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IDENTIFICATION OF A CRACK IN A ROD BASED ON CHANGES IN A PAIR OF NATURAL FREQUENCIES

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This paper deals with the identification of a single crack in a vibrating rod based on the knowledge of the damage-induced shifts in a pair of natural frequencies. The crack is simulated by an equivalent linear spring connecting the two segments of the bar. The analysis is based on an explicit expression of the frequency sensitivity to damage and enables non-uniform bars under general boundary conditions to be considered. The inverse problem is generally "ill-posed", because even if the system is not symmetrical, cracks in different locations can still produce identical changes in a pair of natural frequencies. In spite of this, it is found that there are certain situations concerning uniform rods in which the effects of the non-uniqueness of the solution may be considerably reduced by means of a careful choice of the data. The theoretical results are confirmed by a comparison with dynamic measurements on steel rods with a crack. Some of the results are also valid for cracked beams in bending.

1. INTRODUCTION

This paper focuses on detecting a single crack in a vibrating rod from the knowledge of damage-induced shifts in a pair of natural frequencies.

In most of the studies of dynamic methods for damage identification, researchers have resorted to the changes in natural frequency as the diagnostic tool. Frequencies can be measured more easily than mode shapes, and as a rule they are less seriously affected by experimental errors. In damage-detection problems, two objectives have to be attained: the location of the damage, and its magnitude or severity. It is known that in beam-like structures the change in a natural frequency produced by a small single crack may be represented as the product of two terms, of which the first is proportional to the severity and the second depends solely on the location of the damage (see equations (11) and (51) for a rod and a beam respectively). This results in an important consequence: the ratios of the change in different natural frequencies depend only on the damage location, not on its severity, see Adams *et al.* [1], Gudmundson [2] and Morassi [3]. Hearn and Testa [4] and Liang *et al.* [5] have used this property for a damage localization analysis in beam-like structures.

Two inverse problems related to damage detection may be posed: (1) determine the location of a crack from the ratios of the changes in the natural frequencies; and (2) determine the location and the severity of a crack from the changes in natural frequencies. In particular, if the undamaged system is completely defined and if the damage is simulated by a linear spring, only two parameters need to be determined, namely the stiffness K of the spring and the abscissa s of the cracked cross-section. Such a peculiarity in damage detection has been noted more or less explicitly elsewhere in the literature and has been recently emphasized by Vestroni and Capecchi [6] and Vestroni *et al.* [7]. Therefore, it is reasonable to investigate the extent to which the measurement of the crack-induced changes in a pair of natural frequencies can be useful for identifying the damage. By using this set of data, both problems (1) and (2) are generally poorly defined: if the system is symmetrical, then a crack located at any one of a set of symmetrically placed points will produce identical changes in natural frequencies. Even if the system is not symmetrical, cracks in different locations can still produce identical changes to a pair of natural frequencies. In spite of this poor definition of the diagnostic problem, there are certain situations in which the effects of the non-uniqueness of the solution may be considerably reduced thanks to a careful choice of the data. In a recent paper [8], Narkis has shown that if the damaged system is a perturbation of the virgin system, namely whenever the crack is very small, the only information required for accurate crack localization is the variation of the first two natural frequencies caused by the crack. The results were shown for uniform free–free vibrating rods and for uniform simply supported beams in bending. Narkis has defined a closed-form solution for the crack location. Adams *et al.* [1] have come to the same conclusions for uniform free–free rods but have not obtained an explicit relation for the crack location.

The present work develops the results presented in reference [8] in different directions. It is found that for uniform free–free beams under axial vibration, knowledge of the ratio between the variations of the $2m$th and mth frequencies uniquely determines the position variable $S = \cos 2m\pi s/L$, where s stands for the abscissa of the cracked cross-section and L is the length of the rod. Thus, the ensuing particular case equal to $m = 1$ agrees with the result achieved by Narkis [8]. Furthermore, the variations of the $2m$th and mth frequencies also allows the stiffness K of the damage-simulating elastic spring to be uniquely determined. In both cases, simple closed-form expressions are deduced for S and for K. As for cantilevers or beams with fixed ends it is borne out that by simultaneously employing axial frequencies related to different boundary conditions it is still possible to determine uniquely the damage parameters S and K. The explicit expression for the damage sensitivity of natural frequencies given in reference [3] plays a crucial role in this analysis. In fact, the methodology that has been employed does not call for the explicit solution of the eigenvalue problem for the damaged system, as it focuses only on the knowledge of the eigensolutions corresponding to the integral configuration. Such a method is substantially different from Narkis' [8] and, as a result, the procedure that has been proposed can be also extended to the analysis of initially non-uniform cracked beams in axial vibration.

The dynamic tests performed on cracked steel rods supported the proposed method for the solution of the diagnostic problem. Analytical results agree well with experimental tests. Finally, part of the results above are also valid for cracked beams in bending under simply supported or sliding–sliding boundary conditions.

2. THEORETICAL RESULTS FOR GENERAL RODS WITH A CRACK

It is assumed that the spatial variation of the free vibration of an undamaged straight rod of length L is governed by the differential equation

$$(a(x)u'(x))' + \omega^2 \rho(x) u(x) = 0, \quad x \in (0, L), \tag{1}$$

where $u(x)$ describes the mode and ω is the associated natural frequency. The rod is assumed to have no material damping, since its effect on the natural frequencies is known to be negligible. The quantities $a(x) = EA(x)$ and $\rho(x)$ denote the axial stiffness and the linear-mass density of the rod. E is the Young's modulus of the material and $A(x)$ the cross-section area of the rod. This analysis is concerned with rods for which $a(x)$ is strictly positive and continuously differentiable function of x; $\rho(x)$ will be assumed to be continuous and strictly positive function. The end conditions are taken as

$$a(0)u'(0) - hu(0) = 0, \qquad a(L)u'(L) + Hu(L) = 0, \tag{2, 3}$$

where h and H are the stiffnesses of the elastic end supports. Three important cases can be distinguished:

$$\text{Supported (S):} \quad h = \infty = H, \quad u(0) = 0 = u(L), \tag{4}$$

$$\text{Free (F):} \quad h = 0 = H, \quad u'(0) = 0 = u'(L), \tag{5}$$

$$\text{Cantilever (C):} \quad h = \infty, H = 0, \quad u(0) = 0 = u'(L). \tag{6}$$

Modes and frequencies are the eigensolutions of the boundary value problem (1)–(3), and the mth eigenpair of the undamaged rod, $m \geqslant 0$, is denoted by $(u_m(x), v_m \equiv \omega_m^2)$. It is well known that for such $a(x)$ and $\rho(x)$, and end conditions (2) and (3), there is an infinite sequence $\{v_m\}_{m=0}^{\infty}$ such that $0 \leqslant v_0 < v_1 < \cdots$ with $\lim_{m\to\infty} v_m = \infty$. It is decided to designate the elements v_0 and $u_0(x)$ of the 0th eigenpair, respectively, *fundamental* eigenvalue and *fundamental* vibrating mode.

Suppose that a crack appears at the cross-section of abscissa $s \in (0, L)$. Assuming that the crack remains always open during the longitudinal vibration, by modelling it as a massless translational spring, at $x = s$, see references [9, 10], the eigenvalue problem for the damaged rod is the following:

$$(a(x)w'(x))' + \omega_d^2 \rho(x) w(x) = 0, \quad x \in (0, s) \cup (s, L), \tag{7}$$

where, in addition to the boundary conditions (2) and (3) it is necessary to consider the jump conditions

$$[w'(s)] = 0, \qquad K[w(s)] = a(s)w'(s) \tag{8,9}$$

that are to hold at the cross-section where the crack occurs. In equations (8) and (9) $[\phi(s)] \equiv (\phi(s^+) - \phi(s^-))$ denotes the jump of the function $\phi(s)$ at $x = s$. The expression K is the spring stiffness and can be related to the crack geometry as suggested, for example, by Freund and Herrmann [9] or by Dimarogonas and Paipetis [10]. The undamaged system corresponds to $K \to \infty$ or $\varepsilon \equiv 1/K \to 0$.

