Journal of Sound and Vibration (2001) **242**(5), 749–765
doi:10.1006/jsvi.2000.3196, available online at http://www.idealibrary.com on IDEAL®

AN INVERSE NON-LINEAR FORCE VIBRATION PROBLEM OF ESTIMATING THE EXTERNAL FORCES IN A DAMPED SYSTEM WITH TIME-DEPENDENT SYSTEM PARAMETERS

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(*Received* 20 *October* 1999, *and in final form* 12 *June* 2000)

An inverse non-linear force vibration problem based on the iterative regularization method, i.e., the conjugate gradient method (CGM), is used to estimate the unknown time-dependent external forces in a damped system having time-dependent system parameters by using the measured system displacement.

It is assumed that no prior information is available on the functional form of the unknown external forces in the present study, thus, it is classified as the function estimation in inverse calculation. The accuracy of the inverse analysis is examined by using the simulated exact and inexact measurements.

The numerical simulations are performed to test the validity of present algorithm by using different types of external forces and measurements. Results show that an excellent estimation on the external forces can be obtained with any arbitrary initial guesses within a couple of seconds of CPU time at Pentium II-300 MHz PC.

1. INTRODUCTION

The direct solutions for a non-linear damped force vibration problem are concerned with the determination of the system displacement, velocity and acceleration at time t when the initial conditions, external forces and time-dependent system parameters are specified. In contrast, the inverse vibration solutions for a damped system that we are going to discuss here involve the determination of the time-dependent external forces from the knowledge of the measured displacement and velocity (or just displacement measurements) at different time t.

External force estimation is the process of determining the applied loadings from the measurements of the system responses. This can be found in many engineering applications, especially for the structure responses on those excitations of wind, wave, seismic, explosion, noise and so on. However, the inverse problem tends to be ill-posed, in the sense that small variations in the measured data can excite large excursion in the estimated values. For this reason a suitable algorithm should be chosen to avoid ill-posed phenomena.

The techniques of inverse problems were applied in many different area of engineering research. Many difficulties but practical inverse heat transfer problems in thermal sciences were solved by using a very powerful algorithm, i.e., the conjugate gradient method (CGM). For instance, Huang and Chen [1] used boundary element method and conjugate gradient method to estimate the growth of boundary thickness of a multiple region domain. Huang

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et al. [2] used the CGM to estimate the contact conductance for the plat-finned tube heat exchangers. Huang *et al.* [3] used the same technique in estimating the unknown internal shapes of the material. Moreover, it has also been used in engineering fracture mechanics. For example, Huang and Shih [4] used CGM to estimate the interfacial cracks in bimaterials, etc.

For the inverse vibration problems, the textbook by Gladwell [5] contains a general presentation of the inverse problem for undamped vibrating system. Starek and Inman [6–8] have analyzed an inverse eigenvalue problem in estimating the coefficient matrices. Stevens [9] has shown an overview in identifying the forces for the case of linear vibratory system. Desanghere and Snoeys [10] used a condition number in force identification problems and observed it is a reliable indicator for ill-conditioned matrix. Bateman *et al.* [11] presented two force reconstruction techniques, i.e., the sum of the weighted acceleration and the deconvolution techniques to evaluate the impact test. Michaels and Pao [12] presented an iterative method of deconvolution, which determined the inverse source problem for an oblique force on an elastic plate. More recently, Ma *et al.* [13] used the Kalman filter with a recursive estimator to determine the impulsive loads in a single-degree-of-freedom (SDOF) as well as for a multiple-degree-of-freedom (MDOF) lumped-mass systems.

In all the above references the system parameters are all assumed constants. The discussions of the inverse non-linear force vibration problems (i.e., the system parameters are function of time) in estimating the external forces using the conjugate gradient method have never seen, to the author's best knowledge, in the literature. For this reason, the purpose of the present paper is to establish an algorithm based on CGM to estimate the unknown external forces in the inverse non-linear force vibration problems.

The CGM is also called an iterative regularization method, which means the regularization procedure is performed during the iterative processes. The CGM derives basis from the perturbational principles [14] and transforms the inverse problem to the solution of three problems, namely, the direct problem, the sensitivity problem and the adjoint problem, which will be discussed in detail in the text.

