Journal of Sound and Vibration (2001) **242**(3), 553–558
doi:10.1006/jsvi.2000.3371, available online at http://www.idealibrary.com on IDEAL®

A TWO-DIMENSIONAL ANALYSIS OF ANISOTROPIC VIBRATING BEAMS

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(*Received* 14 *June* 2000 *and in final form* 25 *August* 2000)

1. INTRODUCTION

Composite materials are increasingly used in several types of structural applications. Advantages that motivate their use are high strength and stiffness, low weight, improved fatigue life, etc. Many structural components made of composites have the form of beams. Accordingly, some refined theories were developed in order to consider the special structural properties of composite beams such as the important role of the shear deformability and the effects of anisotropy.

A theory of shear deformable orthotropic beams was developed by Nowinsky [1]. On the other hand, Dharmarajan and McCutchen [2] have discussed a method for obtaining shear correction factors for these types of beams. An orthotropic beam theory including normal deformability along with the shear effect was presented by Soldatos and Elishakoff [3]. Recently, Murakami and Yamakawa developed an anisotropic beam theory of the Timoshenko type from a mixed variational principle [4]. The model was used to calculate vibration frequencies of cantilever and simply supported beams.

In this article, a two-dimensional vibration analysis for anisotropic beams is presented in order to verify the Murakami–Yamakawa beam theory. The beam is modelled by means of a plane state of stress corresponding to an anisotropic elastic body. Natural frequencies are determined by means of the finite element system FLEXPDE [5]. The results are compared against the values obtained in reference [4]. Additional results are given for clamped beams.

2. GOVERNING EQUATIONS

An anisotropic beam with a narrow rectangular cross-section is considered (see Figure 1). According to the elasticity theory the problem is governed by means of the following equations:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = \rho \frac{\partial^2 u}{\partial t^2}, \qquad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = \rho \frac{\partial^2 v}{\partial t^2}, \tag{1a,b}$$

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad 2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \tag{2a–c}$$

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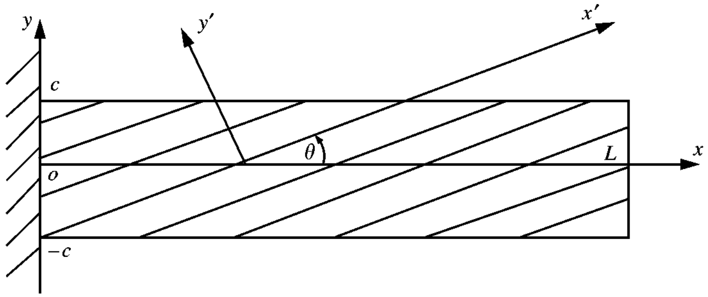

Figure 1. Analyzed anisotropic beam and reference system.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ 2\varepsilon_{xy} \end{Bmatrix}. \tag{3}$$

The expressions of the coefficients $\bar{Q}_{ij}$ in terms of the material moduli $\mathbf{E}_{11}$, $\mathbf{E}_{22}$, $\mathbf{G}_{12}$ and v_{12} are detailed in reference [6].

The present equations along with the corresponding boundary conditions were solved by means of the FLEXPDE system. This is a flexible solver of partial differential equations based on the finite element method. Among other notable features, this system has an adaptive grid refinement controlled by an estimation of the maximum relative error required. More details about the program are given in reference [5].

The cases analyzed in this paper are shown in Table 1. There, it is possible to observe the boundary conditions of the beam type and the equivalent boundary conditions according to the two-dimensional model used in this article.

3. NUMERICAL RESULTS

Figures 2–5 show the non-dimensional natural frequencies of vibration given by

$$\bar{\omega} = \frac{c\omega}{(c/L)^2\sqrt{E_{11}/3\rho}}.$$

TABLE 1

Boundary conditions analyzed

Case	Boundary conditions of the beam type	Present boundary conditions
Cantilever	$U = \phi = V = 0\ (x = 0)$ $N = M = Q = 0\ (x = L)$	$u = v = 0\ (x = 0)$ $\sigma_x = \sigma_{xy} = 0\ (x = L)$ $\sigma_y = \sigma_{xy} = 0\ (y = \pm c)$
Simply supported	$U = M = V = 0\ (x = 0)$ $N = M = V = 0\ (x = L)$	$u = 0\ (x = 0,\ y = 0)$ $v = 0\ (x = 0)$ $\sigma_x = 0\ (x = 0)^\dagger$ $\sigma_x = v = 0\ (x = L)$ $\sigma_y = \sigma_{xy} = 0\ (y = \pm c)$
Clamped	$U = \phi = 0\ (x = 0)$ $U = \phi = 0\ (x = L)$	$u = v = 0\ (x = 0,\ L)$ $\sigma_y = \sigma_{xy} = 0\ (y = \pm c)$

Note: U: Longitudinal beam displacement; ϕ: rotational beam displacement; V: transverse beam displacement; N: longitudinal force; M: bending force; Q: shear force.
† This condition does not apply at $x = y = 0$.