If the crack is small, namely ε is small enough, then the first order variation of the natural frequencies with ε may be found as shown in reference [3]. By taking

$$v_{dm} = v_m + \varepsilon(\Delta v_m), \tag{10}$$

the first variation of the mth eigenvalue is given by

$$\delta v_m \equiv \varepsilon(\Delta v_m) = -\frac{(a(s)u_m'(s))^2}{K}, \tag{11}$$

where the normalizing condition $\int_0^L \rho(x) u_m^2(x)\,\mathrm{d}x = 1$ has been taken into account. That is, the change in a natural frequency produced by a single notch may be expressed as the product of two terms, the first of which is proportional to the severity and the second depends only on the location of the damage. In particular, this second term is the square of the *axial force*

$$N_m(s) \equiv a(s)u'_m(s) \tag{12}$$

in the mth mode shape of the undamaged rod evaluated at the cracked cross-section.

Equation (11) has an important consequence; the ratios of the change in two different natural frequencies depend only on the damage location, not on its severity. That is

$$\frac{\delta v_n}{\delta v_m} = \left(\frac{N_n(s)}{N_m(s)}\right)^2 \equiv f(s), \tag{13}$$

where $s \in (0, L)$ and $\delta v_m < 0$ (if $\delta v_m = 0$, the possible crack locations coincide in the node points of the axial force $N_m(x)$ resulting from the mth vibrating mode of the integral rod). It is to be noted that the methodology pursued by Narkis [8] (section 4) has likewise led to a relation similar to equation (13) (see equation (18), (18a) and (18b) in the paper in question). The relation above has been achieved in reference [8] by considering a linearization of the characteristic polynomial explicitly referred to the cracked rod, in case the damaged system should be a perturbation of the undamaged one. Adapting such a method to the analysis of non-uniform beams is likely to be difficult, while the use of expression (13) makes it possible to overcome such hindrance.

The problem related to the crack location lies in determining the solutions of equation (13) for a fixed (measured) value of the ratio $\delta v_n/\delta v_m$. It follows from equation (13) that all, and the only possible, locations of the crack are the abscissas of the points of the $f(x) = (N_n(x)/N_m(x))^2$ diagram intersecting with the horizontal straight line drawn parallel to the abscissa axis at a distance equal to the ratio $\delta v_n/\delta v_m$. On a practical level, once the free vibration problem related to the integral rod is solved, the behaviour of $f(x)$ is known and therefore it is possible, via numerical methods for example, to determine the solutions of equation (13).

In order to infer some qualitative properties of the damage location problem, in the remainder of this section the behaviour of the function $f(x)$ will be investigated. In order to simplify the analysis it is decided to investigate in detail the case of a free rod (F) under the assumption that $\rho(x) \equiv \gamma A(x)$ in the axial motion equation (1), where γ is the (uniform) volume mass density. The method to be accounted for can be easily extended in such a way as to take general boundary conditions.

Firstly, the case $m = 1$ and $n \geqslant 2$ is considered, which means the variations in the first and in the nth frequencies are taken as data. In the following, it is worth pointing out that if $(u(x), \omega^2)$ is an eigenpair of the eigenvalue problem (1)–(5) for the undamaged rod, then $(N(x) = au'(x), \tilde{\omega}^2 \equiv \omega^2\gamma/E)$ is an eigenpair of Dirichlet's eigenvalue problem

$$(\tilde{a}(x)N'(x))' + \tilde{\omega}^2\tilde{a}(x)N(x) = 0, \quad x \in (0, L),$$

$$N(0) = 0 = N(L), \tag{14}$$

wherein $\tilde{a}(x) \equiv 1/a(x)$. Through dividing by $a(x) = EA(x)$ and subsequently differentiating equation (1) once the result obtained is

$$\left(\frac{1}{a(x)} N'(x)\right)' + \omega^2 \frac{\gamma}{E} u'(x) = 0.$$

Multiplying and dividing the term with $u'(x)$ by $a(x)$ gives the expression we have in equation $(14)_1$. The boundary conditions $(14)_2$ directly derived from the conditions for $u(x)$ for the free–free bar. By virtue of well-known properties of the solutions of the Sturm–Liouville operator $(14)_1$, $N_n(x)$ has $(n-1)$ simple zeroes $\{x_n^{(i)}\}_{i=1}^{n-1}$ in the interval $(0, L)$, say $x_n^{(1)} < x_n^{(2)} < \cdots < x_n^{(n-1)}$. In addition, $x_n^{(0)} = 0$, $x_n^{(n)} = L$. Clearly, $f(x) \geqslant 0$ in $(0, L)$ and $f(x_n^{(i)}) = 0$, $i = 1, n-1$.

It can now be shown that $f(x)$ is a well-defined function in the interval $[0, L]$. For this purpose it is sufficient to note that $N_1(x) \neq 0$ in $(0, L)$ and the following limits prove to be finite: $\lim_{x\to 0^+} f(x)$, $\lim_{x\to L^-} f(x)$. Consider as an example the limit $\lim_{x\to 0^+} f(x)$ which in terms of its being determined is exactly like the other limit. By applying twice the de l'Hôspital's rule it is determined that

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{(N_n'(x))^2 + N_n(x)N_n''(x)}{(N_1'(x))^2 + N_1(x)N_1''(x)}. \tag{15}$$

As a result, the indeterminate form has been solved. By using the differential equation (1) to determine $N_n'(x)$, $N_1'(x)$, while bearing in mind that by virtue of one eigensolutions property of (1)–(5) $u_n(0) \neq 0$ for each n, it follows that

$$\lim_{x\to 0^+} f(x) = \left(\frac{\omega_n^2}{\omega_1^2}\right)^2 \left(\frac{u_n(0)}{u_1(0)}\right)^2. \tag{16}$$

Likewise

$$\lim_{x\to L^-} f(x) = \left(\frac{\omega_n^2}{\omega_1^2}\right)^2 \left(\frac{u_n(L)}{u_1(L)}\right)^2. \tag{17}$$

At this stage it can be asserted that: (A) *$f(x)$ is strictly decreasing in the interval* $(x_n^{(0)}, x_n^{(1)})$; *$f(x)$ has a single relative maximum point in each interval* $(x_n^{(i)}, x_n^{(i+1)})$, $i = 1, n-2$, *which can be designated as* $\xi_n^{(i)}$, *and* $f(\xi_n^{(i)}) = (\omega_n^2/\omega_1^2)^2\,(u_n(\xi_n^{(i)})/u_1(\xi_n^{(i)}))^2$; *$f(x)$ is strictly increasing in the interval* $(x_n^{(n-1)}, x_n^{(n)})$.

The existence and the number of solutions of equation (13) rely naturally on $\delta v_n/\delta v_1$ as well as on the values that $f(x)$ takes at the local maximum points. The former assertion (A), which shall be demonstrated hereinafter, allows some qualitative properties of the solutions to be proved. In fact, from the hypothesis that $\delta v_n < 0$,

(1) there is exactly one crack location, if there exists one, in $(x_n^{(0)}, x_n^{(1)})$ which corresponds to the ascribed ratio $\delta v_n/\delta v_1$;
(2) there are exactly two crack locations, if they exist, in $(x_n^{(i)}, x_n^{(i+1)})$ (which may coincide if $f(\xi_n^{(i)}) = \delta v_n/\delta v_1$), $i = 1, n-2$, corresponding to the ascribed ratio $\delta v_n/\delta v_1$;
(3) the property stated in (1) is thus valid for the interval $(x_n^{(n-1)}, x_n^{(n)})$.