Finally, the inverse solutions for a damped vibration problem with different types of external forces will be illustrated to show the validity of using the CGM in the present inverse vibration problem.

2. THE DIRECT PROBLEM

To illustrate the methodology for developing expressions for use in determining unknown time-dependent external forces in a non-linear damped vibration system with time-dependent system parameters (i.e., the damping coefficient $C(t)$ and spring constant $K(t)$ are both functions of time), we consider the following damped force vibration problem.

The initial displacement and velocity conditions of the system are $x(0) = x_0$ and $\mathrm{d}x(0)/\mathrm{d}t = y(0) = y_0$ respectively. When $t > 0$, the system parameters $K(t)$ and $C(t)$ are given; moreover, the time-dependent external forces $f(t)$ are also assumed to be known.

The system under consideration here is shown in Figure 1 and the mathematical formulation of this damped force vibration problem is given by

$$M\,\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = -\,C(t)\,\frac{\mathrm{d}x(t)}{\mathrm{d}t} - K(t)x(t) + f(t), \quad t > 0 \tag{1a}$$

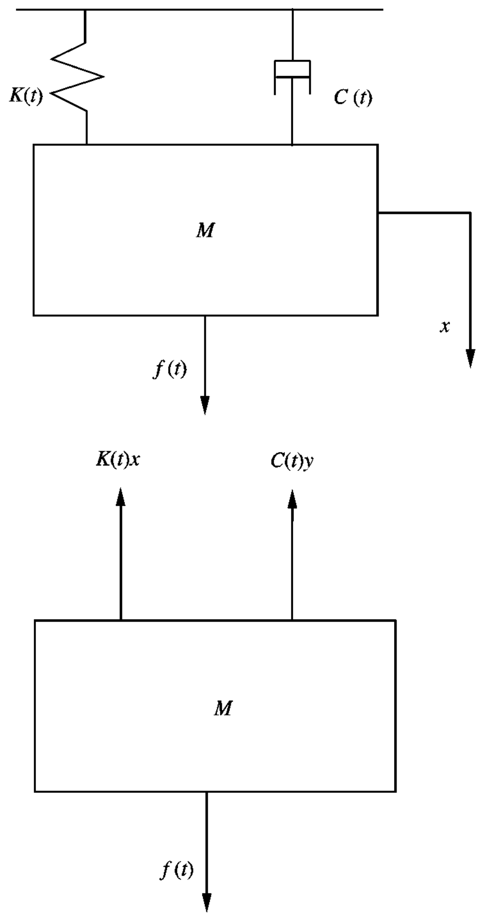

Figure 1. A non-linear force vibration system considered in the present study.

with initial conditions

$$x(0) = x_0 \quad \text{and} \quad \frac{dx(0)}{dt} = y(0) = y_0. \tag{1b}$$

Here M represents the mass of the system. There exists no exact solution for equation (1) for any arbitrary function of $K(t)$ and $C(t)$. For this reason, the numerical solution with the technique of the fourth order Runge–Kutta method will be applied to solve equation (1) by reducing it into two coupled first order differential equations as follows:

$$\frac{dx(t)}{dt} = y(t), \quad t > 0, \quad x(0) = x_0, \tag{2a}$$

$$\frac{dy(t)}{dt} = -\frac{C(t)}{M} y - \frac{K(t)}{M} x(t) + \frac{f(t)}{M}, \quad t > 0, \quad y(0) = y_0. \tag{2b}$$

The direct problem considered here is concerned with the determination of the system displacement $x(t)$ and velocity $y(t)$ when the initial conditions, the system parameters and $K(t)$ and $C(t)$ and the time-dependent external forces $f(t)$ are all given.

Here the fourth order Runge–Kutta method is used to solve the system of equations (2a) and (2b).

3. THE INVERSE PROBLEM

For the inverse problem, the time-dependent external forces $f(t)$ are regarded as being unknown, but everything else in equations (2a) and (2b) are known. In addition, system displacement and velocity measured at some appropriate time are considered available.

Let the measured system displacement and velocity with time be denoted by $X(t)$ and $Y(t)$, respectively; here $t = 0$ to t_f, where t_f represents the final time of the measurements. Then the inverse problem can be stated as follows: by utilizing the above-mentioned measured system displacement, $X(t)$, and velocity, $Y(t)$, data to estimate the unknown time-dependent external forces $f(t)$.

