r_4 \frac{\partial w(x_1, b)}{\partial x_2} = -\left(D_{12} \frac{\partial^2 w(x_1, b)}{\partial x_1^2} + D_{22} \frac{\partial^2 w(x_1, b)}{\partial x_2^2} + 2D_{26} \frac{\partial^2 w(x_1, b)}{\partial x_1 \partial x_2} \right). \tag{6e}$$

In equations (6b)–(6e), r_i denotes the rotational stiffness per unit length along the edge i. Since boundary conditions containing derivatives of orders higher than $m-1$ are called unstable for a differential equation of order 2 m [5], the boundary conditions (6b)–(6e) are unstable if $0 \leqslant r_i < \infty$, while the conditions (6a) and (6b)–(6e) with $r_i = \infty$, are geometric or stable. The geometric and natural boundary conditions are of different nature so in order to clearly distinguish them, it is useful to introduce the space V of elements of the Sobolev space $H^2(G)$ which satisfy the corresponding stable homogeneous boundary conditions [6].

A weak solution of the equation $Au = f$ of order 4 is a function from the Sobolev space $H^2(G)$, in consequence, the space V is given by $V = \{v,\ v \in H^2(G),\ v|_\Gamma = 0$ in the sense of traces$\}$

2. THE WEAK SOLUTION OF THE BOUNDARY VALUE PROBLEM

Now the boundary value problem (5)–(6) is transformed into one that leads to the concept of weak solution. First assume that $q(x) \in C(\bar{G})$ and let $w \in C^{(4)}(\bar{G})$ be the classical solution of the problem (5)–(6) with the operator A rewritten as

$$\begin{aligned} Aw = {} & \frac{\partial^2}{\partial x_1^2}\left(D_{11} \frac{\partial^2 w}{\partial x_1^2} + D_{12} \frac{\partial^2 w}{\partial x_2^2} + 2D_{16} \frac{\partial^2 w}{\partial x_1 \partial x_2} \right) + \frac{\partial^2}{\partial x_2^2}\left(D_{12} \frac{\partial^2 w}{\partial x_1^2} + D_{22} \frac{\partial^2 w}{\partial x_2^2} + 2D_{26} \frac{\partial^2 w}{\partial x_1 \partial x_2} \right) \\ & + \frac{\partial^2}{\partial x_1 \partial x_2}\left(2D_{16} \frac{\partial^2 w}{\partial x_1^2} + 2D_{26} \frac{\partial^2 w}{\partial x_2^2} + 4D_{66} \frac{\partial^2 w}{\partial x_1 \partial x_2} \right). \end{aligned} \tag{7}$$

If we take an arbitrary function $v \in V$, and multiply equation (5) by this function and integrate the result over the domain G we get

$$\iint_G A(w(x_1, x_2))v(x_1, x_2)\,\mathrm{d}x = \iint_G q(x_1, x_2)\,v(x_1, x_2)\,\mathrm{d}x \quad \text{with } \mathrm{d}x = \mathrm{d}x_1\,\mathrm{d}x_2. \tag{8}$$

We will use the Green formula $\iint_G u(\partial v/\partial x_i)\,\mathrm{d}x = \oint_\Gamma uvn_i\,\mathrm{d}s - \iint_G v(\partial u/\partial x)\mathrm{d}x$, $u, v \in H^{(1)}(G)$, where n_i denotes the components of the normal exterior to the boundary of G. If we apply the Green formula to the left-hand side of equation (8) we obtain

$$\iint_G \left[P(v)\frac{\partial^2 w}{\partial x_1^2} + Q(v)\frac{\partial^2 w}{\partial x_2^2} + R(v)\frac{\partial^2 w}{\partial x_1\partial x_2}\right]\mathrm{d}x + \oint_\Gamma \frac{\partial}{\partial x_1}P(w)vn_1\,\mathrm{d}s - \oint_\Gamma P(w)\frac{\partial v}{\partial x_1}n_1\,\mathrm{d}s$$
$$+ \oint_\Gamma \frac{\partial}{\partial x_2}Q(w)\,vn_2\,\mathrm{d}s - \oint_\Gamma Q(w)\frac{\partial v}{\partial x_2}n_2\,\mathrm{d}s + \oint_\Gamma \frac{\partial}{\partial x_2}R(w)vn_1\,\mathrm{d}s - \oint_\Gamma R(w)\frac{\partial v}{\partial x_1}n_2\,\mathrm{d}s,$$

