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NATURAL FREQUENCIES OF A NON-UNIFORM BEAM WITH MULTIPLE CRACKS VIA MODIFIED FOURIER SERIES

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A new method is presented in this paper for computing the natural frequencies of a non-uniform beam with an arbitrary number of transverse open cracks. The essence of this new method lies in the use of a kind of modified Fourier series (MFS) which is specially developed for a beam with transverse open cracks. Unlike conventional Fourier series, modified Fourier series can approach a function with internal geometrical discontinuities. Based on the modified Fourier series, one can treat the cracked beam *in the most usual way* and thus reduce the problem to a simple one. The beam can be of non-uniform cross section and the number of cracks can be arbitrary. By using the present method, only standard linear eigenvalue equations, rather than non-linear algebraic equations, need to be solved. All the formulae are expressed in matrix form which renders the programming quite straightforward.

1. INTRODUCTION

Knowing the dynamic behaviour of a structure with cracks is of significant importance in engineering. There are two types of problems related to this topic: the first may be called "direct problem" and the second called "inverse problem". The "direct problem" is to determine the effect of damages on the structural dynamic characteristics, while the "inverse problem" is to detect, locate and quantify the extent of the damages. In the past two decades, both the direct and the inverse problems have attracted many researchers and many relevant literatures have been published. Dimarogonas [1] presented a state-of-the-art review of various methods in tackling the cracked structure problem.

Although many researchers [2–12] studied the effect of damages on the structural dynamic characteristics, their studies were often limited to a uniform cantilever beam or a uniform simply supported beam. Moreover, there are only one or at most two cracks presented in their beams. Recently, Shifrin and Ruotolo [13] developed a method which can be used to tackle a beam with multiple cracks, but their studies are still restricted to uniform beams.

Gudmunson [14] employed a theory based on first order perturbation to study the effect of cracks, notches and other geometrical discontinuities on the eigenfrequencies of slender structures. By employing the Euler–Bernoulli beam theory, Christides and Barr [15] established a set of differential equation for one-dimensional cracked beams.

On the whole, the available methods can be classified into two main categories. In the first category, the beam is modelled as an assembly of a number of sub-beams connected by massless rotational springs. Subsequently, the vibrational differential equations are established and then solved individually [2–9, 13]. The second category falls within the regime of the finite element method [10–12, 14–18]. The former is a kind of continuous method while the latter is a kind of discrete method. Compared to the finite element method, the continuous method yields more accurate solutions since no discretization was made. On the other hand, the continuous method has some limits and drawbacks. Firstly, its applications are usually restricted to uniform beams. Secondly, the formulae are more complex and not unified. Thirdly, to obtain the natural frequencies, it usually needs to search the roots of a non-linear algebraic equation (i.e., the determinant of an eigenmatrix). To overcome these drawbacks and to retain the accuracy are the main objectives of developing the present method.

Conceptually, the simulation of a cracked beam is analogous to that of a beam with stepped changes of cross-sections and/or with intermediate point supports. Recently, the modified Fourier series (MFS) and the modified beam vibration functions (MBVF) were developed and have been successfully used to solve the vibrational problems for structures with stepped cross-sections and/or intermediate point supports [19–24].

In this paper, a new method is developed for computing the natural frequencies of a non-uniform beam with an arbitrary number of transverse open cracks. The essence of this new method lies in the use of a kind of modified Fourier series that is developed specially for the analysis of a beam with arbitrary number of transverse open cracks. Unlike the conventional Fourier series, the modified series is able to approach a function with internal geometrical discontinuities effectively. Based on the present modified Fourier series, one can treat the cracked beam *in the most usual way* and thus reduce the problem to be a *simple* one. As can be seen in Equation (45), the extra effort needed is just to add the $\mathbf{K}_3$ matrix to the stiffness matrix of the beam. The beam can be of non-uniform cross-section and the number and depth of cracks can be arbitrary. In the present method, only standard linear eigenvalue equations, rather than non-linear algebraic equations, need to be solved. Since this new method falls within the frame of continuous methods, its capability of achieving higher accuracy is expected. Moreover, all the formulae can be expressed in a unified way and in matrix form, which renders the programming quite straightforward. To demonstrate the effectiveness and accuracy of the present method, several numerical examples are shown. The results show good agreement with other available published results.

2. THEORY AND FORMULATION

2.1. MODIFIED FOURIER SERIES $Y_m(y)$

Figure 1 shows a beam having $(Q-1)$ number of transverse open cracks located at $y = y_2, y_3, \ldots, y_Q$ and with N point-spring supports located at $y = s_1, s_2, \ldots, s_N$ respectively. The beam can have non-uniform cross-sectional areas $A(y)$ and various second moment of area $I(y)$ along the longitudinal direction y. The depth of the cracks are $\{a_i,\ i = 1, 2, \ldots, Q-1\}$, $a_i \geqslant 0$, and the translational and rotational stiffness of the point-springs are $\{k_i, \chi_i, i = 1, 2, \ldots, N\}$. The point-springs are introduced here for the purpose of modelling the boundary supports and the intermediate point supports, if any.

The transverse deflection of the beam is denoted by $w(y, t)$ where y stands for the location and t stands for the time. By using the modified Fourier series expansion, we have

$$w(y, t) = \sum_{m=1}^{R} q_m(t) Y_m(y) \quad (R = 2r + 1), \tag{1}$$

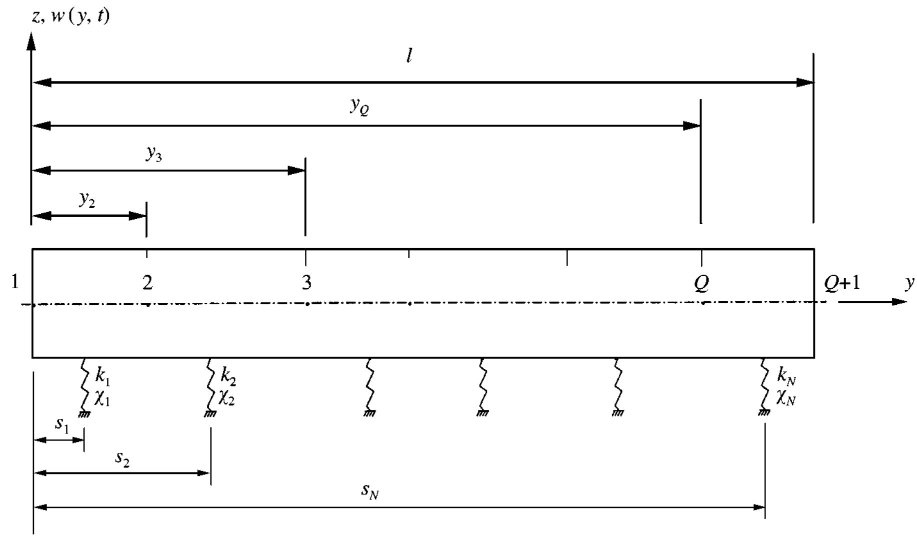

Figure 1. A beam having $(Q-1)$ number of cracks located at $y = y_2, y_3, \ldots, y_Q$ and N spring supports located at $y = s_1, s_2, \ldots, s_N$.