In the present study, we have not used real measured system displacement and velocity, rather, we use the exact external forces to generate the simulated values of $X(t)$ and $Y(t)$, then try to retrieve the time-dependent external forces by using $X(t)$ and $Y(t)$ and initial guesses of the external forces.

The solution of the present inverse vibration problem is to be obtained in such a way that the following functional is minimized:

$$J[f(t)] = \int_{t=0}^{t_f} \{[x(t) - X(t)]^2 + [y(t) - Y(t)]^2\} \, dt. \tag{3}$$

Here, $x(t)$ and $y(t)$ are the estimated or computed displacement and velocity at time t. These quantities are determined from the solution of the direct problem given previously by using an estimated $\hat{f}(t)$ for the exact $f(t)$. Here the hat "ˆ" denotes the estimated quantities.

4. CONJUGATE GRADIENT METHOD FOR MINIMIZATION

The following iterative process based on the conjugate gradient method [14] is now used for the estimation of time-dependent external forces $f(t)$ by minimizing the functional $J[f(t)]$:

$$\hat{f}^{n+1}(t) = \hat{f}^{n}(t) - \beta^n P^n(t) \quad \text{for } n = 0, 1, 2, \ldots, \tag{4}$$

where β^n is the search step size in going from iteration n to iteration $n + 1$, and $P^n(t)$ is the directions of descent (i.e., search direction) given by

$$P^n(t) = J'^n(t) + \gamma^n P^{n-1}(t), \tag{5}$$

which are a conjugation of the gradient direction $J'^n(t)$ at iteration n and the direction of descent $P^{n-1}(t)$ at iteration $n - 1$. The conjugate coefficient is determined from

$$\gamma^n = \frac{\int_{t=0}^{t_f} (J'^n)^2 \, dt}{\int_{t=0}^{t_f} (J'^{n-1})^2 \, dt} \quad \text{with } \gamma^0 = 0. \tag{6}$$

We note that when $\gamma^n = 0$ for any n, in equation (6), the direction of descent $P^n(t)$ becomes the gradient direction, i.e., the "steepest descent" method is obtained. The convergence of the above iterative procedure in minimizing the functional J is guaranteed in reference [15].

To perform the iterations according to equation (4), we need to compute the step size β^n and the gradient of the functional $J'^n(t)$. In order to develop expressions for the determination of these two quantities, *a* "sensitivity problem" and an "adjoint problem" are constructed as described below.

4.1. SENSITIVITY PROBLEM AND SEARCH STEP SIZE

The sensitivity problem is obtained from the original direct problem defined by equations (2a) and (2b) in the following manner: it is assumed that when $f(t)$ undergoes a variation $\Delta f(t)$, $x(t)$ and $y(t)$ are perturbed by Δx and Δy. Then replacing in the direct problem f by $f + \Delta f$, x by $x + \Delta x$ and y by $y + \Delta y$, subtracting from the resulting expressions the direct problem and neglecting the second order terms, the following sensitivity problems for the sensitivity functions Δx and Δy are obtained:

$$\frac{\mathrm{d}\Delta x(t)}{\mathrm{d}t} = \Delta y(t), \quad t > 0, \quad \Delta x(0) = 0, \tag{7a}$$

$$\frac{\mathrm{d}\Delta y(t)}{\mathrm{d}t} = -\frac{C(t)}{M}\Delta y - \frac{K(t)}{M}\Delta x(t) + \frac{\Delta f(t)}{M}, \quad t > 0, \quad \Delta y(0) = 0. \tag{7b}$$

The technique of fourth order Runge–Kutta method is used to solve this sensitivity problems.

