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A CLOSED-FORM SOLUTION OF A BERNOULLI–EULER BEAM ON A VISCOELASTIC FOUNDATION UNDER HARMONIC LINE LOADS

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Fourier transform is used in this paper to solve the problem of steady state response of a beam on a viscoelastic foundation subjected to a harmonic line load. The solution is constructed in the form of the convolution of the Green function of the beam. The theorem of residue is employed to evaluate the generalized integral such that a closed-form solution can be achieved. All the different combinations of damping and vibration frequency are discussed and analytical solutions are presented. As a special case, the solution of the beam on a Winkler foundation is also discussed. The validation of the solution is verified by considering the static solution of the beam and comparing the degraded solution to a well-known result. The closed-form expression of the result can be used to construct algorithms for the inverse problems of non-destructive testing of pavement structures using vibration devices.

1. INTRODUCTION

Non-destructive testing and evaluation have received much attention in the field of pavement engineering since the 1980s [1–3]. As one of the most commonly used non-destructive testing devices for pavement structural evaluation, vibratory devices, including the Dynaflect, Road Rater, and the US Army Engineer Waterways Experiment Station 16-kip (71 kN) vibrator, exert steady state dynamic loads on pavement surfaces [3]. The deflections of the pavement surface at different locations are measured using geophones. Pavement structural evaluation is then achieved based on the inverse analysis of the recorded response of the pavement structure to the steady state load.

The mathematical problem involved in interpreting the deflection response is to estimate the parameters of the governing equations of pavement structures assuming that the applied load is known as a harmonic line load. Given the complexity involved in the inverse problem, the most widely used techniques for the parameter estimation of pavement structures are to use the static analysis of pavement structure and then compare the measured response with the calculated response in the context of optimization with certain objective functions. Typical models for representing pavements are an infinite beam or plate resting on a Winkler foundation [4–8]. The pavement structural parameters are eventually determined by identifying the parameters of a pavement structure whose calculated response is most close to the measured response in terms of certain objective functions [9].

Clearly, there are two fatal defects involved in the current approach. One is that the calculated response of pavement is based on a static analysis, which ignores the time,

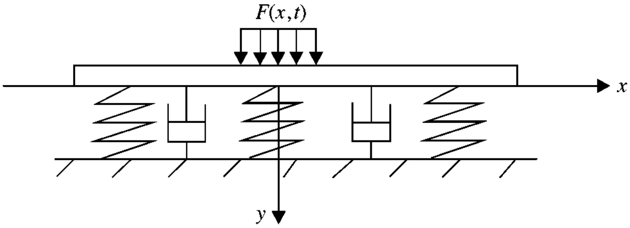

Figure 1. A beam on a viscoelastic foundation subjected to a harmonic line load.

inertial and damping effects. Therefore, the theoretical model is apparently inconsistent with the realistic loading condition. The other problem is that the uniqueness is always in conjunction with inverse problems. Consequently, even though the comparison between the calculated and the measured responses may identify a set of structural parameters that fit the experimental data well, these parameters may not be the actual parameters of the pavement structure. These problems are essentially caused by a lack of awareness of the dynamic response of pavement structures to the steady state loads. To better understand the dynamic response of pavement structures to the steady state loading condition, it is necessary to investigate the forward problem, that is, the response of pavement structures under harmonic line loads.

This paper focuses on the analytical solution of an infinite Bernoulli–Euler beam on a viscoelastic foundation subjected to a harmonic line load. Similar to the method used in reference [10], the Fourier transform is utilized to simplify the governing equation of the beam to an algebraic equation. It is found that the response properties are different for low- and high-frequency vibration, which confirms the results obtained in reference [11].

The content of this paper is organized as follows. The related problem is formulated in section 2 and the associated viscoelastic foundation model and its derivative, i.e., the Winkler foundation model, are also discussed. In section 3, the Green function of the beam is derived using the integral transform method. In section 4, the solution is constructed using a convolution in terms of the Green function obtained in section 3. In section 5, a closed-form solution is developed using complex function techniques. Special cases, such as the static solution and Winkler foundation, are discussed in section 6.

2. PROBLEM FORMULATION

Figure 1 depicts the co-ordinate system and significant dimensions. The infinite length of the beam, respectively, runs along the x-axis.