where

$$P(u) = D_{11}\frac{\partial^2 u}{\partial x_1^2} + D_{12}\frac{\partial^2 u}{\partial x_2^2} + 2D_{16}\frac{\partial^2 u}{\partial x_1\partial x_2}, \quad Q(u) = D_{12}\frac{\partial^2 u}{\partial x_1^2} + D_{22}\frac{\partial^2 u}{\partial x_2^2} + 2D_{26}\frac{\partial^2 u}{\partial x_1\partial x_2},$$

$$R(u) = 2D_{16}\frac{\partial^2 u}{\partial x_1^2} + 2D_{26}\frac{\partial^2 u}{\partial x_2^2} + 4D_{66}\frac{\partial^2 u}{\partial x_1\partial x_2} \quad \text{with } u = v \text{ or } w.$$

Taking into account the boundary conditions (6b)–(6e) and the fact that since $v \in V$, is $v = 0$ in $L^2(\Gamma)$, we get (see Figure 1)

$$\oint_\Gamma \frac{\partial}{\partial x_1}P(w)vn_1\,\mathrm{d}s - \oint_\Gamma P(w)\frac{\partial v}{\partial x_1}n_1\,\mathrm{d}s + \oint_\Gamma \frac{\partial}{\partial x_2}Q(w)vn_2\,\mathrm{d}s - \oint_\Gamma Q(w)\frac{\partial v}{\partial x_2}n_2\,\mathrm{d}s$$
$$+ \oint_\Gamma \frac{\partial}{\partial x_2}R(w)vn_1\,\mathrm{d}s - \oint_\Gamma R(w)\frac{\partial v}{\partial x_1}n_2\,\mathrm{d}s = r_1\oint_{\Gamma_1}\frac{\partial v}{\partial x_1}\frac{\partial w}{\partial x_1}\mathrm{d}s + r_2\oint_{\Gamma_2}\frac{\partial v}{\partial x_1}\frac{\partial w}{\partial x_1}\mathrm{d}s$$
$$+ r_3\oint_{\Gamma_3}\frac{\partial v}{\partial x_1}\frac{\partial w}{\partial x_2}\mathrm{d}s + r_4\oint_{\Gamma_2}\frac{\partial v}{\partial x_2}\frac{\partial w}{\partial x_2}\mathrm{d}s$$

Then, we have

$$B(v, w) = A(v, w) + a(v, w) = \iint_G \left\{P(v)\frac{\partial^2 w}{\partial x_1^2} + Q(v)\frac{\partial^2 w}{\partial x_2^2} + R(v)\frac{\partial^2 w}{\partial x_1\partial x_2}\right\}\mathrm{d}x$$
$$+ r_1\oint_{\Gamma_1}\frac{\partial v}{\partial x_1}\frac{\partial w}{\partial x_1}\mathrm{d}s + r_2\oint_{\Gamma_2}\frac{\partial v}{\partial x_1}\frac{\partial w}{\partial x_1}\mathrm{d}s + r_3\oint_{\Gamma_3}\frac{\partial v}{\partial x_2}\frac{\partial w}{\partial x_2}\mathrm{d}s + r_4\oint_{\Gamma_4}\frac{\partial v}{\partial x_2}\frac{\partial w}{\partial x_2}\mathrm{d}s. \tag{9}$$

The double integral in equation (9) constitutes the bilinear form $A(v, w)$ associated with the differential operator A defined in equation (7), and the curvilinear integrals constitute the boundary bilinear form $a(v, w)$. The equality (8) now assumes the form

$$B(v, w) = \iint_G q\,v\,\mathrm{d}x = (q, v)_{L^2(G)}.$$

Now it is possible to weaken the assumptions. Let $q(x) \in L^2(G)$ and the bilinear form $B(v, w)$ continuous. A function $w \in H^2(G)$ is called a weak solution of the boundary value problem (5), (6) if

$$\begin{aligned} & w \in V, \\ & B(v, w) = (v, q)_{L^2(G)} \forall v \in V. \end{aligned} \tag{10}$$