The functional $J(\hat{f}^{n+1})$ for iteration $n + 1$ is obtained by rewiting equation (3) as

$$J[\hat{f}(t)] = \int_{t=0}^{t_f} \{[x(\hat{f}^n - \beta^n P^n) - X(t)]^2 + [y(\hat{f}^n - \beta^n P^n) - Y(t)]^2\}\,\mathrm{d}t, \tag{8}$$

where we replaced $\hat{f}^{n+1}(t)$ by the expression given by equation (4). If estimated displacement $x(\hat{f}^n - \beta^n P^n)$ and velocity $y(\hat{f}^n - \beta^n P^n)$ are linearized by a Taylor expansion, equation (8) takes the form

$$J[\hat{f}^{n+1}(t)] = \int_{t=0}^{t_f} \{[x(\hat{f}^n) - \beta^n \Delta x(P^n) - X(t)]^2 + [y(\hat{f}^n) - \beta^n \Delta y(P^n) - Y(t)]^2\}\,\mathrm{d}t, \tag{9}$$

where $x(\hat{f}^n)$ and $y(\hat{f}^n)$ are the solutions of the direct problem by using estimate $\hat{f}^n(t)$ for exact $f(t)$ at time t. The sensitivity functions $\Delta x(P^n)$ and $\Delta y(P^n)$ are taken as the solutions of problems (7a) and (7b) at time t by letting $\Delta f(t) = P^n(t)$ in equation (7b) [14].

Equation (9) is differentiated with respect to β^n and equated them to zero. Finally, the step size can be obtained as

$$\beta^n = \frac{\int_{t=0}^{t_f} \{\Delta x(P^n)[x(\hat{f}^n) - X] + \Delta y(P^n)[y(\hat{f}^n) - Y]\}\,\mathrm{d}t}{\int_{t=0}^{t_f} [\Delta x^2(P^n) + \Delta y^2(P^n)]\,\mathrm{d}t}. \tag{10}$$

4.2. ADJOINT PROBLEM AND GRADIENT EQUATION

To obtain the adjoint problems when perturbing $f(t)$, equations (2a) and (2b) are multiplied by the Lagrange multipliers (or adjoint functions) $\lambda_1(t)$ and $\lambda_2(t)$ respectively. The resulting expression is integrated over the corresponding time domain, then the result is added to the right-hand side of equation (3) to yield the following expression for the functional $J[f(t)]$:

$$J[f(t)] = \int_{t=0}^{t_f} \{[x(t) - X(t)]^2 + [y(t) - Y(t)]^2\} \, \mathrm{d}t + \int_{t=0}^{t_f} \lambda_1(t) \left[\frac{\mathrm{d}x(t)}{\mathrm{d}t} - y(t) \right] \mathrm{d}t$$

$$+ \int_{t=0}^{t_f} \lambda_2(t) \left[\frac{\mathrm{d}y(t)}{\mathrm{d}t} + \frac{C(t)}{M} y(t) + \frac{K(t)}{M} x(t) - \frac{f(t)}{M} \right] \mathrm{d}t. \quad (11)$$

The variation ΔJ is obtained by perturbing f by $f + \Delta f$, x by $x + \Delta x$ and y by $y + \Delta y$ in equation (11), subtracting from the resulting expression the original equation (11) and neglecting the second order terms. We thus find

$$\Delta J[f(t)] = 2 \int_{t=0}^{t_f} \{[x(t) - X(t)] \Delta x + [y(t) - Y(t)] \Delta y\} \, \mathrm{d}t$$

$$+ \int_{t=0}^{t_f} \lambda_1(t) \left[\frac{\mathrm{d}\Delta x(t)}{\mathrm{d}t} - \Delta y(t) \right] \mathrm{d}t$$

$$+ \int_{t=0}^{t_f} \lambda_2(t) \left[\frac{\mathrm{d}\Delta y(t)}{\mathrm{d}t} + \frac{C(t)}{M} \Delta y(t) + \frac{K(t)}{M} \Delta x(t) - \frac{\Delta f(t)}{M} \right] \mathrm{d}t. \quad (12)$$

In equation (12), the second and third integral terms are integrated by parts; the initial conditions of the sensitivity problem are utilized. The vanishing of the integrands leads to the following adjoint problems for the determination of $\lambda_1(t)$ and $\lambda_2(t)$:

$$-\frac{\mathrm{d}\lambda_1(t)}{\mathrm{d}t} = -\frac{K(t)}{M} \lambda_2(t) - 2(x - X), \quad t > 0, \quad \lambda_1(t_f) = 0, \quad (13a)$$

$$-\frac{\mathrm{d}\lambda_2(t)}{\mathrm{d}t} = -\frac{C(t)}{M} \lambda_2(t) + \lambda_1(t) - 2(y - Y), \quad t > 0, \quad \lambda_2(t_f) = 0. \quad (13b)$$

The adjoint problems are different from the standard initial value problems in that the final time conditions at time $t = t_f$ is specified instead of the customary initial condition. However, this problem can be transformed to an initial value problem by the transformation of the time variables as $\tau = t_f - t$. Then the standard techniques of fourth order Runge–Kutta method can be used to solve the above adjoint problems.