It is to be noted that the assertion (A) directly results in an *a priori* estimate of $\delta v_n/\delta v_1$, that is

$$\frac{\delta v_n}{\delta v_1} \leqslant \max\{f(\xi_n^{(i)}), \quad i = 1, n-2; f(0); f(L)\}, \tag{18}$$

the variation in the nth frequency cannot be on some accounts much greater than the variation in the first frequency. Owing to the uniqueness of the solution of equation (13) the most "unfavourable" situation coincides when $\delta v_n/\delta v_1 < \min\{f(\xi_n^{(i)}), i = 1, n-2; f(0); f(L)\}$: in fact, in this case there are $(n-2)2+2 = 2n-2$ possible locations for the crack and they all exactly lead to the same ratio $\delta v_n/\delta v_1$. What is described above explains why as a rule the employment of a pair of high frequencies increases the number of possible solutions of the damage location problem. In detail, it has been shown that the number of the possible damage locations corresponding to the same ratio $\delta v_n/\delta v_1$ between the variations in the pair of frequencies goes on increasing the number of the points of zero-sensitivity to the damage for the nth frequency. Such points coincide with the nodes of the axial force $N_n(x)$ and this way their number increases as the mode order increases. It is to be noted that if $n = 2$, and on condition that the beam is symmetrical, the crack location can be uniquely determined (except for symmetrical positions), as it can be read in the next section dealing with the perticular case of the uniform beam.

The present analysis is completed by demonstrating the assertion (A). Without affecting the character of generality it can be assumed that $N_1(x) > 0$ in $(0, L)$ and $N_n(x) > 0$ in $(x_n^{(0)}, x_n^{(1)})$. In order to prove the monotonicity of $f(x)$ in $(x_n^{(0)}, x_n^{(1)})$, $f'(x)$ is calculated and it is thus demonstrated that $f'(x) < 0$ within this interval. Accordingly, it follows that

$$f'(x) = \frac{2N_1(x)N_n(x)}{\tilde{a}(x)N_1^4(x)}\, g(x), \tag{19}$$

wherein $g(x)$ is given by the expression

$$g(x) \equiv (N_1(\tilde{a}N_n') - N_n(\tilde{a}N_1'))(x). \tag{20}$$

Taking into consideration equation $(14)_1$ it is determined that

$$g'(x) = (\tilde{\omega}_1^2 - \tilde{\omega}_n^2)\tilde{a}(x)N_1(x)N_n(x). \tag{21}$$

On considering that $N_1(x)N_n(x) > 0$ in $(x_n^{(0)}, x_n^{(1)})$ and $\tilde{\omega}_1^2 < \tilde{\omega}_n^2$ it is verified at once that $g'(x) < 0$ in $(x_n^{(0)}, x_n^{(1)})$. But $g(x_n^{(0)}) = 0$ and then $g(x) < 0$ in the entire interval $(x_n^{(0)}, x_n^{(1)})$. It emerges that $f'(x) < 0$ in $(x_n^{(0)}, x_n^{(1)})$, which was to be demonstrated.

At this point the attention must be diverted to the subsequent interval $(x_n^{(1)}, x_n^{(2)})$. A direct calculation shows that $g(x_n^{(1)}) = \tilde{a}N_1N_n'(x_n^{(1)}) < 0$ and $g(x_n^{(2)}) = \tilde{a}N_1N_n'(x_n^{(2)}) > 0$. Bearing in mind that $N_1(x)N_n(x) < 0$ in $(x_n^{(1)}, x_n^{(2)})$, then $g'(x) > 0$ in the interval in question. As a consequence exactly one single point of $(x_n^{(1)}, x_n^{(2)})$ wherein $g(x)$ vanishes can be found, which can be designated as $\xi_n^{(1)}$. It is therefore possible to draw the conclusion that $f'(x) > 0$ in $(x_n^{(1)}, \xi_n^{(1)})$, $f'(\xi_n^{(1)}) = 0$ and $f'(x) < 0$ in $(\xi_n^{(1)}, x_n^{(2)})$, which is the thesis to be demonstrated solely for the second interval.

It suffices to repeat the steps of the procedure described above to deal with the remaining intervals. Finally, taking into consideration that $g(\xi_n^{(i)}) = 0$ and bearing in mind definition (12) of the axial force N, it can be shown that $f(\xi_n^{(i)}) = (\omega_n^2/\omega_1^2)^2\,(u_n(\xi_n^{(i)})/u_1(\xi_n^{(i)}))^2$, which establishes assertion (A).

For reasons of completeness, the case in which the data are represented by the variations in the m and nth frequencies, wherein $n > m > 1$, is briefly discussed. Without affecting the character of generality, consider $m = 2$ and assume that $\delta v_2 < 0$. The only relevant difference from the former case in which $m = 1$ is due to the fact that the second vibrating mode in the integral rod turns out to have a point of zero-sensitivity to the crack, e.g., $N_2(x_2^{(1)}) = 0$ for $x_2^{(1)} \in (0, L)$. It is easy to verify that the former procedure can also be adapted for this situation, provided there should be a distinction between the first case wherein $x_2^{(1)}$ does not coincide with any node point of $N_n(x)$ (which occurs, for instance, when $n = 3$), and the second case where $x_2^{(1)} = x_n^{(i)}$ for a certain value of i. Leaving aside the details of the demonstration, it follows from the first case that:

(1)′ There are always two crack locations in the interval whose ends coincide with the nodes of $N_n(x)$ which are adjacent to $x_2^{(1)}$.
(2)′ The other intervals $(x_n^{(i)}, x_n^{(i+1)})$, $1 \leqslant i \leqslant n-2$, have exactly two locations of the crack, if they exist, which correspond to the ascribed ratio $\delta v_n/\delta v_2$ (which may coincide if $f(\xi_n^{(i)}) = \delta v_n/\delta v_2$).
(3)′ As for the intervals at the ends of the bar $(x_n^{(0)}, x_n^{(1)})$ and $(x_n^{(n-1)}, x_n^{(n)})$ what was specified in (1) and (3) can be applied.

It follows from the second case that:

(1)″ Two locations of the crack, if they exist (which may coincide if $f(\xi_n^{(i)}) = \delta v_n/\delta v_2$) belong to the interval $(x_n^{(i-1)}, x_n^{(i+1)})$.
(2)″ As for the other intervals, including those at the ends, what was stated in (1)′ and (2)′ can be applied.

It is worth noticing that, since the second mode has a point of zero sensitivity to damage, the ensuing consequence is a lack of control over $\delta v_n/\delta v_2$ as exemplified in equation (18).

3. THEORETICAL RESULTS FOR UNIFORM RODS WITH A CRACK

In section 2, it was shown that the problem of locating a crack in a vibrating rod from knowledge of the damage-induced shifts in a pair of natural frequencies is generally poorly defined: if the system is symmetrical, then a crack at any one of the set of symmetrically placed points will produce identical changes in natural frequencies. Even if the system is not symmetrical, cracks in different locations can still produce identical changes in a pair of natural frequencies. In spite of this, it will be shown that there are certain situations in which the effects of the non-uniqueness of the solution may be considerably reduced by means of a careful choice of the data.

The simple but very common case of uniform rods, e.g., rods for which $a = a(x)$ and $\rho = \rho(x)$ are constants on the interval $(0, L)$ will be considered here. The most exhaustive results concern free vibrating rods (F). C_m^F is denoted by

$$C_m^F = -\frac{\delta v_m^F}{Bm^2}, \tag{22}$$

where m is a positive integer and B is the constant

$$B = \left(a\sqrt{\frac{2}{\rho L}}\frac{\pi}{L}\right)^2. \tag{23}$$

It can be proved that, *if* $C_m^F > 0$, *the measurement of the pair* $\{C_m^F, C_{2m}^F\}$, $m \geqslant 1$, *uniquely determines the severity of the damage*, e.g., *the spring stiffness* K, *and the variable* $S = \cos 2m\pi s/L$ *of the damage location* s.