2.1. THE CONTINUITY OF THE BILINEAR FORM

We must prove the continuity of the bilinear form ($B(v, w)$. Considering that the terms of $A(v, w)$ have the form $D_{kl} \iint_G D^\alpha w D^\beta v \, \mathrm{d}x$ up to a numerical factor, by the Schwarz inequality we get $|D_{kl} \iint_G D^\alpha v D^\beta w \, \mathrm{d}x| \leqslant D_{kl} \iint_G |D^\alpha v||D^\beta w| \mathrm{d}x \leqslant D_{kl} \|v\|_{H^2(G)} \|w\|_{H^2(G)}$.

Combining the estimates, the existence of constant $C_1 > 0$ follows such that

$$|A(v, w)| \leqslant C_1 \|v\|_{H^2(G)} \|w\|_{H^2(G)} \forall v,\ w \in H^2(G). \tag{11}$$

On the other hand, since $w,\ v \in H^2(G)$, then $(\partial w/\partial x_i)$, $(\partial v/\partial x) \in H^1(G)$, and consequently, these functions, have traces which belong to $L^2(\Gamma)$. Moreover, from the trace theorem [5], these exist a constant $C_2 > 0$ such that

$$\left\| \frac{\partial v}{\partial x_i} \right\|_{L^2(\Gamma)} \leqslant C_2 \left\| \frac{\partial v}{\partial x_i} \right\|_{H^1(G)} \leqslant C_2 \|v\|_{H^2(G)} \forall v \in H^{(2)}(G)$$

and we have

$$\left| r_j \oint_{\Gamma_j} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i} \, \mathrm{d}s \right| \leqslant r_j \oint_{\Gamma_j} \left| \frac{\partial v}{\partial x_i} \right| \left| \frac{\partial w}{\partial x_i} \right| \mathrm{d}s \leqslant r_j \sqrt{\oint_{\Gamma_j} \left| \frac{\partial v}{\partial x_i} \right|^2 \mathrm{d}s} \sqrt{\oint_{\Gamma_j} \left| \frac{\partial w}{\partial w_i} \right|^2 \mathrm{d}s} \leqslant r_j C_2 \|v\|_{H^2(G)} \|w\|_{H^2(G)}.$$

We finally get

$$\begin{aligned} |a(v, w)| \leqslant & r_1 \oint_{\Gamma_1} \left| \frac{\partial v}{\partial x_1} \right| \left| \frac{\partial w}{\partial x_1} \right| \mathrm{d}s + r_2 \oint_{\Gamma_2} \left| \frac{\partial v}{\partial x_1} \right| \left| \frac{\partial w}{\partial x_1} \right| \mathrm{d}s + r_3 \oint_{\Gamma_3} \left| \frac{\partial v}{\partial x_2} \right| \left| \frac{\partial w}{\partial x_2} \right| \mathrm{d}s \\ & + \Gamma_4 \oint_{\Gamma_4} \left| \frac{\partial v}{\partial x_2} \right| \left| \frac{\partial w}{\partial x_2} \right| \mathrm{d}s \leqslant C_3 \|v\|_{H^2(G)} \|w\|_{H^2(G)}, \text{ with } C_3 > 0. \end{aligned} \tag{12}$$

From equations (11) and (12) we have that $B(v, u)$ is continuous on $H^2(G) x H^2(G)$, i.e. there exists a constant $C_4 > 0$ such that

$$|B(v, w)| \leqslant C_4 \|v\|_{H^2(G)} \|w\|_{H^2(G)} \forall v,\ w \in H^2(G). \tag{13}$$