Finally, the integral term left is

$$\Delta J = \int_{t=0}^{t_f} -(\lambda_2 \Delta f / M) \, \mathrm{d}t. \quad (14)$$

From definition [14], the functional increment can be presented as

$$\Delta J = \int_{t=0}^{t_f} (J' \Delta f) \, \mathrm{d}t. \quad (15)$$

A comparison of equations (14) and (15) leads to the following expression for the gradient of functional J':

$$J'[f(t)] = -\lambda_2(t)/M. \quad (16)$$

4.3. STOPPING CRITERION

If the problem contains no measurement errors, the traditional check condition is specified as

$$J[\hat{f}^{n+1}(t)] < \varepsilon, \tag{17}$$

where ε is a small specified number. However, the measured displacement and velocity data may contain measurement errors. Therefore, we do not expect the functional equation (3) to be equal to zero at the final iteration step. Following the experiences of the authors [1–4, 14], we use the discrepancy principle as the stopping criterion, i.e., we assume that the residuals for the displacement and velocity may be approximated by

$$x(t) - X(t) \approx \sigma_1, \tag{18a}$$

$$y(t) - Y(t) \approx \sigma_2, \tag{18b}$$

where σ_1 and σ_2 are the standard deviation of the displacement and velocity measurements, respectively, which are assumed to be a constant. Substituting equations (18a) and (18b) into equation (3), the following expression is obtained for stopping criteria ε.

$$\varepsilon = (\sigma_1^2 + \sigma_2^2)t_f. \tag{19}$$

Then, the stopping criterion is given by equation (17) with ε determined from equation (19).

5. COMPUTATIONAL PROCEDURE

The computational procedure for the solution of this inverse problem using conjugate gradient method may be summarized as follows:

Suppose $\hat{f}^n(t)$ is available at iteration n.

Step 1. Solve the direct problems given by equations (2a) and (2b) for $x(t)$ and $y(t)$ respectively.

Step 2. Examine the stopping criterion given by equation (19). Continue the iteration if not satisfied.

Step 3. Solve the adjoint problems given by equations (13a) and (13b) for $\lambda_1(t)$ and $\lambda_2(t)$ respectively.

Step 4. Compute the gradient of the functional $J'(t)$ from equations (16).

Step 5. Compute the conjugate coefficients γ^n and the direction of descent P^n from equations (6) and (5) respectively.

Step 6. Set $\Delta f = P^n$. The solve the sensitivity problems given by equations (7a) and (7b) for Δx and Δy respectively.

Step 7. Compute the search step size β^n from equation (10).

Step 8. Computed the new estimations for $\hat{f}^{n+1}(t)$ from equation (4) and return to step 1.

6. RESULTS AND DISCUSSION

The objective of this work is to show the validity of the CGM in estimating the external forces $f(t)$ in the inverse non-linear force vibration problems with no prior information on the functional form of the unknown quantities.

To illustrate the accuracy of the conjugate gradient method in predicting external forces $f(t)$ in a damped vibration problem from the knowledge of transient displacement and velocity recordings, two specific examples having different form of external forces are considered here.

In order to compare the results for situations involving random measurement errors, we assume normally distributed uncorrelated errors with zero mean and constant standard deviation. The simulated inexact measurement displacement and velocity data **X** and **Y** can be expressed as

$$\mathbf{X} = \mathbf{X}_{exact} + \omega\sigma_1, \tag{20a}$$

$$\mathbf{Y} = \mathbf{Y}_{exact} + \omega\sigma_2, \tag{20b}$$

where $\mathbf{X}_{exact}$ and $\mathbf{Y}_{exact}$ are the solution of the direct vibration problem with an exact external forces $f(t)$; σ_1 and σ_2 are the standard deviation of the measured displacement and velocity, respectively, and ω is a random variable that is generated by subroutine DRNNOR of the IMSL [16] and will be within $-2{\cdot}576$–$2{\cdot}576$ for a 99% confidence bound.