The eigenpairs of a free uniform rod (F) are given by

$$v_m^F = \frac{a}{\rho}\left(\frac{m\pi}{L}\right)^2, \quad u_m^F(x) = \sqrt{\frac{2}{\rho L}}\cos m\pi x/L, \tag{24}$$

$m = 0, 1, 2, \ldots$. The rigid mode $u_0^F(x)$ obviously is always insensitive to damage. Putting the expressions of v_m^F and $u_m^F(x)$ for $m \geqslant 1$ into equation (11) gives

$$C_m^F = \frac{1}{K} \sin^2 m\pi \frac{s}{L} \tag{25}$$

and, using standard trigonometric identities, gives

$$K(4C_m^F - C_{2m}^F) = 4K^2(C_m^F)^2. \tag{26}$$

Since $C_1^F > 0$, from the identity above it follows that

$$4C_1^F - C_2^F > 0, \tag{27}$$

and

$$\text{if } C_m^F > 0 \quad \text{then } 4C_m^F - C_{2m}^F > 0, \quad m \geqslant 2. \tag{28}$$

Inequalities (27) and (28) represent a particular case of inequality (18) in uniform free–free rods. Let it be assumed that $C_m^F > 0$. Equation (26) can be solved for the damage severity:

$$K = \frac{1 - C_{2m}^F/4C_m^F}{C_m^F}. \tag{29}$$

Note that conditions (27) and (28) guarantee that K takes positive values. By inserting expression (29) of K into equation (25) the damage can be localized:

$$S = \frac{C_{2m}^F}{2C_m^F} - 1, \tag{30}$$

where $S \in [-1, 1)$ because of the inequalities (27) and (28). Note that the ratio of the first two natural frequency changes is sufficient to localize the damage (except for symmetrical positions). Finally, if $C_m^F = 0$ for a certain $m \geqslant 2$, then from equation (25) it follows that $S = 1$; that is the crack is located in one of the points of zero-sensitivity of the mth mode, and K remains undetermined. This establishes the assertion.

The preceding result improves existing results about crack localization in different directions. One of the results by Narkis [8] is a particular case concerning the unique localization of the damage based on knowledge of the first two frequencies of a free–free vibrating rod (see equation (26) in reference [8]). However, it is to be noted that the existence condition on S for $m = 1$, which corresponds to inequality (27) in the present paper, is not explicitly acknowledged in the work by Narkis (e.g., our condition (27) is equivalent to the inequality $R_{A21} \leqslant 4$, where R_{A21} is defined by equation (24) in reference [8]).

Expressions (29) and (30) for the damage parameters indicate that the pair of natural frequencies mth and $2m$th plays a crucial role when localizing the damage. In fact, provided that the mth frequency proves to be sensitive to damage, that is $C_m^F > 0$, the pair $\{C_m^F, C_{2m}^F\}$ uniquely determines the damage severity, namely the stiffness K. Quite surprisingly the expression for K turns out to be the same for all pairs of values $\{C_m^F, C_{2m}^F\}$. Finally, it is shown that the number of the possible crack locations, corresponding to the same ratio

C^F_{2m}/C^F_m, increases as the order m of the modes assessed increases, which accounts for the recourse to "low" frequencies for the problem of damage localization.

The preceding result does not consider the problem that occurs when a pair of values such as $\{C^F_k, C^F_l\}$ with $l \neq 2k$ is chosen as data. It can be shown that in these cases the solution of the inverse problem is generally non-unique (even by leaving symmetrical positions aside). Attention is focused on the crack localization problem and, as an example, the values of the pairs $\{C^F_1, C^F_3\}$ and $\{C^F_1, C^F_4\}$ are regarded as data.

Case 1. *Initially uniform rod* (F) *with data* $\{C^F_1, C^F_3\}$. Equation (25) can be rewritten in function of the variable $y \equiv \cos 2\pi s/L$ for $m = 1, 3$. Then one has the following non-linear system:

$$1 - y = 2KC^F_1, \qquad 1 - 4y^3 + 3y = 2KC^F_3, \tag{31}$$

to be solved with respect to $K > 0$ and $y \in [-1, 1)$ for given data $\{C^F_1, C^F_3\}$. A direct calculation shows that

$$9C^F_1 - C^F_3 > 0. \tag{32}$$

In fact, equation (31) yields

$$2K(9C^F_1 - C^F_3) = 4(y-1)^2(y+2)$$

and the right-hand side of the expression above is always strictly greater than zero for $y \in [-1, 1)$. Denoting by η the ratio

$$\eta \equiv \frac{C^F_3}{C^F_1}, \tag{33}$$

the damage location problem consists of solving the polynomial equation

$$(2y+1)^2 = \eta \tag{34}$$

in the interval $[-1, 1)$ for a given $\eta \in [0, 9)$. For reasons of symmetry, consider only the damage location s in the interval $(0, L/2]$. If $\eta \in (1, 9)$ there is one single solution $\tilde{y}_1 \in (0, 1)$ of equation (34), which corresponds to $s_1 \in (0, L/4)$. If $\eta \in (0, 1]$ there are two distinct solutions $y_1 \in [-1, -1/2)$, $y_2 \in (-1/2, 0]$, which, respectively, correspond to $s_1 \in (L/3, L/2]$ and $s_2 \in [L/4, L/3)$. Finally, if $\eta = 0$ it follows that $y_1 = y_2 = -1/2$ and $s_1 = s_2 = L/3$. Therefore, should the crack be located within the first quarter of the beam adjacent to the free end, the measure of the first and third frequencies determines uniquely the location of the damaged cross-section; should the crack be located within the quarter of the beam $(L/4, L/2)$ and $s \neq L/3$, there are two different locations corresponding to the same ratio η.

Case 2. *Initially uniform rod* (F) *with data* $\{C^F_1, C^F_4\}$. Following the same procedure used for Case 1 and adopting the same notation, the solutions of the polynomial equation are

$$r(y) \equiv 8y^2(y+1) = \eta, \tag{35}$$

where $\eta \equiv C^F_4/C^F_1 < 16$ and $y \in [-1, 1)$. $r(y)$ has a local maximum at $y_{max} = -2/3$ ($r(y_{max}) = 32/27$) and a local minimum at $y_{min} = 0$ ($r(y_{min}) = 0$). Focusing attention on one-half of the beam for reasons of symmetry, if η proves to be "adequately sized", $\eta > 32/27$, the measurement of the first and fourth frequencies localize the crack in a unique manner; if $0 < \eta < 32/27$ there are three different locations of the crack that correspond to the same ratio η; finally, if $\eta = 32/27$ and if $\eta = 0$ there are two distinct possible locations.

The analysis has hitherto been related to uniform beams under axial vibration with free ends and it has been borne out that the first two frequencies allow the crack (except for symmetrical positions) to be uniquely identified. Such a result does not prove true if different boundary conditions, for example (C) or (S), are being considered. For instance, in case (C), when the crack is located in the half of the rod adjacent to the fixed end, there are two distinct locations corresponding to the same ratio between the variations in the first two frequencies, as already asserted by Narkis [8] (section 4.2, equation (25)). Within these situations it is possible to apply the same procedure used for analyzing Cases 1 and 2.

An alternative method of proceeding lies in resorting at the same time to frequency measurements on the cracked rod which derive from different boundary conditions. This way it is entirely feasible to recover the character of uniqueness for the solution of the diagnostic problem.