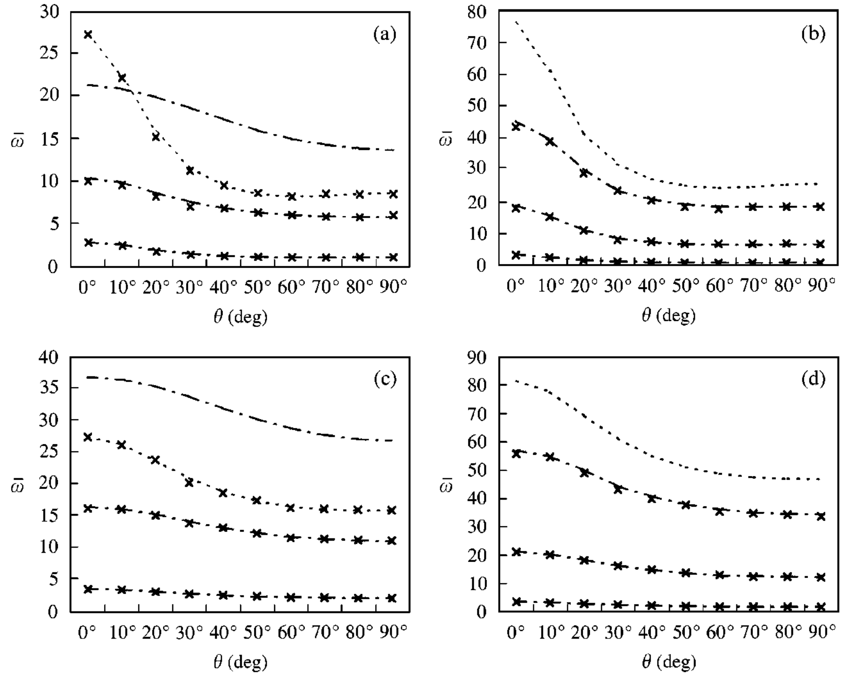

Figure 2. Non-dimensional natural frequencies $\bar{\omega}$ versus angle θ for cantilever beams. (a) material 1 with $c/L = 1/10$; (b) material 1 with $c/L = 1/30$; (c) material 2 with $c/L = 1/10$; (d) material 2 with $c/L = 1/30$: x, Murakami–Yamakawa beam theory; ----, dominant axial mode; —-—; dominant flexural mode.