One of the advantages of using the conjugate gradient method to solve the inverse problems is that the initial guesses of the unknown quantities can be chosen arbitrarily. In all the test cases considered here the initial guesses of $\hat{f}(t)$ is taken as $\hat{f}(t)_{initial} = 0{\cdot}0$.

6.1. NUMERICAL TEST CASE 1

We now present below the numerical experiments in determining $f(t)$ by the inverse analysis using the CGM.

The parameters that used in the test case 1 are taken as

$$M = 1{\cdot}0,\ K(t) = 2{\cdot}0 + 0{\cdot}01t \quad \text{and} \quad C(t) = 3{\cdot}0\text{–}0{\cdot}01t.$$

The initial conditions for displacement and velocity are both assumed zero, i.e., $x(0) = 0$ and $y(0) = 0$. Time interval is chosen as 120, i.e., $t_f = 120$, and a time step $\Delta t = 1$ is used. Therefore, a total of 120 unknown discretized external forces are to be determined in the present study. The number of measured displacement and velocity data are both 120.

The unknown transient external forces are assumed as

$$f(t) = \begin{cases} 10 \times \mathrm{SIN}\left(\dfrac{2\pi t}{50}\right) & \text{for } 0 < t \leqslant 40 \\ 0{\cdot}0 & \text{for } 41 < t \leqslant 50 \\ 20 & \text{for } 51 < t \leqslant 70 \\ 0{\cdot}0 & \text{for } 71 < t \leqslant 80 \\ 10 \times \mathrm{COS}\left(\dfrac{2\pi t}{50}\right) & \text{for } 81 < t \leqslant 110 \\ 0{\cdot}0 & \text{for } 111 < t \leqslant 120. \end{cases} \tag{21}$$

The inverse analysis is first performed by using both the displacement and velocity measurements and assuming no measurement errors, i.e., $\sigma_1 = \sigma_2 = 0{\cdot}0$. When the stopping criteria is set as $\varepsilon = 0{\cdot}1$, after only five iterations the inverse solutions are converged, J is

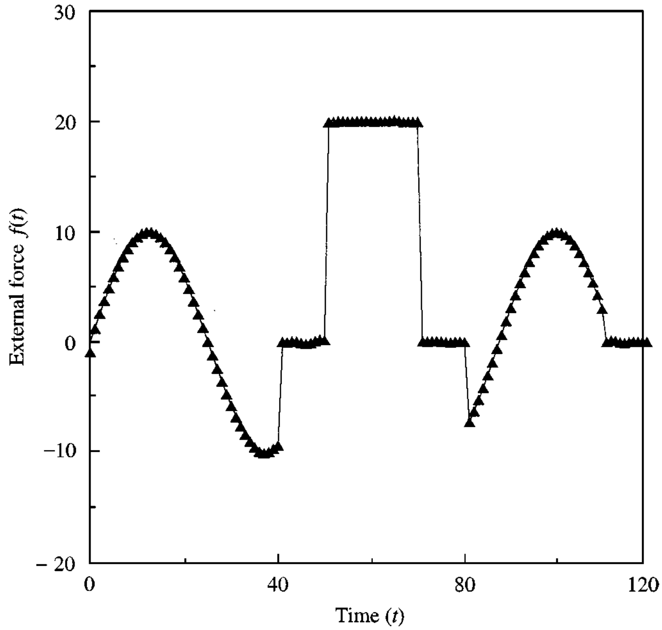

Figure 2. The exact (f) and estimated ($\hat{f}$) external forces using both displacement and velocity measurements with $\sigma_1 = \sigma_2 = 0{\cdot}0$ in numerical test case 1: ——, exact; ▲, estimated.