Attention is again concentrated on initially uniform rods and the data resulting from boundary conditions of the types (S) and (F) are first considered. There emerges that: *the measurement of the* $(m+1)$*th frequency in the cracked rod under boundary conditions of the type* (F) *and of the mth frequency under boundary conditions of type* (S), *for* $m \geqslant 0$, *determines uniquely the severity of the crack and the location variable* $S' = \cos 2(m+1)\pi s/L$, *where s stands for the abscissa of the cracked cross-section.*

The eigenpairs of a supported uniform rod (S) are given by

$$v_m^S = \frac{a}{\rho}\left(\frac{(m+1)\pi}{L}\right)^2, \quad u_m^S(x) = \sqrt{\frac{2}{\rho L}}\sin(m+1)\pi x/L, \tag{36}$$

$m = 0, 1, 2, \ldots$ Define $C_m^S \equiv -\delta v_m^S / B(m+1)^2$. By using a standard trigonometric identity in the system

$$C_{m+1}^F = \frac{1}{K}\sin^2(m+1)\pi s/L, \qquad C_m^S = \frac{1}{K}\cos^2(m+1)\pi s/L, \tag{37}$$

it follows that

$$K = \frac{1}{C_{m+1}^F + C_m^S}. \tag{38}$$

Taking into consideration the expression of K if, for example, $C_m^S > 0$, then

$$S' = -1 + \frac{2}{1 + C_{m+1}^F / C_m^S}. \tag{39}$$

Otherwise, if $C_m^S = 0$ then $S' = -1$. It turns out that the damage is uniquely determined (except for symmetrical positions) by the measurement of the pair $\{C_0^S, C_1^F\}$. The same result is valid when the pair $\{C_m^S, C_{2(m+1)}^F\}$, $m \geqslant 0$, is considered as data. In this case, if $C_m^S > 0$ then $4C_m^S - C_{2(m+1)}^F > 0$ and one has the following expressions for the damage parameters:

$$K = \frac{1 - C_{2(m+1)}^F / 4C_m^S}{C_m^S}, \qquad S' = 1 - \frac{C_{2(m+1)}^F}{2C_m^S}. \tag{40, 41}$$

A similar result holds true also when frequency measurements are derived from (F) and (C) boundary conditions. In this case: *from the knowledge of the mth frequency in the cracked rod under boundary conditions* (C) *and of the* $(1+2m)$*th frequency under boundary conditions*

(F) *it is possible to uniquely determine the stiffness* K *and the position variable* $S'' = \cos(1 + 2m)\pi s/L$, $m = 0, 1, 2, \ldots$.

Indeed, bearing in mind that the eigenpairs in case (C) are given by

$$v_m^C = \frac{a}{\rho}\left(\frac{(1 + 2m)\pi}{2L}\right)^2, \quad u_m^C(x) = \sqrt{\frac{2}{\rho L}}\,\sin(1 + 2m)\pi x/2L, \tag{42}$$

$m = 0, 1, 2, \ldots$, the system corresponding to equation (37) is as follows:

$$C_{1+2m}^F = \frac{1}{K}\sin^2(1 + 2m)\pi s/L, \qquad C_m^C = \frac{1}{K}\cos^2(1 + 2m)\pi s/2L, \tag{43}$$

where $C_m^C \equiv -\dfrac{\delta v_m^C}{B((1 + 2m)/2)^2}$. By proceeding as exemplified above, it follows that

$$K(4C_m^C - C_{1+2m}^F) = 4K^2(C_m^C)^2. \tag{44}$$

If $C_m^C > 0$ then $4C_m^C - C_{1+2m}^F > 0$, thus obtaining

$$K = \frac{1 - C_{1+2m}^F/4C_m^C}{C_m^C}, \qquad S'' = 1 - \frac{C_{1+2m}^F}{2C_m^C}. \tag{45, 46}$$

4. EXPERIMENTS

The preceding sections have shown how to employ the measurement of a pair of axial frequencies of a cracked rod so as to assess the location as well as the severity of the damage. Aiming to account for the prospective practical use of the results above within the analysis of real cases, the present section is devoted to outlining some applications of experimental character.

Before bringing forward the results it is appropriate to make some remarks. The conclusions drawn in the preceding sections, and the respective identification technique, have been inferred from qualitative and quantitative properties underlying the analytical model ruled by equations (7)–(9) for the cracked rod in the case of minor damage. At this point, it is known that as a rule the mono-dimensional analytical model, which is based on the classical theory regarding beams under axial vibration and on the macroscopic description of the notch, provides an efficient assessing of the frequencies in the lower section of the spectrum, whereas it gradually lacks accuracy on increasing of the order of vibrating modes. Such a feature suggests that the employment of lower frequencies in the application, for example, of formulae (29) and (30) should guarantee a more accurate assessing of the damage parameters. From this point of view, choosing the first pair of frequencies turns out to be optimal, even though, as shall be highlighted within the second experiment, under certain circumstances one of the frequencies in question may be subject to serious modelling errors. If such is the case, it is safer to resort to frequency pairs of higher order also.

The second aspect concerns the perturbative character of the present analysis. The work hypothesis, as stated beforehand in section 2, is that the crack should be small, namely the cracked configuration should be a perturbation of the undamaged one. Such a hypothesis is reasonable from the practical point of view, in that it is crucial to be in a position to identify correctly the damage as it arises, whilst on the other hand the crack-induced variations in the frequencies prove to be small only in the case of low vibrating modes. In fact, it can be noted in expression (11) that the frequency variation caused by the crack increases as the order of the mode increases. Such a mutual relation is in agreement with a general property proved in reference [11] according to which the high-frequency spectrum of the cracked rod splits into two branches corresponding to the asymptotic course of the spectrum of the two rod segments adjacent to the cracked cross-section (under adequate boundary conditions). The asymptotic separation of the spectrum is more heavily marked in the case of severe damage. Also, the latter aspect suggests the use of lower frequencies for the damage identification.

Summing up, the technique when applied to real cases is expected to yield the more reliable results when the damage is less severe and the lower the order of the frequencies considered. Furthermore, it is worth noting that a relevant factor is represented by possible measurement and modelling errors to which the technique, as shall be seen, seems to be sensitive. Some applications to axially vibrating bars shall be considered here. All the experimental models consisted of steel bars under free–free (F) boundary conditions. Every specimen was damaged by saw-cutting the transversal cross-section. The width of each notch was equal to 1·5 mm and, because of the small level of the excitation, during the dynamic tests each notch remains always open.

The first model (rod 1), in Figure 1(a), is a steel rod of square solid cross-section. By using an impulsive dynamic technique, the first 30 natural frequencies of the undamaged bar and of the bar under a series of three damage configurations (D1, D2 and D4) were determined. The rod was suspended by two steel wire ropes to simulate free–free boundary conditions. The excitation was introduced at one end by means of an impulse force hammer, while the axial response was measured by a piezoelectric accelerometer fixed in the centre of an end cross-section of the rod. Vibration signals were acquired by a dynamic analyser and then determined in the frequency domain to measure the relevant frequency response term (inertance). The well-separated vibration modes and the very small damping allowed indentification of the natural frequencies by means of the single mode technique; see reference [11] (section 5, second experiment) for a complete account of the experiment. The damage configurations were obtained by introducing a notch of increasing depth at $s = 1{\cdot}00$ m from one end. Table 1 compares the experimental natural frequencies and their corresponding analytical estimates for the undamaged and damaged rod. For the definition of the analytical model for the damaged rod, the theoretical value of the stiffness K, for each damage configuration, was obtained by assuming that the position s of the damage is known and by taking the measured value for the fundamental elastic frequency of the damaged rod. The analytical model turns out to be extremely accurate for all the configurations under investigation and the percentage discrepancy between the measured and the analytical values of the natural frequencies is lower than 1% within the 30th vibrating mode. The severity and the location of the damage have been achieved by applying the formulae (29) and (30). The results of the identification are summed up in Tables 2 and 3. With reference to the localization of the notched cross-section the accuracy of the method proves to be satisfactory, even though the discrepancy between assessed damage location and actual damage location becomes more relevant when experimental data are resorted to (see Table 2). It is pointed out that the inaccuracy resulting from the pair $\{C_3^F, C_6^F\}$ is due to the location of the notch in proximity to a zero-sensitivity point of the third vibrating mode (and, as a result, of all the vibrating modes with an order which is a multiple of three). This aspect also prejudices the reliability in assessment of the constant K whenever it is used for the identification of a frequency that is associated with a vibrating

TABLE 1

Experimental and analytical frequencies f_n of rod 1 (data from Morassi [11]). Abscissa of the cracked cross-section: $s = 1{\cdot}000$ m. (1) $EA = 9{\cdot}9491 \times 10^7$ N, $\rho = 3{\cdot}735$ kg/m, $L = 2{\cdot}925$ m ($K = \infty$). (2) $K_{anal} = 3{\cdot}09119 \times 10^{10}$ N/m. (3) $K_{anal} = 7{\cdot}84984 \times 10^9$ N/m. (4) $K_{anal} = 4{\cdot}37183 \times 10^8$ N/m. Frequency values in Hz.