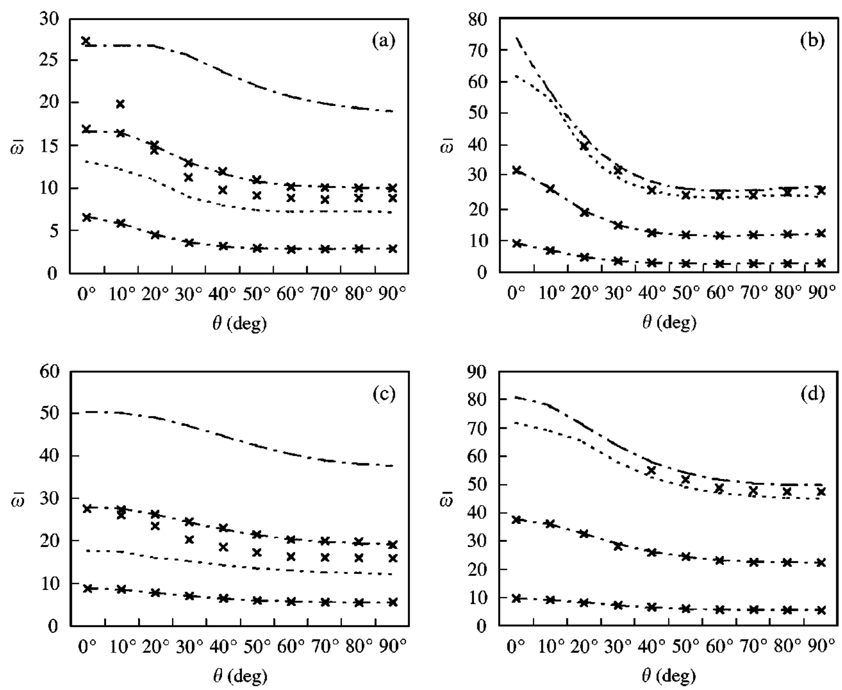

Figure 3. Non-dimensional natural frequencies $\bar{\omega}$ versus angle θ for simply supported beams. (a) material 1 with $c/L = 1/10$; (b) material 1 with $c/L = 1/30$; (c) material 2 with $c/L = 1/10$; (d) material 2 with $c/L = 1/30$: x, Murakami–Yamakawa beam theory; ----, dominant axial mode; —-—; dominant flexural mode.

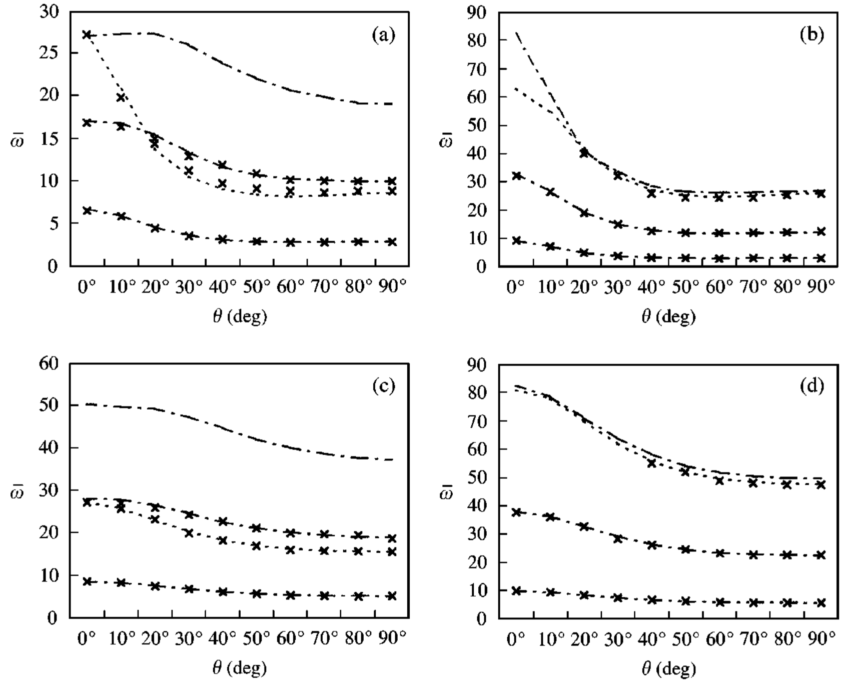

Figure 4. Non-dimensional natural frequencies $\bar{\omega}$ versus angle θ for simply supported beams with rigid left end. (a) material 1 with $c/L = 1/10$; (b) material 1 with $c/L = 1/30$; (c) material 2 with $c/L = 1/10$; (d) material 2 with $c/L = 1/30$: x, Murakami–Yamakawa beam theory; ----, dominant axial mode; —-—; dominant flexural mode.