Denote $y(x, t)$ as the deflection of the beam in the y direction, in which x represents the travelling direction of the pavement structure, and t the time. The well-known governing equation of a Bernoulli–Euler beam on a foundation is (see references [12–14])

$$\mathrm{EI}\,\frac{\partial^4 y}{\partial x^4} + m\,\frac{\partial^2 y}{\partial t^2} = F(x, t) - Q(x, t), \tag{1}$$

where EI is the rigidity of the beam, E is Young's modulus of elasticity, I is the moment of inertia of the beam, m is the unit mass of the beam, $F(x, t)$ and $Q(x, t)$ are applied external loads and restoring forces from the foundation, respectively.

One of the most commonly used foundation models in pavement design is the Winkler foundation model. It performs well in many circumstances (see references [5, 7, 14]). The

Winkler foundation model assumes that the reactive pressure is proportional to the deflection of the beam, that is $Q = Ky$. The term K is called the modulus of subgrade reaction. The assumption that K is constant implies the linear elasticity of the subgrade. In reality, damping effects always exist in any dynamic system. If the damping effect of the subgrade is considered, the restoring force $Q = Ky + C\partial y/\partial t$. This is a viscoelastic foundation model consisting of a spring of stiffness K and a dashpot of viscosity, C, placed parallel, as shown in Figure 1. Substitution of the restoring force into equation (1) gives

$$\mathrm{EI}\,\frac{\partial^4 y}{\partial x^4} + Ky + C\,\frac{\partial y}{\partial t} + m\,\frac{\partial^2 y}{\partial t^2} = F(x, t). \tag{2}$$

A harmonic line load can be expressed by

$$F(x, t) = P\,\frac{H(r_0^2 - x^2)}{2r_0}\exp(\mathrm{i}\Omega t), \tag{3}$$

where r_0 is the halfwidth of the line load, Ω and P are frequency and amplitude of the applied steady state load, respectively, and $H(\cdot)$ is the Heaviside step function, which is defined by

$$H(x - x_0) = \begin{cases} 0 & \text{for } x < x_0, \\ \frac{1}{2} & \text{for } x = x_0, \\ 1 & \text{for } x > x_0. \end{cases} \tag{4}$$

Assume the beam is at rest initially. This means that the initial condition of the beam displacement is zero. Equations (2) and (3) constitute the mathematical description of the problem considered here.

3. THE GREEN FUNCTION OF THE BEAM

According to the theory of mathematical–physical equations, the Green function of a partial differential equation represents the fundamental solution of the equation as the load condition is given in the form of the Dirac-delta function. For the current problem, the Green function of the beam is defined as the solution of equation (2), given that the external load is characterized by

$$F_\delta(x, t) = \delta(x - x_0)\delta(t - t_0), \tag{5}$$

in which $\delta(\cdot)$ is the Dirac-delta function. It is defined by

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x)\,\mathrm{d}x = f(x_0). \tag{6}$$

Define the two-dimensional (2-D) Fourier transform and its inversion as

$$\tilde{f}(\xi, \omega) = \mathbf{F}[f(x, t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, t)\exp[-\mathrm{i}(\xi x + \omega t)]\,\mathrm{d}x\,\mathrm{d}t, \tag{7a}$$

$$f(x, t) = \mathbf{F}^{-1}[\tilde{f}(\xi, \omega)] = (2\pi)^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}(\xi, \omega)\exp[\mathrm{i}(\xi x, \omega t)]\,\mathrm{d}\xi\,\mathrm{d}\omega, \tag{7b}$$

where $\mathbf{F}[\cdot]$ and $\mathbf{F}^{-1}[\cdot]$ are the Fourier transform and its inversion respectively. To solve the Green function, take the 2-D Fourier transform of both sides of equation (2),

$$\mathrm{EI}\,\xi^4 \tilde{G}(\xi,\omega;x_0,t_0) + K\tilde{G}(\xi,\omega;x_0,t_0) + \mathrm{i}C\omega\tilde{G}(\xi,\omega;x_0,t_0) - m\omega^2\tilde{G}(\xi,\omega;x_0,t_0) = \tilde{F}(\xi,\omega), \tag{8}$$

in which $\tilde{F}(\xi,\omega)$ is the Fourier transform of $F_\delta(x,t)$, and the deflection response $y(x,t)$ has been replaced by the symbol $\tilde{G}(\xi,\omega;x_0,t_0)$ to indicate the Fourier transform of the Green function. Also, the following property of Fourier transform is used in the derivation of equation (8):

