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LETTERS TO THE EDITOR

VIBRATION ANALYSIS OF A TAPERED BAR BY DIFFERENTIAL TRANSFORMATION

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1. INTRODUCTION

A promising, new, exact series, analytical method for solving differential equations is known as the differential transformation (DT) method. It was introduced in 1986 [1] and is based on the Taylor series expansion. It has recently been applied to vibration analysis of uniform beams [1, 2] and plates [3].

The purpose of the present note is to apply the DT procedure to obtain exact solutions for the free axial vibration of a tapered bar. The governing differential equation for the mode shapes of this problem is a second order equation with varying coefficients

$$xu_{,xx} + u_{,x} + \bar{\omega}^2 xu = 0, \tag{1}$$

where $(\ )_{,x} = \mathrm{d}(\ )/\mathrm{d}x$, $u(x)$ is the mode shape, x is the dimensionless axial position co-ordinate (normalized by the bar's length L), and $\bar{\omega}$ is the dimensionless natural frequency defined by

$$\bar{\omega} = \omega(m_0 L^2/A_0 E)^{1/2}. \tag{2}$$

Here A_0 is a characteristic area [in $A(x) = 2A_0 x$], E is the elastic modulus, m_0 is a characteristic mass [in $m(x) = 2m_0 x$], and ω is the circular natural frequency.

The boundary conditions are that the bar is fixed at $x = 1$ and free at $x = 0$:

$$u(1) = 0, \qquad \lim_{x \to 0} xu_{,x}(0) = 0, \tag{3}$$

which implies

$$u_{,x}(0) = 0. \tag{4}$$

2. DIFFERENTIAL TRANSFORMATION METHOD AND NUMERICAL RESULTS

An arbitrary function $u(x)$ can be expanded in Taylor series at point $x = 0$ as

$$u(x) = \sum_{k=0}^{\infty} x^k \frac{1}{k!} \left[\frac{\mathrm{d}^k u}{\mathrm{d}x^k}\right]_{x=0}. \tag{5}$$

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By defining a differential transformation of function $u(x)$ as

$$U(k) = \frac{1}{k!}\left[\frac{\mathrm{d}^k u}{\mathrm{d}x^k}\right]_{x=0}, \tag{6}$$

one obtains the inverse differential transformation as

$$u(x) = \sum_{k=0}^{\infty} x^k U(k). \tag{7}$$

Some useful operations needed in this problem can be derived from definitions (6) and (7) as shown in Table 1.

Substituting equation (7) into equation (1), with the use of the basic operations, equation (1) can be rewritten as

$$\sum_{l=0}^{k} \delta(l-1)(k-l+1)(k-l+2)U(k-l+2) + (k+1)U(k+1)$$

$$+ \bar{\omega}^2 \sum_{l=0}^{k} \delta(l-1)U(k-1) = 0, \quad \text{for } k = 0, 1, \ldots, \infty. \tag{8}$$

Using equation (7), the boundary conditions, equations (3) and (4), become

$$\sum_{k=0}^{\infty} U(k) = 0 \tag{9}$$

and

$$\sum_{k=0}^{\infty} kx^{k-1} U(k) = U(1) = 0. \tag{10}$$

To illustrate the procedure for eigenvalues, for $k = 0, \ldots, 5$, with the use of equation (10), equation (8) becomes

$$U(2) = -\tfrac{1}{4}\bar{\omega}^2 U(0), \quad U(3) = 0, \quad U(4) = \tfrac{1}{64}\bar{\omega}^4 U(0), \quad U(5) = 0, \quad U(6) = \tfrac{1}{2304}\bar{\omega}^6 U(0). \tag{11}$$

TABLE 1

Basic operations

Original function	DT
$u(x)$	$U(k)$
$f(x)g(x)$	$\sum_{l=0}^{k} F(l)G(k-l)$
$u_{,x}(x)$	$(k+1)U(k+1)$
$u_{,xx}(x)$	$(k+1)(k+2)U(k+2)$
x	$\delta(k-1) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k \neq 1 \end{cases}$

TABLE 2

Dimensionless frequencies for various summation numbers

First mode (*exact analytical solution*: 2.4048)				
N	5	7	9	10
DT (first mode)	2·8284	2·3916	2·4056	2·4048
Difference (%)	17·62	0·55	0·03	0
Second mode (*exact analytical solution*: 5·5201)				
N	10	12	20	21
DT (first mode)	5·9893	5·4059	5·5183	5·5201
Difference (%)	0·09	0·02	0·0003	0

Substituting equation (11) into equation (9), one obtains

$$\left(-\tfrac{1}{4}+\tfrac{1}{64}\bar{\omega}^2-\tfrac{1}{2304}\bar{\omega}^4\right)\bar{\omega}^2 U(0)=0. \tag{12}$$

The roots of equation (12) are 0, $\pm$ 2·8284. The positive real root 2·8284 is the eigenvalue for $N = 5$. More terms are needed for more accurate results. The results for more terms are listed in Table 2.

The exact solutions listed here are obtained by exact analytical solution in terms of Bessel functions.

3. CONCLUSIONS

It has been shown that the DT method is a convenient and exact method for solving the problem of free axial vibration of a tapered bar.

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