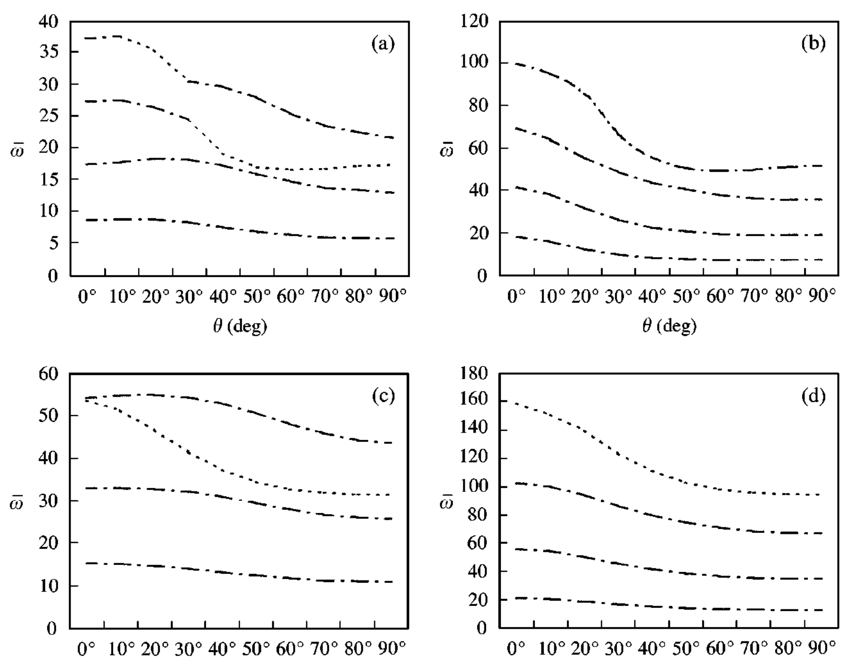

Figure 5. Non-dimensional natural frequencies $\bar{\omega}$ versus angle θ for clamped beams. (a) material 1 with $c/L = 1/10$; (b) material 1 with $c/L = 1/30$; (c) material 2 with $c/L = 1/10$; (d) material 2 with $c/L = 1/30$: x, Murakami–Yamakawa beam theory; ----, dominant axial mode; —-—; dominant flexural mode.

The following material properties were considered. Material 1: $\mathbf{E}_{22}/\mathbf{E}_{11} = 0{\cdot}1$, $\mathbf{G}_{12}/\mathbf{E}_{11} = 0{\cdot}0333$, $\nu_{12} = 0{\cdot}3$. Material 2: $\mathbf{E}_{22}/\mathbf{E}_{11} = 0{\cdot}333$, $\mathbf{G}_{12}/\mathbf{E}_{11} = 0{\cdot}1667$, $\nu_{12} = 0{\cdot}25$. Figure 2 shows the first four natural frequencies of vibration for cantilever beams, with the slenderness ratios $c/L = 1/10$ and 1/30, as a function of the θ angle between the strong fiber direction of the material and the longitudinal x-axis. Except for $\theta = 0$ or $\theta = 90°$, the flexural and axial motions are strongly coupled. However, in the figures, the modes have been classified as flexural- or axial-dominant according to the value of the ratio $\alpha = u_{max}/v_{max}$ (flexural modes when $\alpha < 1$ and axial modes when $\alpha > 1$).

The present results are compared with those obtained by Murakami and Yamakawa (M–Y). It may be noted that the M–Y results are practically coincident with the present ones for all the cases analyzed. However, in the short beam case ($c/L = 0{\cdot}1$) with material 1 (more anisotropic), the third frequency obtained with the present model corresponds to a flexural mode for $\theta < 10°$. This fact was not detected by the M–Y calculation.

Figure 3 shows the results corresponding to simply supported beams. The comparison with the M–Y theory shows a good agreement for the long beam cases for both materials. However, there are notable discrepancies in the frequencies corresponding to axial modes for the short beam cases.

These discrepancies are due to the fact that the left boundary condition corresponding to the restraint of the longitudinal displacement at $x = y = 0$, yields a notable warping of the cross-section. This behavior is very different from the assumption, used in the beam theory, that plane cross-sections remain plane. In Figure 4, results are shown for simply supported beams again, but with the left end modelled as a very rigid lamina. This is obtained by defining a very narrow region placed at the left side of the beam ($0 < x < 0{\cdot}01\ L$, $-c < y < c$) with very high rigidities ($\mathbf{E}_{11} = \mathbf{G}_{12} = \mathbf{E}_{22} = \nu_{12} = 10^{16}$). In this situation, the agreement among the present theory and the M–Y theory is complete.

Finally, natural frequencies corresponding to clamped beams are given in Figure 5. The general behavior is similar to the above-mentioned cases in the sense that the frequencies decrease with the increase of the θ angle.

4. CONCLUSIONS

Natural frequencies of vibration corresponding to anisotropic beams were calculated by means of a two-dimensional plane stress model. These results were compared, for the cases of cantilever and simply supported beams, with the values given by Murakami and Yamakawa using their anisotropic beam theory. The comparison shows a very good agreement for cantilever beams.

From the point of view of the two-dimensional theory, there exist several ways of modelling a simply supported end. The numerical values of the frequencies corresponding to axial modes are highly sensitive to the detailed form in which the simple support is materialized as it was shown. This means that the anisotropic beam theory should be applied with caution for short simply supported beams.

Numerical determinations of vibration frequencies of clamped beams were done, in order to illustrate their dynamic behavior.

ACKNOWLEDGMENTS

The present research was sponsored by Secretaría de Ciencia y Tecnología, Universidad Tecnológica Nacional and CONICET.

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