where $q_m(t)$ are the generalized co-ordinates of the beam and $Y_m(y)$ are the so-called modified Fourier series which is constructed specifically such that it can approach a function with internal discontinuities. In the present formulation, $Y_m(y)$ is expressed as the sum of a basic Fourier series $\tilde{Y}_m$ [25] and an augmenting piece-wise linear function $\tilde{Y}_m(y)$ as follows:

$$Y_m(y) = \bar{Y}_m(y) + \tilde{Y}_m(y), \tag{2}$$

$$\bar{Y}_m(y) = \begin{cases} 1, & m = 1, \\ \cos(k\omega_0 y), & m = 2k, \quad k = 1, 2, \ldots, r, \\ \sin(k\omega_0 y), & m = 2k+1, \quad k = 1, 2, \ldots, r, \end{cases} \tag{3}$$

$$\tilde{Y}_m(y) = \sum_{j=1}^{Q+1} f_j l_j(y), \tag{4}$$

where $\omega_0 = \pi/l$ is the basic frequency and $l_j(y)$ are the piece-wise linear-interpolation base functions:

$$l_j(y) = \begin{cases} \dfrac{y - y_{j-1}}{y_j - y_{j-1}}, & y_{j-1} \leqslant y \leqslant y_j \qquad (\text{omitted if } j = 1), \\ \dfrac{y - y_{j+1}}{y_j - y_{j+1}}, & y_j \leqslant y \leqslant y_{j+1} \qquad (\text{omitted if } j = Q+1), \\ 0, & y \notin [y_{j-1}, y_{j+1}]. \end{cases} \tag{5}$$

By adding the piece-wise linear functions $\{\tilde{Y}_m(y), m = 1, 2, \ldots, R\}$ (see Figure 2) onto the basic Fourier series $\{\bar{Y}_m(y), m = 1, 2, \ldots, R\}$, we can *force* the whole function $[Y_m(y), m = 1, 2, \ldots, R\}$ to satisfy the geometrical discontinuity conditions at the locations

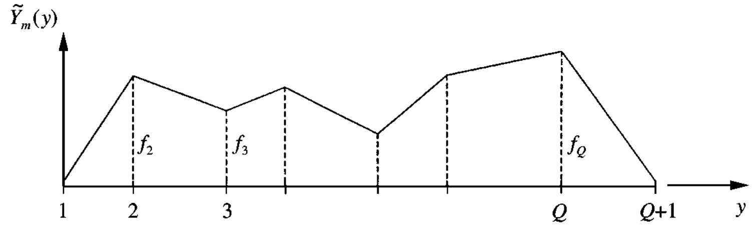

Figure 2. Augmenting piece-wise linear function $\tilde{Y}_m(y)$.

of cracks. Thus, in the following analysis, we can treat the cracked beam in the most usual way and need not bother further about the internal geometrical discontinuities.

The geometrical discontinuity condition at the crack's location $y = y_j$ $(j = 2, 3, \ldots, Q)$ is [18]

$$Y'_m(y_j + 0) - Y'_m(y_j - 0) = c_{j-1} Y''_m(y \to y_j), \tag{6}$$

where c_{j-1} is the flexibility coefficient of the cracks having a depth of a_{j-1}. For one-sided cracks, it can be expressed as

$$c_{j-1} = 5{\cdot}346 h(y_j) f(\xi_{j-1}) \quad (j = 2, 3, \ldots, Q), \tag{7}$$

where $h(y_j)$ is the depth of the cross-section of the beam at the $y = y_j$ and

$$\xi_{j-1} = a_{j-1}/h(y_j), \tag{8}$$

$$f(\xi) = 1{\cdot}8624\xi^2 - 3{\cdot}95\xi^3 + 16{\cdot}375\xi^4 - 37{\cdot}226\xi^5 + 76{\cdot}81\xi^6 - 126{\cdot}9\xi^7 + 172\xi^8$$
$$- 143{\cdot}97\xi^9 + 66{\cdot}56\xi^{10}. \tag{9}$$

Substituting equation (2) into equation (6) and considering that $\bar{Y}_m(y)$ is a smooth harmonic function, we have

$$\tilde{Y}'_m(y_j + 0) - \tilde{Y}'_m(y_j - 0) = c_{j-1} \bar{Y}''_m(y_j). \tag{10}$$

Substituting equations (4) and (5) into equation (10), we have

$$- h_j f_{j-1} + (h_{j-1} + h_j) f_j - h_{j-1} f_{j+1} = - h_{j-1} h_j c_{j-1} \bar{Y}''_m(y_j) \quad (j = 2, 3, \ldots, Q), \tag{11}$$

where

$$\begin{cases} h_{j-1} = y_j - y_{j-1}, \\ h_j = y_{j+1} - y_j. \end{cases} \tag{12}$$

At the two ends, $y = y_1$ and $y = y_{(Q+1)}$, we set

$$f_1 = 0, \tag{13}$$

$$f_{Q+1} = 0. \tag{14}$$

Equations (11), (13) and (14) can be expressed in matrix form as follows:

$$\mathbf{A}\mathbf{f} = \mathbf{b}, \tag{15}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -h_2 & h_1 + h_2 & -h_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -h_3 & h_2 + h_3 & -h_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & \cdots & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -h_Q & h_{Q-1} + h_Q & -h_{Q-1} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \tag{16}$$

$$\mathbf{f} = [f_1 \; f_2 \; f_3 \; \cdots \; \cdots \; f_{Q-1} \; f_Q \; f_{Q+1}]^{\mathrm{T}}, \tag{17}$$

$$\mathbf{b} = [0 \;\; -h_1 h_2 c_1 \bar{Y}''_m(y_2) \; \cdots \; \cdots \; \cdots \; \cdots -h_{Q-1} h_Q c_{Q-1} \bar{Y}''_m(y_Q) \;\; 0]^{\mathrm{T}}. \tag{18}$$

By solving equation (15), we can determine the coefficients f_j ($j = 1, 2, \ldots, Q+1$) and thus determine the augmenting piece-wise linear functions $\tilde{Y}_m(y)$ and the modified Fourier series $Y_m(y)$.