2.2. THE V-ELLIPTICITY OF THE BILINEAR FORM

If the bilinear form $B(v, w)$ is also V-elliptic, then the problem under consideration has exactly one weak solution w [5]. It is known from the theory of elasticity that the quadratic form which represents twice the potential energy density of a elastic body is positive definite, i.e., there exists a constant $C_5 > 0$ so that

$$2W(\mathbf{u}) = \sum_{i,k,l,m=1}^{3} c_{iklm} \, \varepsilon_{ik}(\mathbf{u}) \varepsilon_{lm}(\mathbf{u}) \geqslant C_5 \left(\sum_{i,k=1}^{3} \varepsilon_{ik}^2(\mathbf{u}) \right),\ C_5 > 0, \tag{14}$$

where c_{iklm} are the stiffness matrix coefficients, ε_{ik} the strains and $\mathbf{u}$ is the displacement vector in terms of a x_1, x_2, x_3 co-ordinate system. Under the assumptions of the anisotropic plate theory [1] inequality (14) reduces to

$$x_3^2\left[B_{11}\left(\frac{\partial^2 w}{\partial x_1^2}\right)^2+2B_{12}\frac{\partial^2 w}{\partial w_1^2}\frac{\partial^2 w}{\partial x_2^2}+4B_{16}\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 w}{\partial x_1\partial w_2}+B_{22}\left(\frac{\partial^2 w}{\partial x_2^2}\right)^2\right.$$
$$\left.+4B_{26}\frac{\partial^2 w}{\partial x_2^2}\frac{\partial^2 w}{\partial x_1\partial x_2}+4B_{66}\left(\frac{\partial^2 w}{\partial x_1\partial x_2}\right)^2\right]\geqslant C_5x_3^2\left[\left(\frac{\partial^2 w}{\partial x_1^2}\right)^2+2\left(\frac{\partial^2 w}{\partial x_1\partial x_2}\right)^2+\left(\frac{\partial^2 w}{\partial x_2^2}\right)^2\right].$$

where the constants B_{ij} are expressed by the coefficients of the stiffness matrix. The integration in the plate volume, in the case of constant thickness h, leads to

$$\iint_G\left[D_{11}\left(\frac{\partial^2 w}{\partial x_1^2}\right)^2+2D_{12}\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 w}{\partial x_2^2}+4D_{16}\frac{\partial^2 w}{\partial x_1\partial x_2}+D_{22}\left(\frac{\partial^2 w}{\partial x_2^2}\right)^2\right.$$
$$\left.+4D_{26}\frac{\partial^2 w}{\partial x_2^2}\frac{\partial^2 w}{\partial x_1\partial x_2}+4D_{66}\left(\frac{\partial^2 w}{\partial x_1\partial x_2}\right)^2\right]\mathrm{d}x_1\,\mathrm{d}x_2$$
$$\geqslant\frac{C_5h^3}{12}\iint_G\left[\left(\frac{\partial^2 w}{\partial x_1^2}\right)^2+2\left(\frac{\partial^2 w}{\partial x_1\partial x_2}\right)^2+\left(\frac{\partial^2 w}{\partial x_2^2}\right)^2\right]\mathrm{d}x_1\,\mathrm{d}x_2.$$

where the constants D_{ij} are given by $D_{ij}=B_{ij}h^3/12$.

From this and equation (9) it is inferred that a constant $C_6>0$ exists such that

$$B(v,v)\geqslant A(v,v)\geqslant C_6\iint_G\left[\left(\frac{\partial^2 v}{\partial x_1^2}\right)^2+\left(\frac{\partial^2 v}{\partial x_2^2}\right)^2+\left(\frac{\partial^2 v}{\partial x_1\partial x_2}\right)^2\right]\mathrm{d}x.$$

By applying Friedrichs inequality [6].

$$\|u\|^2_{H^2(G)}\leqslant K\left(\sum_{|i|=2}\iint_G(D^iu)^2\,\mathrm{d}x+\int_\Gamma u^2(s)\,\mathrm{d}s\right),\ K>0,\ \forall u\in H^2(G),$$

we obtain

$$B(v,v)\geqslant C_6\iint_G\left[\left(\frac{\partial^2 v}{\partial x_1^2}\right)^2+\left(\frac{\partial^2 v}{\partial x_2^2}\right)^2+\left(\frac{\partial^2 v}{\partial x_1\partial x_2}\right)^2\right]\mathrm{d}x\geqslant\frac{C_6}{K}\|v\|^2_{H^2(G)}\forall v\in V.\tag{15}$$

The inequality (15) implies that $B(v,w)$ is V-elliptic.