$$\mathbf{F}[f^{(n)}(t)] = (\mathrm{i}\omega)^n\mathbf{F}[f(t)]. \tag{9}$$

Since $\tilde{F}_\delta(\xi,\eta)$ is the representation of $F_\delta(x,t)$ in the frequency domain, it is also needed to evaluate the Fourier transform of $F_\delta(x,t)$. This can be implemented by taking the 2-D Fourier transform of both sides of equation (5):

$$\tilde{F}_\delta(\xi,\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-x_0)\delta(t-t_0)\exp[-\mathrm{i}(\xi x+\omega t)]\,\mathrm{d}x\,\mathrm{d}t = \exp[-\mathrm{i}(\xi x_0+\omega t_0)], \tag{10}$$

in which the property of the Dirac-delta function, i.e., equation (6), is utilized for evaluating the above integral. Substitute this result (10) into equation (8) and realize that (8) is an algebraic equation. It is straightforward to see that

$$\tilde{G}(\xi,\omega;x_0,t_0) = \exp[-\mathrm{i}(\xi x_0+\omega t_0)]\,[\mathrm{EI}\xi^4 + K + iC\omega - m\omega^2]^{-1}. \tag{11}$$

The Green function given by equation (11) is in the frequency domain and must be converted to the time domain. To this end, take the inverse Fourier transform on both sides of equation (9),

$$G(x,t;x_0,t_0) = (2\pi)^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp\{\mathrm{i}[\xi(x-x_0)+\omega(t-t_0)]\}}{\mathrm{EI}\xi^4 + K + \mathrm{i}C\omega - m\omega^2}\,\mathrm{d}\xi\,\mathrm{d}\omega. \tag{12}$$

Formula (12) is the Green function of the beam on a viscoelastic foundation. The Green function serves as a fundamental solution of a partial differential equation. It can be very useful when dealing with linear systems.

4. INTEGRAL REPRESENTATION OF THE SOLUTION

According to the theory of linear partial differential equations (see reference [15]), the solution of equation (2) given that $F(x,t)$ is taking the form of formula (3), can be constituted by integrating the Green function in all the dimension, i.e.

$$y(x,t) = \int_{-\infty}^{t}\int_{-\infty}^{\infty} F(x_0,t_0)G(x,t;x_0,t_0)\,\mathrm{d}x_0\,\mathrm{d}t_0. \tag{13}$$

Taking equations (3) and (12) into equation (13) gives

$$y(x,t) = \int_{-\infty}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\mathrm{PH}(r_0^2-x_0^2)\exp\{\mathrm{i}[\xi(x-x_0)+\omega(t-t_0)+\Omega t_0]\}}{(2\pi)^2 2r_0\,\mathrm{EI}(\xi^4+\bar{K}+\mathrm{i}\bar{C}\omega-\bar{m}\omega^2)}\,\mathrm{d}\xi\,\mathrm{d}\omega\,\mathrm{d}x_0\,\mathrm{d}t_0, \tag{14}$$

where $\bar{K} = K/\mathrm{EI}$, $\bar{C} = C/\mathrm{EI}$, and $\bar{m} = m/\mathrm{EI}$ are relative stiffness, damping and mass respectively.

The following integrals exist:

$$\int_{-\infty}^{\infty} \frac{H(r_0^2 - x_0^2)\exp(-\mathrm{i}\xi x_0)}{2r_0}\,\mathrm{d}x_0 = \int_{-r_0}^{r_0} \frac{\exp(-\mathrm{i}\xi x_0)}{2r_0}\,\mathrm{d}x_0 = \frac{\sin r_0\xi}{r_0\xi} \tag{15}$$

and

$$\int_{-\infty}^{t} \exp[\mathrm{i}(\Omega - \omega)t_0]\,\mathrm{d}t_0 = 2\pi\delta(\Omega - \omega). \tag{16}$$

Substituting equations (15) and (16) into equation (14) and reapplying the property (6) of the Dirac-delta function give

$$y(x, t) = \frac{P\exp(\mathrm{i}\Omega t)}{2\pi\mathrm{EI}} \int_{-\infty}^{\infty} \frac{\sin r_0\xi\,\exp(\mathrm{i}\xi x)}{r_0\xi(\xi^4 + \bar{K} + \mathrm{i}\bar{C}\Omega - \bar{m}\Omega^2)}\,\mathrm{d}\xi. \tag{17}$$