n	*Undamaged*		*Damage D1*		*Damage D2*		*Damage D4*	
	Exper.	Model (1)	Exper.	Model (2)	Exper.	Model (3)	Exper.	Model (4)
1	882·25	882·25	881·5	881·5	879·3	879·3	831·0	831·0
2	1764·6	1764·5	1763·3	1763·1	1759·0	1759·2	1679·5	1680·1
3	2645·8	2646·8	2644·0	2646·7	2647·0	2646·7	2646·5	2645·4
4	3530·3	3529·0	3526·8	3525·7	3516·5	3516·2	3306·0	3308·6
5	4411·9	4411·3	4408·8	4408·2	4400·0	4399·5	4250·0	4251·1
6	5293·9	5293·5	5294·3	5293·3	5295·3	5292·9	5287·8	5282·3
7	6175·4	6175·8	6168·8	6169·7	6150·3	6151·8	5808·5	5802·5
8	7056·7	7058·0	7052·0	7053·7	7039·5	7041·6	6864·3	6867·7
9	7937·9	7940·3	7937·5	7939·7	7938·0	7938·2	7909·5	7901·4
10	8819·9	8822·5	8809·8	8813·4	8782·0	8786·5	8340·0	8341·1
11	9702·7	9704·8	9697·3	9699·8	9682·8	9685·7	9503·3	9509·8
12	10583·8	10587·0	10582·8	10585·9	10581·3	10582·3	10514·8	10497·8
13	11464·3	11469·3	11449·0	11457·0	11410·5	11420·9	10933·5	10928·6
14	12345·2	12351·5	12339·5	12346·3	Not available	12331·5	12158	12166
15	13224·4	13233·8	13222·8	13231·5	13322·0	13224·8	13098	13077
16	14 104	14 116	14 087	14 101	14 039	14 056	13 543	13 554
17	14 985	14 998	14 979	14 993	14 964	14 979	14 811	14 829
18	15 862	15 881	15 860	15 877	15 850	15 865	15 676	15 648
19	16 740	16 763	16 721	16 744	16 662	16 691	16 177	16 201
20	17 620	17 645	17 616	17 640	17 596	17 628	17 464	17 497
21	18 496	18 527	18 488	18 521	18 478	18 504	18 237	18 216
22	19 372	19 410	19 351	19 388	19 283	19 328	18 820	18 860
23	20 248	20 292	20 245	20 288	20 227	20 277	20 111	20 168
24	21 124	21 174	21 121	21 166	21 102	21 139	20 801	20 786
25	21 999	22 056	21 978	22 033	21 906	21 966	21 441	21 527
26	22 870	22 939	22 863	22 936	22 872	22 927	22 815	22 841
27	23 744	23 821	23 741	23 809	23 724	23 773	23 357	23 356
28	24 621	24 703	24 599	24 677	24 532	24 606	24 137	24 197
29	25 495	25 585	25 498	25 583	25 512	25 578	Not available	25514
30	26 372	26 468	26 367	26 452	26 344	26 405	25 919	25 929

mode with a multiple-of-three order (see Table 3). Moreover, in this case the accuracy of the estimate for K generally becomes worse when the identification is based on experimental data. Finally, the expectations are further confirmed, as the assessment of K proves un-reliable when the crack is very severe, and such is the case in the configuration D_4 (see Table 3, columns 6 and 7). In this case, the discrepancies could also be caused by the fact that bending vibrations are also excited.

In the second experiment, the steel rod of series HE100B (rod 2) shown in Figure 1(b) was considered. By adopting an experimental technique similar to that used for rod 1, the undamaged bar and two damaged configurations obtained by introducing a notch of increasing severity at the cross-section 0·7 m far from one end were studied (see reference

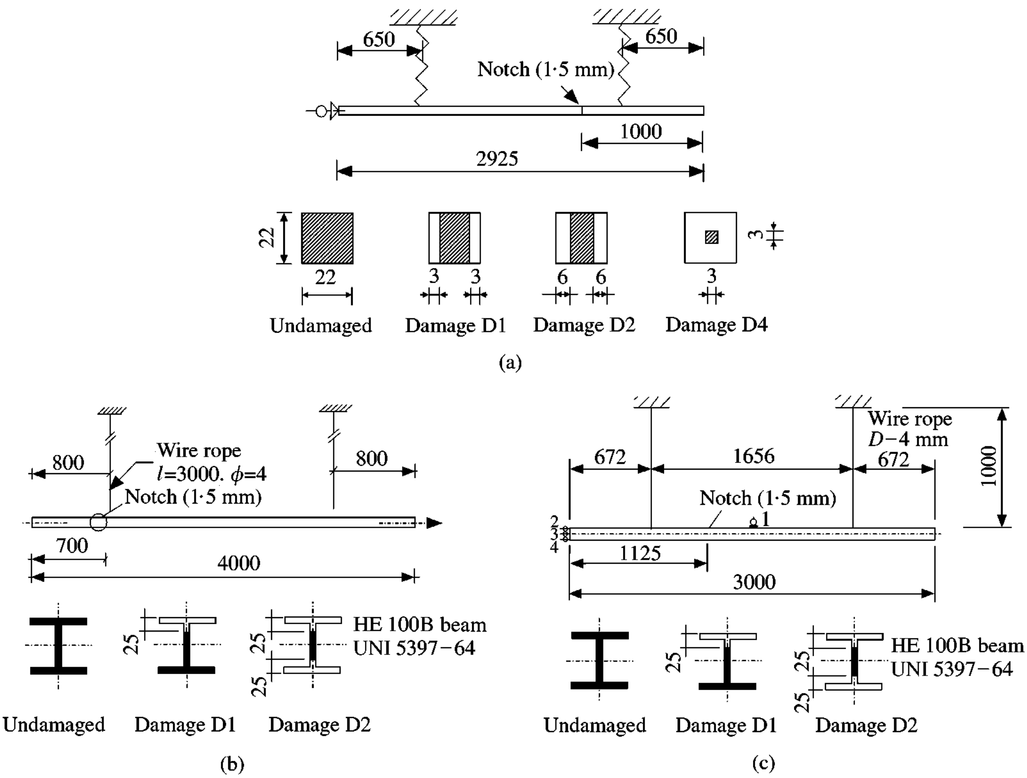

Figure 1. (a)–(c). Experimental models and damage configurations: (a) rod 1; (b) rod 2; (c) rod 3. Length in mm.

[12] for more detail on the experiments). Table 4 shows the measured and analytical values for the first 16 modes. The analytical model for the damaged rod was obtained in the same way as before assuming that the damage location is known and determining K by taking the measured value for the analytical fundamental elastic frequency. The analytical model generally fits well with the real case of the notched rod, even if the percentage errors are more relevant than those of the first experiment. Note that the second and ninth frequencies for D_1 and the seventh frequency for D_2 are affected by relatively large errors. The results of the damage localization are reported in Table 5. It can be noted that the employment of the second frequency, which is more affected by modelling errors, prejudices the reliability of the identification based on the experimental data. In such cases, it is quite advisable to resort to pairs of higher frequency, such as the pair $\{C_3^F, C_6^F\}$, and subsequently intersect the results to identify the beam section which is damaged. Also, the estimates obtained for K confirm this aspect (see Table 6). Moreover, as expected, the estimates for K are already quite rough for damage configuration D_2 even if low frequencies are used in identification. This is because the rod provides an example for which the damage is rather severe from the beginning.