2.2. ENERGY ANALYSIS

2.2.1. *Potential energy*

The potential energy of the cracked beam can be expressed as the summation of three parts:

$$U = U_1 + U_2 + U_3 \tag{19}$$

in which U_1 is the potential energy stored in the cracked beam due to the bending deformation of the beam itself; U_2 is the potential energy stored in the point-springs which are used to model the boundary supports and intermediate supports (if any); U_3 is the potential energy stored in the massless rotational springs which are used to model the existence of crack(s).

To express U_1, U_2 and U_3 in a concise way, firstly we denote

$$\bar{\mathbf{H}}(y) = [\bar{Y}_1(y) \;\; \bar{Y}_2(y) \;\; \cdots \;\; \bar{Y}_R(y)], \tag{20}$$

$$\tilde{\mathbf{H}}(y) = [\tilde{Y}_1(y) \;\; \tilde{Y}_2(y) \;\; \cdots \;\; \tilde{Y}_R(y)], \tag{21}$$

$$\mathbf{H}(y) = [Y_1(y) \;\; Y_2(y) \;\; \cdots \;\; Y_R(y)], \tag{22}$$

$$\mathbf{q}(t) = [q_1(t) \;\; q_2(t) \;\; \cdots \;\; q_R(t)]^{\mathrm{T}}. \tag{23}$$

Thus, we have

$$\mathbf{H}(y) = \bar{\mathbf{H}}(y) + \tilde{\mathbf{H}}(y) \tag{24}$$

and

$$w(y, t) = \mathbf{H}(y)\mathbf{q}(t). \tag{25}$$

Then, we can derive the expressions for the energy terms U_i as follows.

Potential energy U_1:

$$U_1 = \sum_{i=1}^{Q} \frac{1}{2} \int_{y_i}^{y_{i+1}} EI(y) w_{,yy}^2(y, t)\, \mathrm{d}y. \tag{26}$$

Substituting equation (25) into equation (26) we have

$$U_1 = \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{K}_1\mathbf{q}, \tag{27}$$

where $\mathbf{K}_1$ represents the stiffness matrix of the cracked beam corresponding to the potential energy U_1 such that

$$\mathbf{K}_1 = \sum_{i=1}^{Q} \int_{y_i}^{y_{i+1}} EI(y)\mathbf{H}_{,yy}^{\mathrm{T}}(y)\mathbf{H}_{,yy}(y)\, \mathrm{d}y = \int_0^l EI(y)\bar{\mathbf{H}}_{,yy}^{\mathrm{T}}(y)\bar{\mathbf{H}}_{,yy}(y)\, \mathrm{d}y. \tag{28}$$

Potential energy U_2:

$$U_2 = \sum_{i=1}^{N} \frac{1}{2} [k_i w^2(s_i, t) + \chi_i w_{,y}^2(s_i, t)]. \tag{29}$$

Substituting equation (25) into equation (29), we have

$$U_2 = \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{K}_2\mathbf{q}, \tag{30}$$

where $\mathbf{K}_2$ represents the stiffness matrix of the cracked beam corresponding to the potential energy U_2 , such that

$$\mathbf{K}_2 = \sum_{i=1}^{N} [k_i\mathbf{H}^{\mathrm{T}}(s_i)\mathbf{H}(s_i) + \chi_i\mathbf{H}_{,y}^{\mathrm{T}}(s_i)\mathbf{H}_{,y}(s_i)]. \tag{31}$$

Potential energy U_3:

$$U_3 = \sum_{j=2}^{Q} \frac{1}{2} \left[\frac{EI(y_j)}{c_j}\right] [w_{,y}(y_j + 0, t) - w_{,y}(y_j - 0, t)]^2 \tag{32}$$

Substituting equation (25) into equation (32), we have

$$U_3 = \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{K}_3\mathbf{q}, \tag{33}$$

where $\mathbf{K}_3$ represents the stiffness matrix of the cracked beam corresponding to the potential energy U_3 such that

$$\mathbf{K}_3 = \sum_{j=2}^{Q} [c_{j-1} EI(y_j)] \bar{H}^{\mathrm{T}}_{,yy}(y_j) \bar{H}_{,yy}(y_j). \tag{34}$$

Finally, substituting equations (27), (30) and (33) into equation (19), we obtain the total potential energy of the cracked beam system,

$$U = \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{K}\mathbf{q}, \tag{35}$$

where $\mathbf{K}$ is the sum of the three component stiffness matrices such that

$$\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3. \tag{36}$$

2.2.2. *Kinetic energy*

The kinetic energy T of the cracked beam can be expressed as

$$T = \frac{1}{2}\int_0^l \rho A(y) w^2_{,t}(y, t)\, \mathrm{d}y. \tag{37}$$

Substituting equation (25) into equation (37), we have

$$T = \tfrac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}\dot{\mathbf{q}}, \tag{38}$$

where $\mathbf{M}$ is the mass matrix of the cracked beam system such that

$$\mathbf{M} = \int_0^l \rho A(y) \mathbf{H}^{\mathrm{T}}(y) \mathbf{H}(y)\, \mathrm{d}y. \tag{39}$$

2.3. EULER–LAGRANGIAN EQUATIONS

The Euler–Lagrangian equation of the cracked beam is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{0}, \tag{40}$$

where L is the Lagrangian function

$$L = T - U \tag{41}$$

Substituting equations (38) and (35) into equation (41), and then the results into equation (40), we have

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}. \tag{42}$$

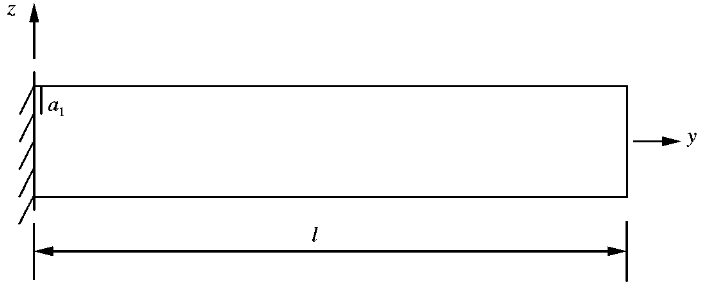

Figure 3. A cantilevered beam with one crack located at the clamped end.