The dynamic deflection corresponding to a harmonic concentrated load $F_{\mathrm{Point}}(\mathbf{x}) = \delta(x)\exp(\mathrm{i}\Omega t)$ can be simply obtained by taking the limit on both sides of equation (17), i.e.,

$$y(x, t) = \frac{P\exp(\mathrm{i}\Omega t)}{2\pi\mathrm{EI}} \int_{-\infty}^{\infty} \frac{\exp(\mathrm{i}\xi x)}{\xi^4 + \bar{K} + \mathrm{i}\bar{C}\Omega - \bar{m}\Omega^2}\,\mathrm{d}\xi. \tag{18}$$

Here, the following limit is used in the derivation of equation (18):

$$\lim_{r_0 \to 0} \frac{\sin r_0\xi}{r_0\xi} = 1. \tag{19}$$

So far, the author has obtained the integral representation of the dynamic deflection of the beam under a harmonic line load. It is noteworthy to point out that only the real part of the integrand of equation (17) should be taken into account. As indicated in equation (17) the integral of form (17) is generally a complex function, which means that a phase difference appears between the frequency of the external excitation and the frequency of the response of the beam. However, if the foundation is considered as a Winkler foundation, that is to say $C = 0$ in equation (17), the integral becomes a real function. In this case, no phase difference exists. Expression (17) can be further developed using complex function techniques. In the following section, the theorem of residue is employed to evaluate the integration of form (17).

5. CLOSED-FORM REPRESENTATION OF THE SOLUTION

5.1. ROOTS OF THE CHARACTERISTIC EQUATION

Before the integration (17) is further evaluated, it is necessary to investigate the roots of the characteristic equation of this type:

$$\xi(\xi^4 + \bar{K} - \bar{m}\Omega^2 + i\bar{C}\Omega) = 0. \tag{20}$$

Characteristic equation (20) is a fifth order algebraic equation with uncertain parameters of the beam, the foundation and the load. The roots of equation (20) depend upon the distribution and combination of these parameters. To identify the roots, the author classifies equation (20) into two categories. One is concerned with a Winkler foundation, where no damping effect is involved. The other is concerned with the viscoelastic foundation, i.e., $C \neq 0$. Furthermore, in each category three cases are, respectively, discussed depending on the relationship between the load frequency and the eigenfrequencies of the structures.

5.1.1. *Without damping* ($C = 0$)

This case corresponds to a beam resting on a Winkler foundation. Define the equivalent stiffness $\hat{K} = |\bar{K} - \bar{m}\Omega^2|$ and the resonance frequency $\Omega_0 = \sqrt{K/m}$.

(1) $\Omega < \Omega_0$. Equation (20) becomes $\xi(\xi^4 + \hat{K}) = 0$. Four roots of this equation possess complex values and can be given by $\xi = \sqrt[4]{\hat{K}}\exp[\mathrm{i}(1 + 2j)\pi/4]$ with $j = 0, 1, 2, 3$. A real root of the equation can be apparently identified as $\xi = 0$.

(2) $\Omega = \Omega_0$. Equation (20) becomes $\xi^5 = 0$. In this case, all five roots of the equation overlap at $\xi = 0$.

(3) $\Omega > \Omega_0$. Equation (20) becomes $\xi(\xi^4 - \hat{K}) = 0$. Two of the five roots of this equation are imaginary values and the other three roots possess real values. They are given by $\xi = \sqrt[4]{\hat{K}}\exp[i(j\pi/2)]$ with $j = 0, 1, 2, 3$ and $\xi = 0$ respectively.

5.1.2. *With damping* ($C \neq 0$)

Define the equivalent damping coefficient $\hat{C} = \bar{C}\Omega$.

(1) $\Omega < \Omega_0$. Equation (20) becomes $\xi(\xi^4 + \hat{K} + i\hat{C}) = 0$. Four roots of this equation possess complex values and can be given by $\xi = \sqrt[8]{\hat{K}^2 + \hat{C}^2}\exp[\mathrm{i}(\vartheta + \pi/2j\pi)/4]$ with $j = 0, 1, 2, 3$ and $\tan\vartheta = \hat{C}/\hat{K} > 0$. One real root is $\xi = 0$.