The third experimental model (rod 3), shown in Figure 1(c), has the same cross-section as the rod assessed in the former experiment (see reference [11] for further details). The damage is a notch of increasing severity in the cross-section located 1·125 m from one end. Table 7 compares the first eight measured frequencies for all test configurations with their analytical estimates. The analytical model of the damaged rod is as before. The results of the

TABLE 2

Determination of the crack location in rod 1 by using the pair $\{C_m^F, C_{2m}^F\}$, $m = 1$–3, as data in formula (30). Analytical (s_{anal}) and experimental (s_{exper}) estimates of the crack location in the interval $(0, L/2)$. Actual crack location $s = 1{\cdot}0$ m. The symbol (*) means: estimate $s \notin (0, L/2)$. Length in meters

	Damage D1		Damage D2		Damage D4	
m	s_{anal}	s_{exper}	s_{anal}	s_{exper}	s_{anal}	s_{exper}
1	0·993	1·012	1·003	0·989	1·023	1·021
2	0·464	0·443	0·460	0·457	0·449	0·448
	0·998	1·019	1·002	1·005	1·013	1·015
3	0·325	(*)	0·163	0·366	0·089	(*)
	0.650	(*)	0·812	0·609	0·966	(*)
	1·300	(*)	1·138	1·341	0·984	(*)

TABLE 3

Determination of the spring stiffness K in rod 1 by using the pair $\{C_m^F, C_{2m}^F\}$, $m = 1$–15, as data in formula (29). Analytical (K_{anal}) and experimental (K_{exper}) estimates of the spring stiffness. The symbol (*) means: $K < 0$. Stiffness values in N/m

	Damage D_1		Damage D_2		Damage D_4	
m	$K_{anal} \times 10^{10}$	$K_{exper} \times 10^{10}$	$K_{anal} \times 10^{9}$	$K_{exper} \times 10^{9}$	$K_{anal} \times 10^{8}$	$K_{exper} \times 10^{8}$
1	3.06884	3.13548	7.90078	7.77130	4.78262	4.77243
2	3.02513	3.06369	7.91843	7.42762	4.92504	4.86351
3	67.52010	5.14409	225.10450	(*)	5.22612	(*)
4	3.04629	2.85448	7.88915	7.35445	5.00392	4.92191
5	3.06546	2.87083	7.88033	7.60444	5.98934	5.97505
6	28.13585	(*)	6.30075	(*)	1.22442	1.89584
7	3.07808	2.84088	7.85500	Not available	5.43360	5.51616
8	3.15008	2.80207	6.06778	7.38615	8.09784	8.07323
9	7.50478	25.36635	6.16743	(*)	17.74320	17.66690
10	3.07268	2.82662	7.85898	7.30876	6.15636	6.16281
11	2.97266	3.14878	8.02601	7.35128	11.09590	10.97391
12	2.98030	28.54265	5.34237	(*)	18.60771	21.92602
13	3.07655	2.40688	7.82641	7.30516	7.21803	7.42754
14	3.03196	3.82819	8.29258	Not available	15.06012	15.35033
15	25.56355	17.18915	6.29049	(*)	16.52263	19.87444
Actual value	3.09119		7.84984		4.37183	

damage identification are summarized in Tables 8 and 9, and they essentially confirm those obtained for the previous case.

5. A BENDING VIBRATION CASE

In the previous sections, the inverse problem of identifying a crack in an axially vibrating beam from frequency measurements has been discussed. Here a cracked beam in bending

TABLE 4

Experimental and analytical frequencies f_n of rod 2 (data from Biscontin et al. [12]). Abscissa of the cracked cross-section: $s = 0{\cdot}700$ m. (1) $EA = 5{\cdot}55078 \times 10^8$ N, $\rho = 20{\cdot}775$ kg/m, $L = 4{\cdot}0$ m ($K = \infty$). (2) $K_{anal} = 2{\cdot}76462 \times 10^9$ N/m. (3) $K_{anal} = 9{\cdot}59952 \times 10^8$ N/m. Frequency values in Hz. Δf_n % $= (f_n(model) - f_n(exp.))/f_n(exp) \times 100$

	Undamaged			Damage D1			Damage D2		
n	Exper.	Model (1)	Δ_n %	Exper.	Model (2)	Δ_n %	Exper.	Model (3)	Δ_n %
1	646·125	646·125	0·00	637·000	637·000	0·00	618·500	618·500	0·00
2	1290·875	1292·250	0·11	1202·375	1239·355	3·08	1142·125	1144·008	0·16
3	1935·125	1938·375	0·17	1846·000	1851·259	0·28	1744·750	1745·710	0·06
4	2579·875	2584·500	0·18	2495·500	2517·129	0·87	2450·875	2452·138	0·05
5	3220·250	3230·625	0·32	3199·625	3210·806	0·35	3180·875	3188·100	0·23
6	3844·625	3876·750	0·84	3834·000	3871·494	0·98	3817·375	3858·825	1·09
7	4544·500	4522·875	−0·48	4407·375	4398·909	−0·19	4054·875	4170·535	2·85
8	5169·875	5169·000	−0·02	4904·500	4945·986	0·85	4801·125	4789·760	−0·24
9	5809·250	5815·125	0·10	5711·375	5632·423	−1·38	5541·750	5544·809	0·06
10	6445·875	6461·250	0·24	6379·250	6363·861	−0·24	6305·000	6312·691	0·12
11	7081·125	7107·375	0·37	7090·125	7092·172	0·03	7057·000	7077·117	0·29
12	7713·625	7753·500	0·52	7694·000	7701·443	0·10	7538·375	7579·852	0·55
13	8342·375	8399·625	0·69	8024·000	8099·616	0·94	7863·750	7902·422	0·49
14	8969·000	9045·750	0·86	8806·250	8747·635	−0·67	8602·250	8659·362	0·66
15	9584·500	9691·875	1·12	9441·250	9491·577	0·53	9351·625	9434·566	0·89
16	10189·875	10338·000	1·45	10128·000	10247·613	1·18	10079·875	10211·333	1·30
17	10777·750	10984·125	1·91	10779·500	10981·189	1·87	10779·500	10977·323	1·84

TABLE 5

Determination of the crack location in rod 2 by using the pair $\{C^F_m, C^F_{2m}\}$, $m = 1$–3, as data in formula (30). Analytical (s_{anal}) and experimental (s_{exper}) estimates of the crack location in the interval $(0, L/2)$. Actual crack location $s = 0{\cdot}7$ m. The symbol (*) means: estimate $s \notin (0, L/2)$. Length in meters

	Damage D1		Damage D2	
m	s_{anal}	s_{exper}	s_{anal}	s_{exper}
1	0·717	(*)	0·811	0·809
2	0·738	0·774	0·779	0·783
	1·262	1·226	1·221	1·217
3	0·629	0·614	0·620	0·609
	0·704	0·719	0·713	0·725
	1·963	1·948	1·953	1·942

will be considered. The physical model, which will be investigated, is a simply supported uniform Euler–Bernoulli beam with an open crack at the cross-section of abscissa s. According to Freund and Herrmann [9] the crack is represented by the insertion of a massless rotational elastic spring at the damaged cross-section. The stiffness K_B of the spring may be related in a precise way to the geometry of the damage as suggested, for example, by Dimarogonas and Paipetis [10]. Denoting the Young's modulus of the

TABLE 6

Determination of the spring stiffness K in rod 1 by using the pair $\{C_m^F, C_{2m}^F\}$, $m = 1$–8, *as data in formula* (29). *Analytical* (K_{anal}) *and experimental* (K_{exper}) *estimates of the spring stiffness. The symbol* (*) *means*: $K < 0$. *Stiffness values in N/m*

m	Damage D1		Damage D2	
	$K_{anal} \times 10^9$	$K_{exper} \times 10^9$	$K_{anal} \times 10^8$	$K_{exper} \times 10^8$
1	2·82224	(*)	11·7369	11·69223
2	2·90583	1·84565	11·3523	11·37091
3	3·13432	3·04760	14·51221	14·60803
4	3·18127	2·63991	17·96205	18·45092
5	8·81534	13·05314	60·00963	63·73511
6	(*)	39·28373	(*)	39·93151
7	3·59458	3·94386	15·94803	12·19434
8	3·11783	2·69218	18·78874	19·40381
Actual value	2·76462		9·59952	