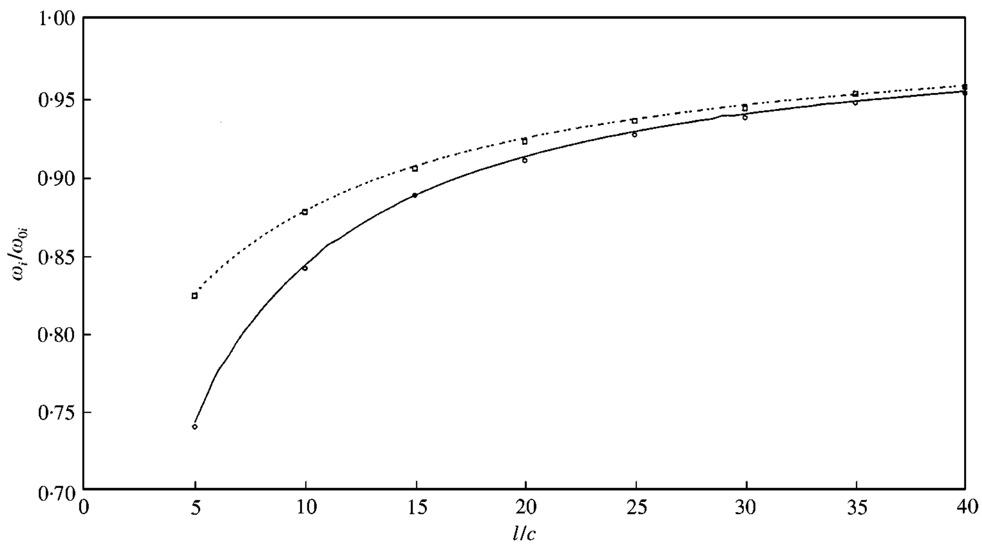

Figure 4. Effect of a single crack at clamped end on the first two natural frequencies: ——, 1st order frequency ratio, Present; - - - - -, 2nd order frequency ratio, Present; ○, 1st order frequency ratio, Shifrin *et al.*; □, 2nd order frequency ratio, Shifrin *et al.*

2.4. FREQUENCY EQUATION

For synchronous vibration, we have

$$\mathbf{q}(t) = \mathbf{q}\cos(\omega t + \phi). \tag{43}$$

Substituting equation (43) into equation (42), we can obtain the frequency equation

$$\mathbf{Kq} = \omega^2\mathbf{Mq} \tag{44}$$

or

$$(\mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3)\mathbf{q} = \omega^2\mathbf{Mq}. \tag{45}$$

Equation (45) is a standard linear eigenvalue equation that can be solved by standard programs. It is worth noting that the matrix $\mathbf{K}_3$ represents the cracks' effect on the stiffness of the beam.

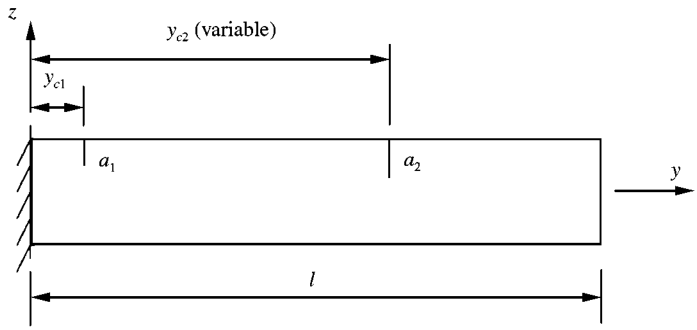

Figure 5. A cantilevered beam with two cracks while the second crack's location is variable.

3. NUMERICAL EXAMPLES

To validate the present theory and to check the correctness of the coded program, we computed the natural frequencies of a uniform beam with 1, 2 or 3 cracks and compared the results with available published results which are based on pure continuous methods. It is then extended to compute the natural frequencies of a non-uniform beam with four cracks.

3.1. EXAMPLE 1: A CANTILEVERED BEAM WITH A CRACK LOCATED AT THE CLAMPED END [13]

Figure 3 shows a cantilevered beam with a crack located at the clamped end. The results obtained by the present method and the results from reference [13] are shown in Figure 4 where the vertical axis stands for the frequency ratio of the natural frequencies of the cracked beam to the natural frequencies of the same but uncracked beam, i.e., the frequency reduction. The horizontal axis stands for the normalized stiffness (l/c_1) of the artificial rotational spring introduced at the crack. It can be seen that the present results agree very well with those from reference [13], in which pure continuous method was employed.

3.2. EXAMPLE 2: A CANTILEVERED BEAM WITH TWO CRACKS [13]

Figure 5 shows a cantilevered beam with two cracks. For the purpose of comparing the results from reference [13], the same geometrical properties of the beam are used, that is length $l = 0{\cdot}8$ m, rectangular cross-section has width $b = 0{\cdot}02$ m and height $h = 0{\cdot}02$ m. The first crack is at fixed location $y_{c1} = 0{\cdot}12$ m and has a depth $a_1 = 2$ mm. The second crack's location varies from the left end to the right end of the beam and its depth also varies ($a_2 = 2$ or 4 or 6 mm). The results obtained by the present method and those from reference [13] are shown in Figures 6–8. Good agreements are observed.

3.3. EXAMPLE 3: A CANTILEVERED BEAM WITH THREE CRACKS [13]

Figure 9 shows a cantilevered beam with three cracks. For the purpose of comparing the results from reference [13], the same geometrical and physical properties of the beam are