(2) $\Omega = \Omega_0$. Equation (20) becomes $\xi(\xi^4 + \mathrm{i}\hat{C}) = 0$. In this case, four roots of the equation possess complex values and can be given by $\xi = \sqrt[4]{\hat{C}}\exp[\mathrm{i}(3\pi + 4j\pi)/8]$ with $j = 0, 1, 2, 3$. A real root is $\xi = 0$.

(3) $\Omega > \Omega_0$. Equation (20) becomes $\xi(\xi^4 - \hat{K} + i\hat{C}) = 0$. Four roots of the equation possess complex values and can be given by $\xi = \sqrt[4]{\hat{C}}\exp[\mathrm{i}(\vartheta + 2j\pi)/4]$ with $j = 0, 1, 2, 3$ and $\vartheta = -\hat{C}/\hat{K} < 0$. A real root is $\xi = 0$.

5.2. CLOSED-FORM REPRESENTATION OF THE SOLUTION

According to the property of symmetry of the problem, the deflection at positions of $x < 0$ is identical to that of $x > 0$. Without loss of generality, only the cases of $x > 0$ and $x = 0$ are considered. Since for each case of C and Ω a real root exists at $\xi = 0$, the integral of type (17) is in the sense of the Cauchy principal value (p.v.) of the integration. According to the theorem of residue, for $x \geqslant 0$, the following integral can be evaluated by residues in the upper half-plane and on the real axis:

$$\text{p.v.}\int_{-\infty}^{\infty}\frac{\sin r_0\xi\exp(\mathrm{i}\xi x)}{r_0\xi(\xi^4 \pm \hat{K} + \mathrm{i}\hat{C})}\,\mathrm{d}\xi = 2\pi\mathrm{i}\sum_{\operatorname{Im}\xi>0}\operatorname{Res}\left[\frac{\sin r_0\xi\exp(\mathrm{i}\xi x)}{r_0\xi(\xi^4 \pm \hat{K} + \mathrm{i}\hat{C})}\right] + \pi\mathrm{i}\sum_{\operatorname{Im}\xi=0}\operatorname{Res}\left[\frac{\sin r_0\xi\exp(\mathrm{i}\xi x)}{r_0\xi(\xi^4 \pm \hat{K} + \mathrm{i}\hat{C})}\right], \tag{21}$$

where p.v. indicates the principal value of the integration, $\operatorname{Im}(\cdot)$ represents imaginary parts of the complex variable ξ, and the positive and negative signs in front of the equivalent stiffness $\hat{K}$ are determined by $\Omega < \Omega_0$ and $\Omega > \Omega_0$ respectively.

TABLE 1

Closed-form deflection of a beam under a harmonic line load ($x \geqslant 0$)