3. APPLICATION OF A DIRECT VARIATIONAL METHOD TO THE CONSTRUCTION OF APPROXIMATIONS OF THE WEAK SOLUTION

We proved that the bilinear form $B(v,w)$ is continuous and V-elliptic. Since it is also symmetric, the function $w(x)$ is the weak solution of equation (10), if and only if it minimizes in the space V, the functional

$$I(v)=B(v,v)-2(v,q)_{L^2(G)}=\iint_G\left\{D_{11}\left(\frac{\partial^2 v}{\partial x_1^2}\right)^2+2D_{12}\frac{\partial^2 v}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2}\right.$$
$$\left.+4D_{16}\frac{\partial^2 v}{\partial x_1^2}\frac{\partial^2 v}{\partial x_1\,\partial x_2}+D_{22}\left(\frac{\partial^2 v}{\partial x_2^2}\right)^2+4D_{26}\frac{\partial^2 v}{\partial x_2^2}\frac{\partial^2 v}{\partial x_1\partial x_2}+4D_{66}\left(\frac{\partial^2 v}{\partial x_1\partial x_2}\right)^2\right]\mathrm{d}x$$
$$\to r_1\oint_{\Gamma_1}\left(\frac{\partial v}{\partial x_1}\right)^2\mathrm{d}S+r_2\oint_{\Gamma_2}\left(\frac{\partial v}{\partial x_1}\right)^2\mathrm{d}s+r_3\oint_{\Gamma_3}\left(\frac{\partial v}{\partial x_2}\right)^2\mathrm{d}s+r_4\oint_{\Gamma_4}\left(\frac{\partial v}{\partial x_2}\right)^2\mathrm{d}s-2\iint_G qv\,\mathrm{d}x.\tag{16}$$

If we apply the Ritz method, the function w which minimizes the functional (16) in the space V is approximately by $w_n(x_1, x_2) = \sum_{i=1}^{n} c_{ni} v_i(x_1, x_2)$, where $v_i(x_1, x_2)$ are elements of a base in V. The coefficients c_{ni} are determined by the condition $I(w_n) = \min$. This procedure leads to the system of linear equations

$$\sum_{j=1}^{n} c_{nj} B(v_i, v_j) = (v_i, q)_{L^2(G)}, \; i = 1, 2, 3, \ldots, n. \tag{17}$$

4. THE EIGENVALUE PROBLEM

Let us consider free vibrations of a rectangular anisotropic plate whose edges are elastically restrained against rotation. The eigenvalue problem is given by equation (4) and boundary conditions (6). We rewrite it as the problem of finding a number Ω and a function w such that

$$\begin{aligned} &w \in V, \; w \neq 0, \\ &B(v, w) - \Omega (v, w)_{L^2(G)} = 0 \forall v \in V. \end{aligned}$$

Since the bilinear form $B(v, w)$ is symmetric, continuous and V-elliptic, it has a countable set of eigenvalues which are given by [6]

$$\Omega_1 = \min \left\{ \frac{B(v, v)}{(v, v)_{L^2(G)}} \, v \in V, v \neq 0 \right\}, \tag{18}$$

$$\Omega_n = \min \left\{ \frac{B(v, v)}{(v, v)_{L^2(G)}} \, v \in V, v \neq 0, (v, v_1)_{L^2(G)} = 0, \ldots, (v, v_n)_{L^2(G)} = 0 \right\}, \quad n = 2, 3, \ldots. \tag{19}$$

In equations (18) and (19) the bilinear form $B(v, v)$ is given by equation (9) and is proportional to the maximum strain energy of the mechanical system under study which is given by