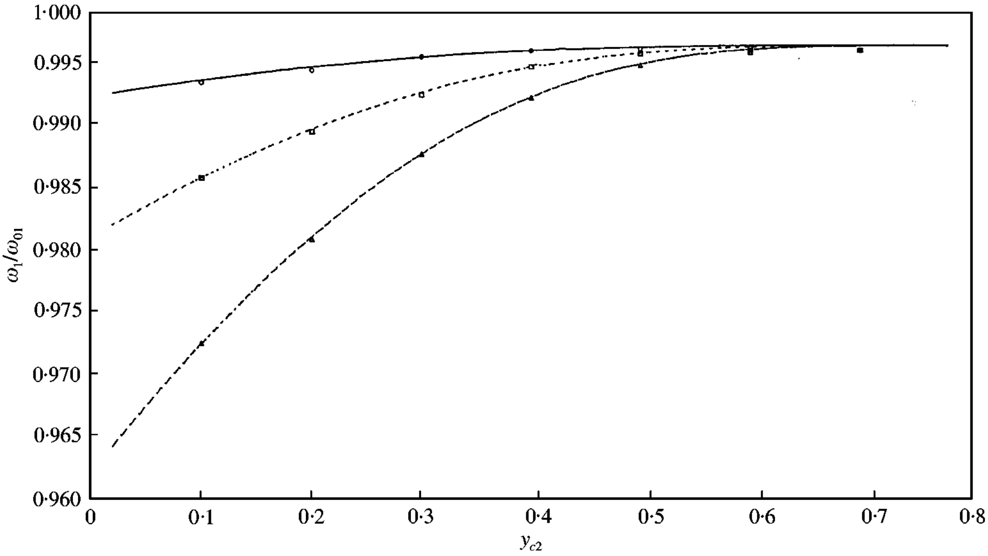

Figure 6. Effect of the second crack on the first order natural frequency: ——, $a_2 = 2$ mm, Present; - - - - -, $a_2 = 4$ mm, Present; — - — -, $a_2 = 6$ mm, Present; ○, $a_2 = 2$ mm, Shifrin *et al.*; □, $a_2 = 4$ mm, Shifrin *et al.*; ▲, $a_2 = 6$ mm, Shifrin *et al.*

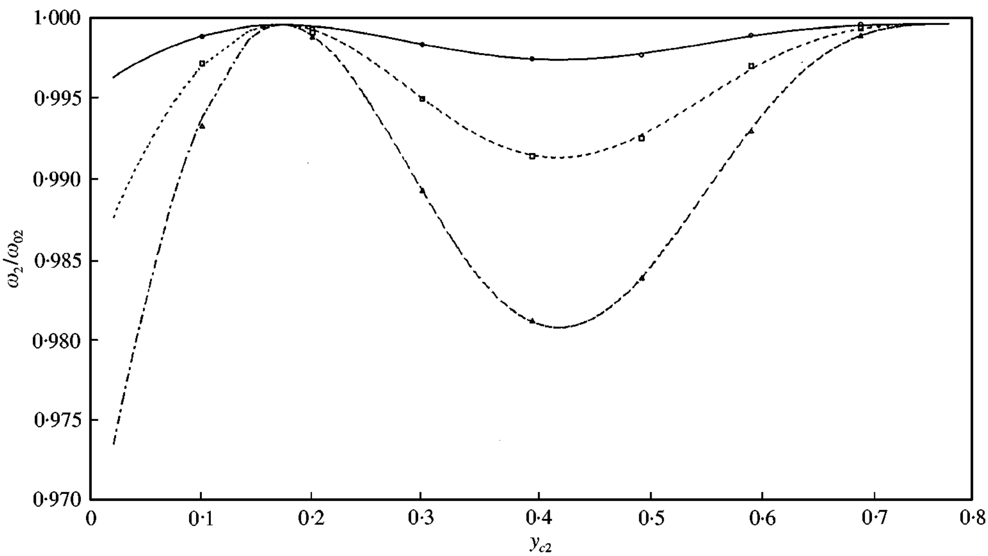

Figure 7. Effect of the second crack on the second order natural frequency: ——, $a_2 = 2$ mm, Present; - - - - -, $a_2 = 4$ mm, Present; — - — -, $a_2 = 6$ mm, Present; ○, $a_2 = 2$ mm, Shifrin *et al.*; □, $a_2 = 4$ mm, Shifrin *et al.*; ▲, $a_2 = 6$ mm, Shifrin *et al.*

used, that is length $l = 0{\cdot}8$ m, rectangular cross-section has width $b = 0{\cdot}02$ m and height $h = 0{\cdot}02$ m. Both the first and the second cracks are at fixed locations ($y_{c1} = 0{\cdot}04$ m, $y_{c2} = 0{\cdot}08$ m) and have a fixed depth ($a_1 = 6$ mm, $a_2 = 4$ mm). The third crack's location varies from $y_3 = 0{\cdot}1$ to $0{\cdot}78$ m and its depth also varies ($a_3 = 2$ or 4 or 6 mm). The results

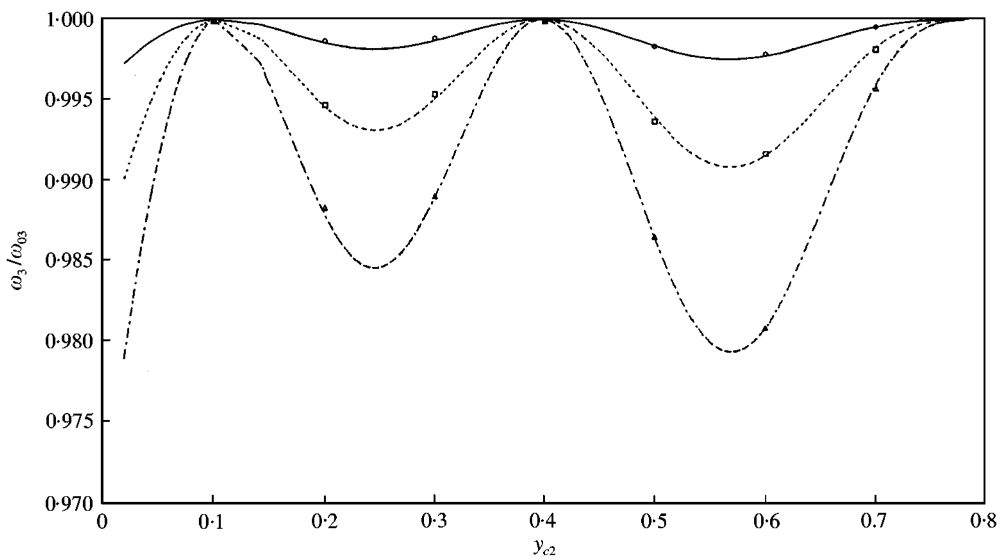

Figure 8. Effect of the second crack on the third order natural frequency: ——, $a_2 = 2$ mm, Present; - - - - -, $a_2 = 4$ mm, Present; – – – -, $a_2 = 6$ mm, Present; ○, $a_2 = 2$ mm, Shifrin *et al.*; □, $a_2 = 4$ mm, Shifrin *et al.*; ▲, $a_2 = 6$ mm, Shifrin *et al.*