Damping	Frequency	Deflection $y(x, t)$
$C = 0$	$\Omega < \Omega_0$	$(4r_0\mathrm{EI}\hat{K})^{-1}\mathrm{i}P\exp(\mathrm{i}\Omega t)\sum_{j=1}^{2}\exp(\mathrm{i}\xi_j x)\sin r_0\xi_j$, where $\xi_1 = \sqrt[4]{\hat{K}}\mathrm{e}^{\mathrm{i}\pi/4}$ and $\xi_2 = \sqrt[4]{\hat{K}}\mathrm{e}^{\mathrm{i}3\pi/4}$
$C = 0$	$\Omega = \Omega_0$	Does not exist
$C = 0$	$\Omega < \Omega_0$	$(4r_0\mathrm{EI}\hat{K})^{-1}\mathrm{i}P\exp(\mathrm{i}\Omega t)\sum_{j=1}^{2}\exp(\mathrm{i}\xi_j x)\sin r_0\xi_j$, where $\xi_1 = i\sqrt[4]{\hat{K}}$ and $\xi_2 = -\sqrt[4]{\hat{K}}$
$C \neq 0$	$\Omega < \Omega_0$	$\dfrac{\mathrm{i}P\exp(\mathrm{i}\Omega t)\sum_{j=1}^{2}\exp(\mathrm{i}\xi_j x)\sin r_0\xi_j}{r_0\mathrm{EI}(-5\sqrt{\hat{K}^2+\hat{C}^2}+\hat{K}+\mathrm{i}\hat{C})}$, where $\xi_1 = \sqrt[8]{\hat{K}^2+\hat{C}^2}\,\mathrm{e}^{\mathrm{i}(\vartheta+\pi)/4}$, $\xi_2 = \sqrt[8]{\hat{K}^2+\hat{C}^2}\,\mathrm{e}^{\mathrm{i}(\vartheta+3\pi)/4}$ and $\tan\vartheta = \hat{C}/\hat{K}$
$C \neq 0$	$\Omega = \Omega_0$	$-(4r_0\mathrm{EI}\hat{C})^{-1}P\exp(\mathrm{i}\Omega t)\sum_{j=1}^{2}\exp(\mathrm{i}\xi_j x)\sin r_0\xi_j$ where $\xi_1 = \sqrt[4]{\hat{C}}\mathrm{e}^{\mathrm{i}3\pi/8}$ and $\xi_2 = \sqrt[4]{\hat{C}}\mathrm{e}^{\mathrm{i}7\pi/8}$
$C \neq 0$	$\Omega > \Omega_0$	$\dfrac{\mathrm{i}P\exp(\mathrm{i}\Omega t)\sum_{j=1}^{2}\exp(\mathrm{i}\xi_j x)\sin r_0\xi_j}{r_0\mathrm{EI}(5\sqrt{\hat{K}^2+\hat{C}^2}-\hat{K}+\mathrm{i}\hat{C})}$, where $\xi_1 = \sqrt[8]{\hat{K}^2+\hat{C}^2}\,\mathrm{e}^{\mathrm{i}(\vartheta+2\pi)/4}$ $\xi_2 = \sqrt[8]{\hat{K}^2+\hat{C}^2}\,\mathrm{e}^{\mathrm{i}(\vartheta+4\pi)/4}$ and $\tan\vartheta = -\hat{C}/\hat{K}$

Substitution of equations (21) into equation (17) gives

$$y(x, t) = \frac{\mathrm{i}P\exp(\mathrm{i}\Omega t)}{2r_0 EI}\left\{2\sum_{\mathrm{Im}\,\xi>0}\mathrm{Res}\left[\frac{\sin r_0\xi\exp(\mathrm{i}\xi x)}{\xi(\xi^4 \pm \hat{K} + \mathrm{i}\hat{C})}\right] + \sum_{\mathrm{Im}\,\xi=0}\mathrm{Res}\left[\frac{\sin r_0\xi\exp(\mathrm{i}\xi x)}{\xi(\xi^4 \pm \hat{K} + \mathrm{i}\hat{C})}\right]\right\}. \quad (22)$$

To further evaluate equation (22), residues at different locations of the complex plane need to be derived. For the residue whose denominator is not zero while the complex variable ξ is taking the poles ξ_j obtained above, its value can be calculated by

$$\mathrm{Res}\left[\frac{\sin r_0\xi\exp(\mathrm{i}\xi x)}{\xi(\xi^4 \pm \hat{K} + \mathrm{i}\hat{C})}\right]\Bigg|_{\xi=\xi_j} = \left[\frac{\sin r_0\xi\exp(\mathrm{i}\xi x)}{(5\xi^4 \pm \hat{K} + \mathrm{i}\hat{C})}\right]\Bigg|_{\xi=\xi_j}. \quad (23)$$

For the residue whose denominator becomes zero while the complex variable ξ is taking the roots ξ_j obtained above, its value can be calculated by

$$\mathrm{Res}\left[\frac{\sin r_0\xi\exp(\mathrm{i}\xi x)}{\xi(\xi^4 \pm \hat{K} + \mathrm{i}\hat{C})}\right]\Bigg|_{\xi=\xi_j} = \lim_{\xi\to\xi_j}(\xi-\xi_j)\frac{\sin r_0\xi\exp(\mathrm{i}\xi x)}{\xi(\xi^4 \pm \hat{K} + \mathrm{i}\hat{C})}. \quad (24)$$

Based on the aforementioned analysis, eventually, the author is able to write the integration (17) in closed-form expressions. Table 1 provides the final results of the deflection for different cases. It is noted that the integration of type (17) does not exist as

$C = 0$ and $\Omega = \Omega_0$. This is reflected by the singularity of the integrand in equation (17) as the complex variable ξ approaches the pole $\xi = 0$. In this case, resonance will occur and the amplitude of deflection becomes infinite.