TABLE 7

Experimental and analytical frequencies f_n *of rod* 3 (*data from Morassi* [11]). *Abscissa of the cracked cross-section*: $s = 1{\cdot}125\ m$. (1) $EA = 5{\cdot}4454 \times 10^8\ N$, $\rho = 20{\cdot}4\ kg/m$, $L = 3{\cdot}0\ m$ ($K = \infty$). (2) $K_{anal} = 2{\cdot}28783 \times 10^9\ N/m$. (3) $K_{anal} = 9{\cdot}43470 \times 10^8\ N/m$. *Frequency values in Hz.* $\Delta f_n\ \% = (f_n(model) - f_n(exp))/f_n(exp) \times 100$

n	Undamaged			Damage D1			Damage D2		
	Exper.	Model (1)	Δ_n %	Exper.	Model (2)	Δ_n %	Exper.	Model (3)	Δ_n %
1	861·4	861·1	0·00	805·7	805·7	0·00	737·6	737·6	0·00
2	1722·2	1722·2	0·00	1664·5	1661·1	−0·20	1600·0	1597·4	−0·16
3	2582·9	2583·3	0·02	2541·9	2552·2	0·41	2505·3	2508·6	0·13
4	3434·2	3444·4	0·30	3162·2	3209·0	1·48	3016·0	3018·8	0·09
5	4353·6	4305·5	−1·10	4332·2	4262·6	−1·60	4310·2	4223·8	−2·00
6	5174·4	5166·6	−0·15	4961·1	4966·7	0·11	4812·6	4805·5	−0·15
7	6020·0	6027·7	0·13	5750·2	5747·3	−0·05	5616·0	5624·3	0·15
8	6870·5	6888·8	0·27	6860·2	6888·8	0·42	6851·3	6888·8	0·55
9	7726·4	7749·9	0·30	7302·3	7325·2	0·31	7095·8	7110·9	0·21

material by E and the volume mass density by γ, the mth eigenpair $(w_m(x), v_{dm}^{S-S} \equiv \omega_{dm}^2)$, $m = 0, 1, 2, \ldots,$ of the bending vibrations of the cracked beam satisfies the following boundary value problem:

$$(EIw''(x))'' = \omega_{dm}^2 \gamma A w(x), \quad x \in (0, s) \cup (s, L), \tag{47}$$

$$w = 0 = w'' \quad \text{at } x = 0 \text{ and } x = L, \tag{48}$$

where the jump conditions

$$[w(s)] = [w''(s)] = [w'''(s)] = 0, \tag{49}$$

$$EIw''(s) = K_B[w'(s)], \tag{50}$$

TABLE 8

Determination of the crack location in rod 3 by using the pair $\{C_m^F, C_{2m}^F\}$, $m = 1$–3, as data in formula (30). Analytical (s_{anal}) and experimental (s_{exper}) estimates of the crack location in the interval $(0, L/2)$. Actual crack location $s = 1{\cdot}125$ m. The symbol (*) means: estimate $s \notin (0, L/2)$. Length in meters

	Damage D1		Damage D2	
m	s_{anal}	s_{exper}	s_{anal}	s_{exper}
1	1·134	1·146	1·146	1·150
2	0·388	0·340	0·416	0·416
	1·112	1·160	1·084	1·084
3	0·151	0·204	0·221	0·227
	0·849	0·796	0·779	0·773
	1·151	1·204	1·221	1·227

TABLE 9

Determination of the spring stiffness K in rod 3 by using the pair $\{C_m^F, C_{2m}^F\}$, $m = 1$–4, as data in formula (29). Analytical (K_{anal}) and experimental (K_{exper}) estimates of the spring stiffness. Stiffness values in N/m

	Damage D1		Damage D2	
m	$K_{anal} \times 10^9$	$K_{exper} \times 10^9$	$K_{anal} \times 10^8$	$K_{exper} \times 10^8$
1	2·50717	2·51753	11·84541	11·88132
2	2·74212	2·34800	15·20423	15·50771
3	3·14460	4·11534	26.00572	26·26323
4	2·74983	2·38855	15·65690	15·86864
	Actual value	2·28783	9·43470	

hold at the cross-section where the crack occurs. In the equations above I and A represent the moment of inertia and the area of the cross-section respectively.

If the crack is small, namely K_B is large enough, on proceeding as in reference [3] and with the above notation, the first order variation of the mth eigenvalue with $1/K_B$ is given by

$$\delta v_m^{S-S} = -\frac{(M_m(s))^2}{K_B}, \tag{51}$$

where $M_m(s) \equiv -EIu_m''(s)$ is the bending moment at the cross-section of abscissa s in the mth (normalized) bending mode of the undamaged beam.

At this stage the problem of identifying the position and severity of the crack from knowledge of the changes in a pair of natural frequencies can be posed. In the above-mentioned paper [8], it was shown that knowledge of the first two frequency changes induced by the damage suffices to identify uniquely the crack location (except for symmetrical positions). Here an improvement of such result is presented. Denote by C_m^{S-S} the quantity

$$C_m^{S-S} = -\frac{\delta v_m^{S-S}}{B(m+1)^4}, \tag{52}$$

where $m \geqslant 0$ is a non-negative integer and B is the constant

$$B = \left(EI\sqrt{\frac{2}{\rho L}}\left(\frac{\pi}{L}\right)^2\right)^2. \tag{53}$$

It can be proved that, *if* $C_m^{S-S} > 0$, *the measurement of the pair* $\{C_m^{S-S}, C_{2m}^{S-S}\}$, $m \geqslant 0$, *determines uniquely the severity of the damage*, e.g., *the spring stiffness* K_B, *and the variable* $S = \cos 2(m+1)\pi s/L$ *of the damage location s.* To show this, it suffices to observe that the eigenpairs of the simply supported uniform beam in bending vibrations are

$$v_m^{S-S} = \frac{EI}{\rho}\left(\frac{(m+1)\pi}{L}\right)^4, \quad u_m^{S-S}(x) = \sqrt{\frac{2}{\rho L}}\sin(m+1)\pi\frac{x}{L}, \tag{54}$$

$m = 0, 1, 2, \ldots$, and then repeat the same procedure used for the free–free axial vibration case to obtain expressions (29) and (30) for K_B and S respectively.

Finally, a result similar to that attained at the end of section 3 is valid for cracked beams in bending also. *If m is a non-negative integer, then the measurement of the mth frequency of the cracked beam under simply supported boundary conditions* (S–S) *and of the* $(m+1)$th *frequency for sliding–sliding boundary conditions* (Sl–Sl) (e.g., $w' = w''' = 0$ *at* $x = 0$ *and* $x = L$) *determines uniquely the stiffness* K_B *of the rotating spring and the position variable* $S' = \cos 2(m+1)\pi s/L$, *wherein s is the abscissa of the cracked cross-section.*

Defining $C_m^{Sl\text{-}Sl} \equiv -\delta v_m^{Sl\text{-}Sl}/Bm^4$ with $m \geqslant 1$, and keeping the conventional meaning of the symbols stable and with $m = 0, 1, 2 \ldots$, it is possible to attain the following expressions for the damage parameters:

$$K_B = \frac{1}{C_{m+1}^{Sl\text{-}Sl} + C_m^{S\text{-}S}}, \tag{55}$$

$$S' = -1 + \frac{2}{1 + C_m^{S\text{-}S}/C_{m+1}^{Sl\text{-}Sl}}. \tag{56}$$

The latter expression is valid if $C_{m+1}^{Sl\text{-}Sl} > 0$. If $C_{m+1}^{Sl\text{-}Sl} = 0$ then $S' = -1$.

6. CONCLUSIONS

This paper has been focused on detecting a single crack from the knowledge of the damage-induced shifts in a pair of natural frequencies of a vibrating rod. In spite of the problem being ill-posed, it was found that there are certain situations concerning uniform rods in which the non-uniqueness of the solution may be considerably reduced by means of a careful choice of the data. The analysis is based on an explicit expression of the frequency sensitivity to damage and allows non-uniform bars under general boundary conditions to be considered. Analytical results agree well with experimental tests on cracked steel rods. Some of the results are also valid for cracked beams in bending.

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