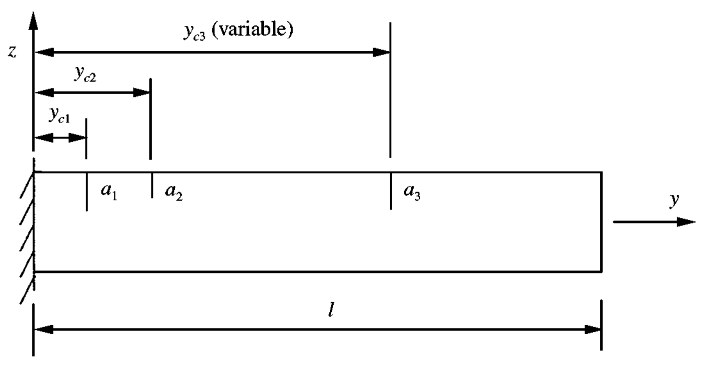

Figure 9. A cantilevered beam with three cracks while the third crack's location is variable.

obtained by the present method and those from reference [13] are shown in Figures 10–12. Good agreements are observed again.

3.4. EXAMPLE 4: A NON-UNIFORM CANTILEVERED BEAM WITH FOUR CRACKS

Figure 13 shows a non-uniform cantilevered beam with four cracks. The beam has the same length ($l = 0{\cdot}8$ m) as those in the examples above. The cross-section is rectangular, having width $b = 0{\cdot}02$ m but the height h changes linearly from 0·02 m at the clamped end to 0·01 m at the free end. The locations and depths of the first, second and third crack are all pre-defined ($y_{c1} = 0{\cdot}04$ m, $a_1 = 6$ mm, $y_{c2} = 0{\cdot}08$ m, $a_2 = 4$ mm and $y_{c3} = 0{\cdot}12$ m,

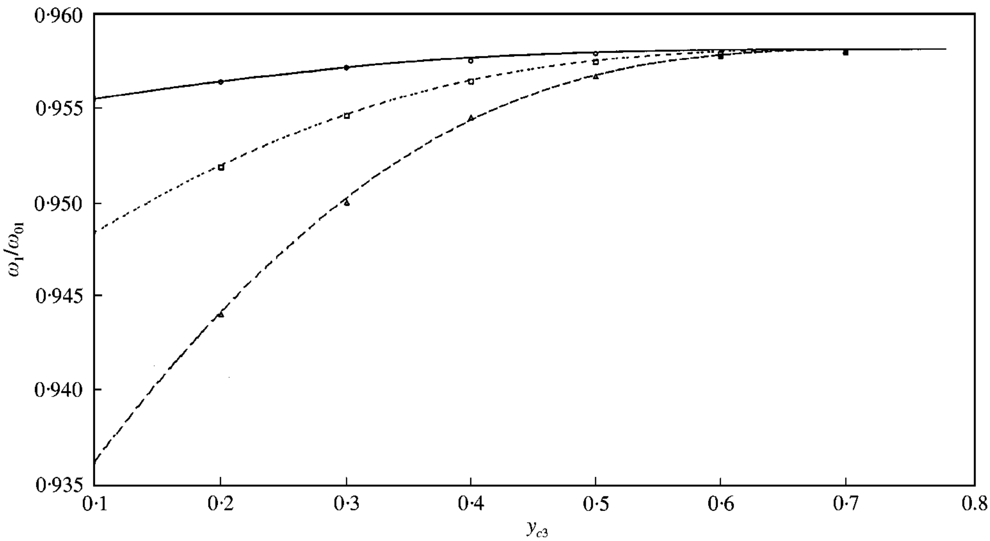

Figure 10. Effect of the third crack on the first order natural frequency: ——, $a_3 = 2$ mm, Present; - - - - -, $a_3 = 4$ mm, Present; – – – -, $a_3 = 6$ mm, Present; ○, $a_3 = 2$ mm, Shifrin *et al.*; □, $a_3 = 4$ mm, Shifrin *et al.*; ▲, $a_3 = 6$ mm, Shifrin *et al.*

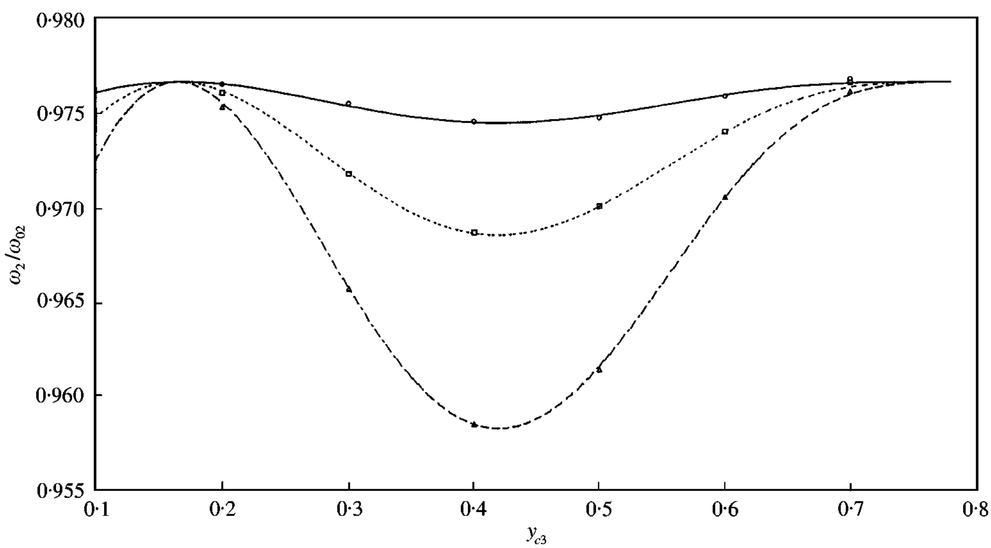

Figure 11. Effect of the third crack on the second order natural frequency: ——, $a_3 = 2$ mm, Present; - - - - -, $a_3 = 4$ mm, Present; – – – -, $a_3 = 6$ mm, Present; ○, $a_3 = 2$ mm, Shifrin *et al.*; □, $a_3 = 4$ mm, Shifrin *et al.*; ▲, $a_3 = 6$ mm, Shifrin *et al.*

$a_3 = 2$ mm). The fourth crack's location varies from $y_{c4} = 0{\cdot}14$ to $0{\cdot}78$ m and its depth also varies ($a_4 = 2$ or 4 or 6 mm). The results are shown in Figures 14–16.