In the case of a Winkler foundation, because damping is not considered in a Winkler foundation, only formulas in the first three rows in Table 1 apply. In practice, the vibratory devices used for pavement non-destructive testing generate harmonic loads with frequencies 5–60 Hz [3]. These frequencies usually fall into the low-frequency range of $\Omega < \Omega_0$. Hence, the dynamic deflection that has the greatest practical value for non-destructive pavement evaluation is the formula of the first row in Table 1. It is helpful to write this expression in a more clear form. Defining a new parameter $\beta = \sqrt[4]{\hat{K}/4}$ and substituting ξ_1 and ξ_2 into that expression, we have

$$y(x, t) = \mathrm{i}(16r_0EI\beta^4)^{-1}P\exp(-\beta x)\exp(\mathrm{i}\Omega t) + \{\exp(\mathrm{i}\beta x)\sin[r_0\alpha(\mathrm{i}+1)] + \exp(-\mathrm{i}\beta x)\sin[r_0\alpha(\mathrm{i}-1)]\}. \tag{25}$$

Utilization of the Euler formula $\exp(\mathrm{i}\theta) = \cos\theta + i\sin\theta$ in equation (25) and rearranging the terms yields

$$y(x, t) = i(16r_0\mathrm{EI}\beta^4)^{-1}P\exp(-\beta x)\exp(\mathrm{i}\Omega t)\{\cos(\beta x)\{\sin[r_0\alpha(i+1)] + \sin[r_0\alpha(i-1)]\} \mathrm{i}\sin(\beta x)\{\sin[r_0\alpha(i+1)] - \sin[r_0\alpha(i-1)]\}\}. \tag{26}$$

Since the sum and the difference of sinusoid functions can be formulated into the form of a product, equation (26) can be finally expressed as

$$y(x, t) = -(8r_0\mathrm{EI}\beta^4)^{-1}P\exp(-\beta x)\exp(\mathrm{i}\Omega t) + [\sin(\beta x)\sin(r_0\beta)\cos(\mathrm{i}r_0\beta) - i\cos(\beta x)\cos(r_0\beta)\sin(\mathrm{i}r_0\beta)]. \tag{27}$$

6. STATIC SOLUTION

It is of interest to examine the static solution corresponding to a concentrated load through applying $\Omega = 0$ and $r_0 = 0$ to the result of $C = 0$ and $\Omega < \Omega_0$ provided in Table 1. In this case, the deflection becomes

$$y(x) = \frac{\mathrm{i}P}{4\mathrm{EI}\bar{K}}\lim_{r_0\to 0}\sum_{j=1}^{2}\frac{\exp(\mathrm{i}\xi_j x)\sin r_0\xi_j}{r_0} = \frac{\mathrm{i}P}{4\mathrm{EI}\bar{K}}\sum_{j=1}^{2}\xi_j\exp(\mathrm{i}\xi_j x). \tag{28}$$

Substitute ξ_1 and ξ_2 of the cases of $C = 0$ and $\Omega < \Omega_0$ into equation (28) and rewrite the expression as

$$y(x) = \frac{\mathrm{i}\sqrt{2}P}{16\mathrm{EI}\beta^3}\left\{\exp\left(\mathrm{i}\frac{\pi}{4}\right)\exp\left[\mathrm{i}\sqrt{2}\beta x\exp\left(\mathrm{i}\frac{\pi}{4}\right)\right] + \exp\left(\mathrm{i}\frac{3\pi}{4}\right)\exp\left[i\sqrt{2}\beta x\exp\left(\mathrm{i}\frac{3\pi}{4}\right)\right]\right.$$
$$= -\frac{P\exp(-\beta x)}{8\mathrm{EI}\beta^3}(\sin\beta x + \cos\beta x). \tag{29}$$

This expression (29) is consistent with the known results provided by references [17, 18].

7. CONCLUSIONS

Fourier transform is used to solve the problem of steady state response of a beam on a viscoelastic foundation subjected to a harmonic line load. The solution is constructed in the form of the convolution of the Green function of the beam. The theorem of residue is employed to evaluate the generalized integral such that a closed-form solution can be achieved. Different cases of damping and frequency are discussed and analytical solutions are presented. The closed-form expression of the result can be used to construct algorithms for harmonic load-based inverse problems of pavement structures.

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