$$\begin{aligned} U_{max} = \frac{1}{2} \iint_G \Bigg\{ & D_{11} \left(\frac{\partial^2 v}{\partial x_1^2} \right)^2 + 2D_{12} \frac{\partial^2 v}{\partial x_1^2} \frac{\partial^2 v}{\partial x_2^2} \\ & + 4D_{16} \frac{\partial^2 v}{\partial x_1^2} \frac{\partial^2 v}{\partial x_1 \partial x_2} + D_{22} \left(\frac{\partial^2 v}{\partial x_2^2} \right)^2 + 4D_{26} \frac{\partial^2 v}{\partial x_2^2} \frac{\partial^2 v}{\partial x_1 \partial x_2} + 4D_{66} \left(\frac{\partial^2 v}{\partial x_1 \partial x_2} \right)^2 \Bigg\} \mathrm{d}x_1 \, \mathrm{d}x_2 \\ & + \frac{r_1}{2} \int_0^b \left(\frac{\partial v}{\partial x_1} (0, x_2) \right)^2 \mathrm{d}x_2 + \frac{r_2}{2} \int_0^b \left(\frac{\partial v}{\partial x_1} (a, x_2) \right)^2 \mathrm{d}x_2 + \frac{r_3}{2} \int_0^a \left(\frac{\partial v}{\partial x_2} (x_1, 0) \right)^2 \mathrm{d}x_1 \\ & + \frac{r_4}{2} \int_0^a \left(\frac{\partial v}{\partial x_2} (x_1, b) \right)^2 \mathrm{d}x_1 . \end{aligned}$$

On the other hand $(v, v)_{L^2(G)} = \iint_G (v)^2 \, \mathrm{d}x$, is proportional to the maximum kinetic energy of the plate, $T_{max} = (\rho h \omega^2/2) \iint_G v^2(x_1, x_2) \, \mathrm{d}x_1 \, \mathrm{d}x_2$.

Let us introduce a new inner product in space V: $((v, w)) = B(v, w) \, \forall w, \; v \in V$. If the sequence $\{v_i(x_1, x_2)\}$ is a base in the space V with the inner product $((v, w))$, the Ritz method leads to the equation for the determinant of the Ritz system:

$$\begin{vmatrix} ((v_1, v_1)) - \Omega(v_1, v_1)_{L^2(G)} \cdots ((v_1, v_n)) - \Omega(v_1, v_n)_{L^2(G)} \\ ((v_n, v_1)) - \Omega(v_n, v_1)_{L^2(G)} \cdots ((v_n, v_n)) - \Omega(v_n, v_n)_{L^2(G)} \end{vmatrix} = 0. \tag{20}$$

Approximate eigenvalues can be obtained from equation (20) when dealing with the dynamical behaviour of the plate under consideration.

5. THE ROLE OF NATURAL BOUNDARY CONDITIONS

When using the Ritz method, we choose a sequence of functions, v_i which constitute a base in the space V, where only the homogeneous stable boundary conditions are included, so there is no need to subject the functions v_i to the natural boundary conditions.

The fact that the natural boundary conditions of a system need not be satisfied by the chosen co-ordinate functions is a very important characteristic of the Ritz method, specially when dealing with problems for which such satisfaction is very difficult to achieve. For instance, this is the case of a rectangular anisotropic, orthotropic or isotropic plate with edges elastically restrained against rotation. This property has been verified in the present work by using orthogonal polynomials in the Ritz method. For using the Ritz procedure we took the approximation of the form

$$w(x_1, x_2) = \sum_{i=0}^{N} \sum_{j=0}^{M} c_{ij} \phi_i(x_1) \varphi_j(x_2). \tag{21}$$

Here the c_{ij} are arbitrary coefficients which are to be determined and $\{\phi_i(x_1), \varphi_j(x_2)\}$ is the set of the orthogonal polynomials. The procedure for constructing the orthogonal polynomials in the variable x_1 is the following:

$$\phi_1(x_1) = (x_1 - B_1)\phi_0(x_1), \quad \phi_i(x_1) = (x_1 - B_i)\phi_{i-1}(x_1) - C_i\phi_{i-2}(x_1) \quad \text{for } i > 1,$$

$$B_i = \frac{\int_0^a x_1 \phi_{i-1}^2(x_1)\,\mathrm{d}x_1}{\int_0^a \phi_{i-1}^2(x_1)\,\mathrm{d}x_1}, \quad C_i = = \frac{\int_0^a x_1 \phi_{i-1}(x_1)\phi_{i-2}(x_1)\,\mathrm{d}x_1}{\int_0^a \phi_{i-1}^2(x_1)\,\mathrm{d}x_1}.$$

The same procedure is applied to the polynomials in x_2. In the present work, all the polynomials satisfy only the geometric (or stable) boundary conditions. Since the set of orthogonal polynomials used is a base in space V, as the upper limits of summation N, M in equation (21) are increased, the exact solution may be approximated as closely as desired.