It is worth noting that the changes of frequency could be quite substantial. Reference [10] reported a case of cantilevered beam having a single crack near the support showing 13% of

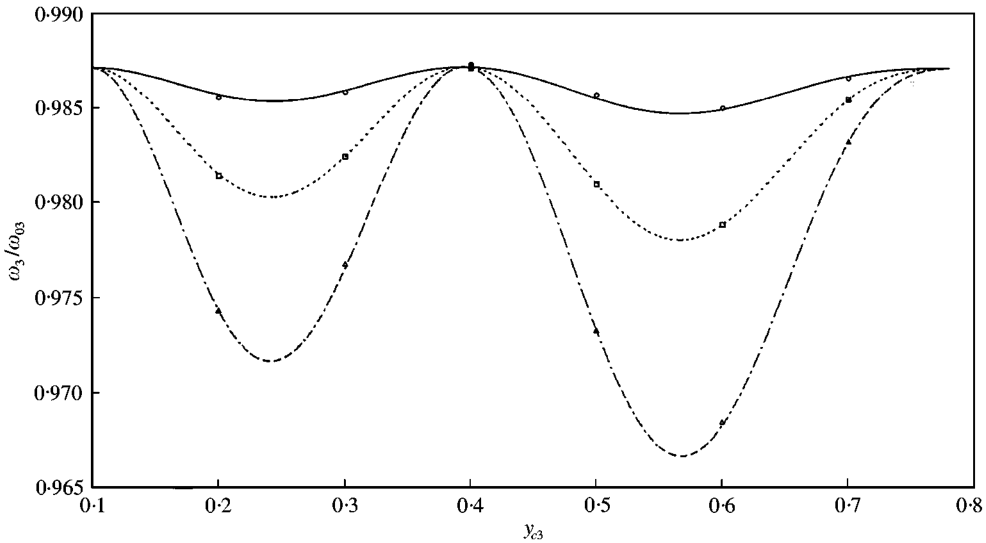

Figure 12. Effect of the third crack on the third order natural frequency: ——, $a_3 = 2$ mm, Present; - - - - -, $a_3 = 4$ mm, Present; – – – -, $a_3 = 6$ mm, Present; ○, $a_3 = 2$ mm, Shifrin *et al.*; □, $a_3 = 4$ mm, Shifrin *et al.*; ▲, $a_3 = 6$ mm, Shifrin *et al.*

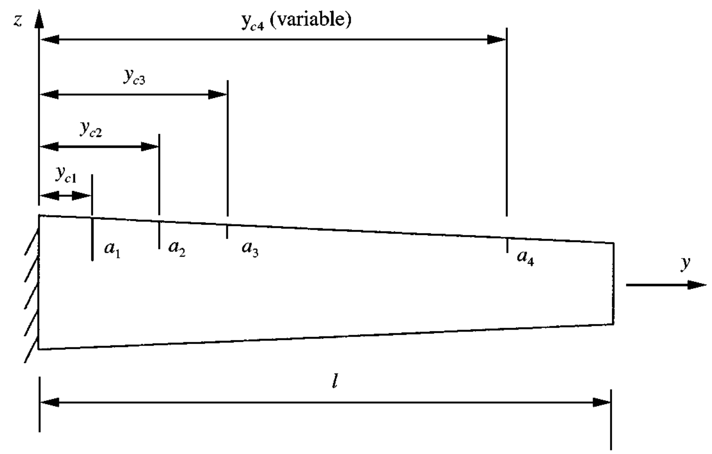

Figure 13. A non-uniform cantilevered beam with four cracks while the fourth crack's location is variable.

changes for a crack with depth ratio of 30%. Therefore, the proposed method in the paper is suitable for the range of crack with reasonable depth.

4. CONCLUSIONS

A new method Fourier series (MFS) was presented. It was developed to tackle the problem in beams with arbitrary number of cracks. The modified Fourier series can approach a function with internal geometrical discontinuities effectively. Applying this

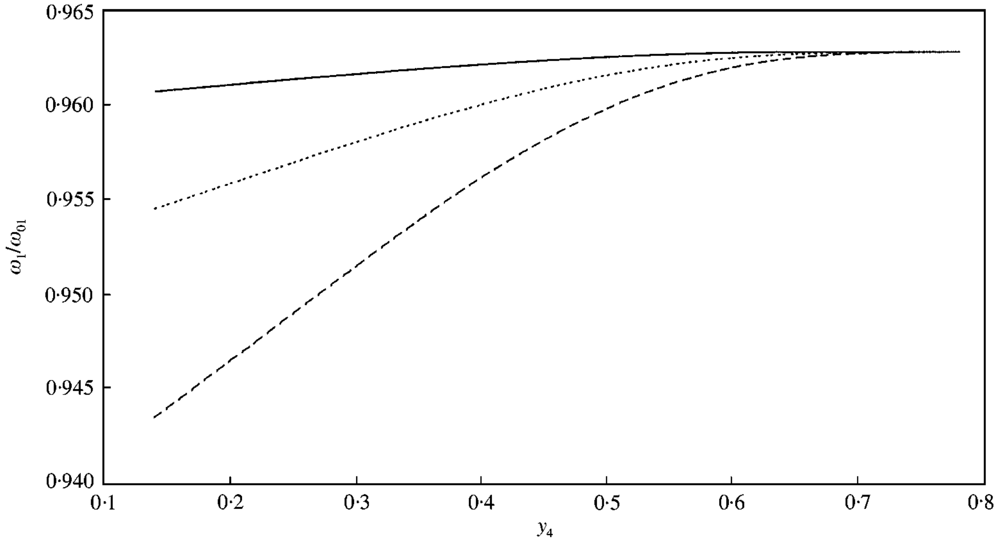

Figure 14. Effect of the fourth crack on the first order natural frequency: ——, $a_4 = 2$ mm; - - - - - -, $a_4 = 4$ mm; – – – –, $a_4 = 6$ mm.

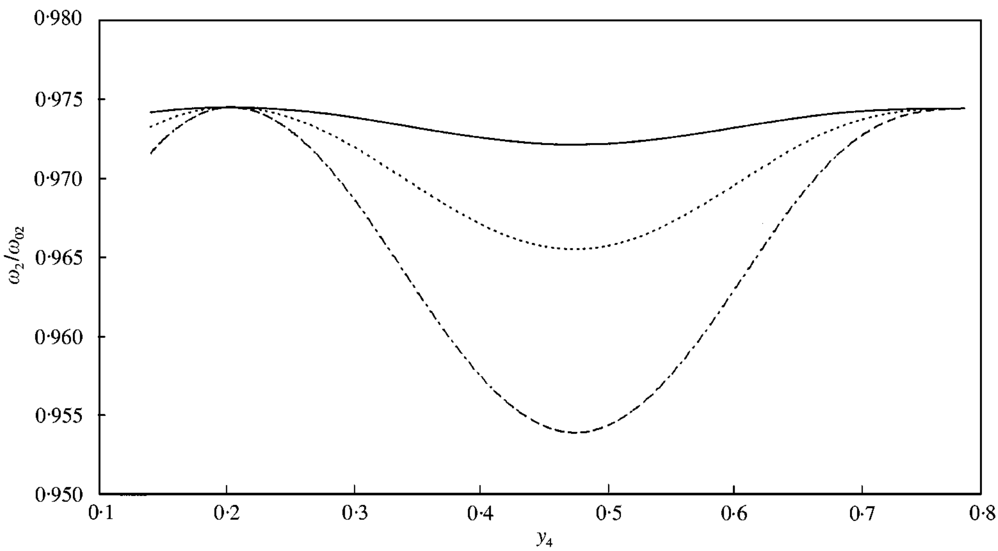

Figure 15. Effect of the fourth crack on the second order natural frequency: ——, $a_4 = 2$ mm; - - - - - -, $a_4 = 4$ mm; – – – –, $a_4 = 6$ mm.

through the Euler–Lagrangian equation, we can treat the vibrational analysis of a cracked beam in the most usual way. It thus renders the problem-solving procedures simple. In the formulation, a crack assumes having stiffness, which is simply added to the stiffness matrix of the beam. The beam can be of non-uniform cross-section and the number of cracks can be arbitrary. In solving the natural frequencies of a cracked beam, only a standard linear eigenvalue equation needs to be solved. All the formulae are expressed in matrix form and computer coding is straightforward. Numerical examples showed that the present method is

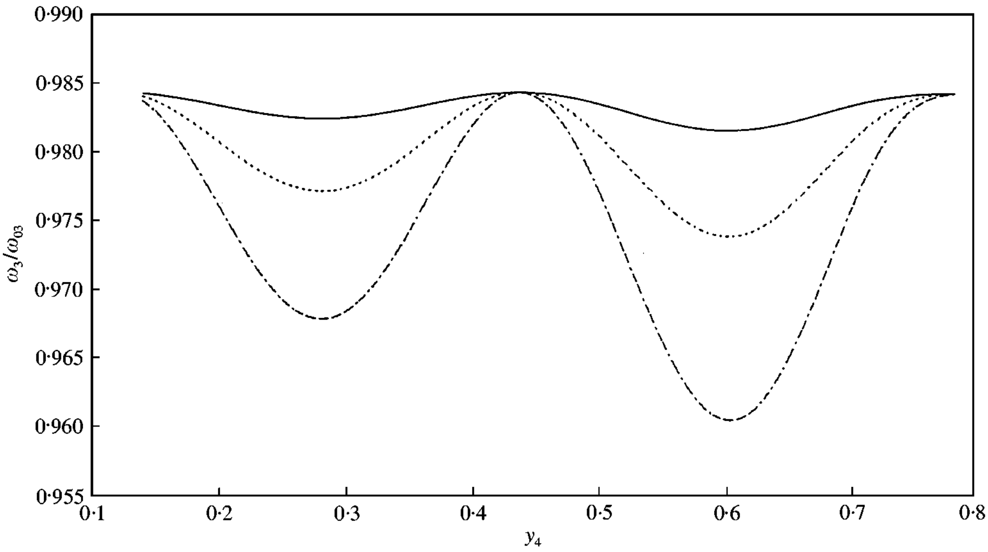

Figure 16. Effect of the fourth crack on the third order natural frequency: ——, $a_4 = 2$ mm; - - - - - -, $a_4 = 4$ mm; – – – –, $a_4 = 6$ mm.

versatile and effective. Nevertheless, the application of the present method is restricted to the cases of small-magnitude vibration with all-time open cracks.

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APPENDIX A: NOMENCLATURE

$\bar{Y}_m(y)$	basic Fourier series
$\tilde{Y}_m(y)$	augmenting piece-wise linear function
$Y_m(y)$	modified Fourier series
$Q-1$	number of transverse open cracks
$\{y_i, i=1,2,\ldots,Q+1\}$	y-ordinate of LHD end-point, cracks and RHD end-point
$\{s_i, i=1,2,\ldots,N\}$	y-ordinate of point-spring supports
$A(y)$	cross-sectional area of the beam
$I(y)$	second moment of area of cross-section
$w(y,t)$	transverse deflection of beam
$q_m(t)$	generalized co-ordinate of a cracked beam
r	highest order of the basic Fourier series
R	number of terms of modified Fourier series
ω_0	basic frequency
$\{l_j(y), j=1,2,\ldots,Q+1\}$	piece-wise linear-interpolation base function
$\{f_j, j=1,2,\ldots,Q+1\}$	value of the augmenting piece-wise linear function
$\{a_j, j=1,2,\ldots,Q-1\}$	depths of cracks
$\{c_j, j=1,2,\ldots,Q-1\}$	flexibility coefficients of cracks
$\mathbf{A}$	coefficient matrix for determining the augmenting function
$\mathbf{f}$	vector of the values of the augmenting piece-wise linear function
$\mathbf{b}$	RHD vector for determining the augmenting function
U	total potential energy of a cracked beam
U_1	potential energy of a beam due to bending deformation
U_2	potential energy stored in the support-springs
U_3	potential energy stored in the equivalent rotational springs which are used to model the existence of cracks

$\bar{\mathbf{H}}$	vector of a basic Fourier series
$\tilde{\mathbf{H}}$	vector of an augmenting function
$\mathbf{H}$	vector of a modified Fourier series
$\mathbf{K}$	total stiffness matrix of a cracked beam
$\mathbf{K}_1$	stiffness matrix corresponding to potential energy U_1
$\mathbf{K}_2$	stiffness matrix corresponding to potential energy U_2
$\mathbf{K}_3$	stiffness matrix corresponding to potential energy U_3
T	kinetic energy of a cracked beam
$\mathbf{M}$	mass matrix of a cracked beam
L	Lagrangian function