6. NUMERICAL RESULTS

Table 1 depicts values of the maximum deflection in terms of the coefficient k where $w_{max} = (ka^4/D_{11})q_0$, of a simply supported anisotropic square plate and a clamped

TABLE 1

Values of the maximum deflection in terms of the coefficient k were $w_{max} = (ka^4/D^{11})q_0$, *of a simply supported square plate and a clamped square plate* ($N = M = 5$ *in equation* (21))

D_{22}/D_{11}	$\frac{(D_{12}+2D_{66})}{D_{11}}$	$\frac{D_{16}}{D_{11}} = \frac{D_{26}}{D_{11}}$	Simply supported square plate		Clamped square Plate
			Reference [2] k	Present study k	Present study k
1	1	0	0·00406	0·0040609	0·0012653
1	1·061	– 0·174	0·00411	0·0041101	0·0012760
1	1·412	– 0·454	0·00444	0·004414	0·0013475
1	1·500	– 0·500	0·00452	0·004491	0·0013676
1	1·690	– 0·587	0·00476	0·004657	0·0014146

anisotropic square plate both subjected to a uniform transverse load $q(x, y) = q_0$. The anisotropy is characterised by the values of the coefficients D_{22}/D_{11}, D_{16}/D_{11}, D_{26}/D_{11} and $(D_{12} + 2D_{66})/D_{11}$, where the notations and material properties follow those of reference [2].

In the case of the simply supported plate, the values obtained with the present method are compared with the exact values reported by Whitney [2]. Excellent agreement was obtained between the present values and the exact results.

Table 2 contains results of the first four frequencies given in terms of λ where $\omega = (\lambda/b^2)\sqrt{D_{11}/\rho}$ for a square-clamped anisotropic plate. The plate considered is composed of an orthotropic material oriented with the principal axis of orthotropy at θ degrees from the x_1-axis. The material properties are $E_L/E_T = 10$, $G_{LT}/E_T = 0{\cdot}25$, $\mu_{LT} = 0{\cdot}3$. The notations and the material properties follow those of reference [2].

7. CONCLUDING REMARKS

The existence and uniqueness of the weak solutions of boundary value problems and eigenvalue problems, which correspond, respectively, to the statical and dynamical behaviour of rectangular anisotropic, orthotropic and isotropic plates with edges elastically restrained against rotation has been demonstrated. The use of the weak solution theory enables a substantial generalisation of assumptions concerning the smoothness of coefficients of the differential equation (1) and of the function q which represents the load in equation (5).

It is also the purpose of the present paper to present some technically interesting results for the deflection and natural frequencies of anisotropic plates. The Ritz method has been employed by using orthogonal polynomials as trial functions which satisfy only the geometric or stable boundary conditions. As it was expected, the convergence of frequencies is monotonic, and upper bounds in the values of the frequency parameters are obtained successively as additional terms are taken in the approximation function (21), in spite of the fact that the co-ordinate functions do not satisfy the natural boundary conditions. Since the combinations of boundary conditions, along with specific values for the stiffness constants for the restraints are prohibitively large in number, results are presented for only a few cases.

TABLE 2

Values of the first four natural frequencies of a square clamped plate, given in terms of λ where $\omega = (\lambda/b^2)\sqrt{D_{11}/\rho}$ ($N = M = 5$ in equation (21))

Stiffness ratios					λ			
D_{22}/D_{11}	D_{12}/D_{11}	D_{66}/D_{11}	D_{16}/D_{11}	D_{26}/D_{11}	Mode 1	Mode 2	Mode 3	Mode 4
0·1	0·03	0·0247750	0	0	23·96642	31·14868	46·4672	62·77512
0·1152032	0·1008125	0·0948811	− 0·243335	− 0·012084	24·60065	33·57042	50·8170	63·3439
0·2482224	0·3448467	0·3361177	− 0·495691	− 0·155368	27·57683	42·8323	65·4984	66·9592
1	0·8425860	0·8259868	− 0·174796	− 0·714796	36·51952	62·3162	83·5668	92·8005

ACKNOWLEDGMENTS

The authors is grateful to Dr Leonid Lebedev for his useful suggestions. The present investigation was sponsored by Consejo de Investigación de la Universidad Nacional de Salta, Project 744.

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