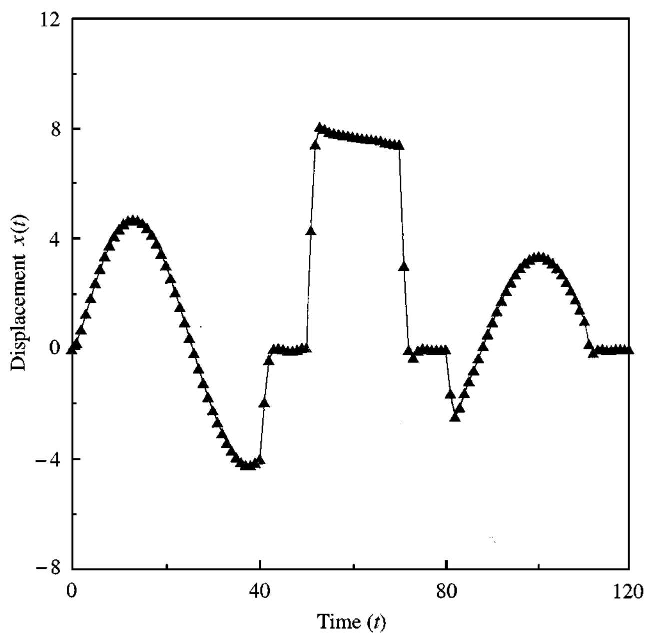

Figure 3. The measured (X) and estimated (x) displacement with $\sigma_1 = \sigma_2 = 0{\cdot}0$ in numerical test case 1: ——, measured $X(t)$; ▲, estimated $x(t)$.

calculated as 0·089 and CPU time at Pentium-II-300 MHz PC is about 3 s. The exact and estimated external forces are shown in Figure 2 while Figure 3 shows the measured and estimated displacement, $X(t)$ and $x(t)$, and Figure 4 shows the measured and estimated velocity, $Y(t)$ and $y(t)$. From these figures we concluded that the present algorithm has been

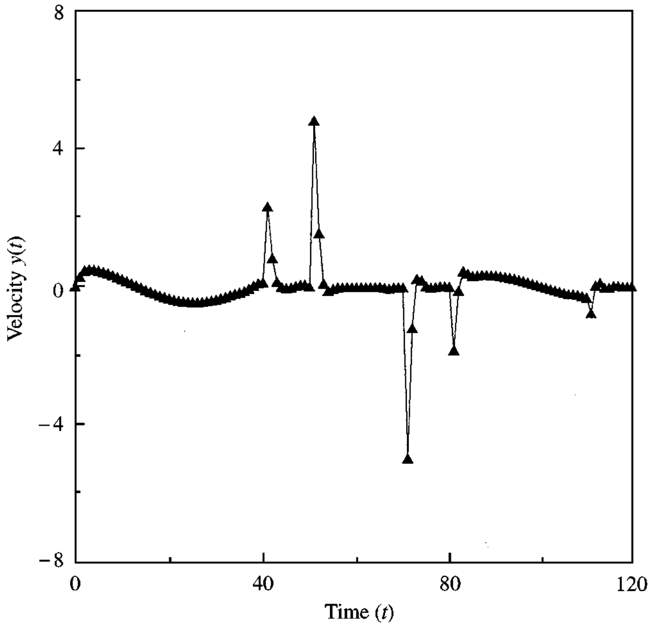

Figure 4. The measured (Y) and estimated (y) velocity with $\sigma_1 = \sigma_2 = 0{\cdot}0$ in numerical test case 1: ——, measured $Y(t)$; ▲, estimated $y(t)$.

applied successfully in the inverse vibration problem in estimating external forces since the estimated results are very accurate.

Next, it is of interest to discuss what will happen when we used only the displacement measurements. This implies that the measurement job can be done easily since the measurement for velocity is not necessary. The same calculation conditions are used except for the term $2(y - Y)$ that appears in equations now vanish. The number of iterations under this situation is 10 and CPU time at Pentium II-300 MHz PC is about 6 s. The estimated $f(t)$ is plotted in Figure 5. It is obvious that the estimated external forces are as accurate as that using both displacement and velocity measurements but the rate of convergence is slower. The measured and estimated displacements $X(t)$ and $x(t)$ are shown in Figure 6. For this reason, we suggested that only the displacement measurements are needed for the rest of the numerical experiments in text.

Finally, let us discuss the influence of the measurement errors on the inverse solutions. When the measurement error for the displacements measured by sensors is taken as $\sigma_1 = 0{\cdot}2$ (about 10% of the average measured displacement) and $\sigma_2 = 0.$ (since only displacement measurements are used). After only two iterations and CPU time is 1 s, the inverse solutions can be obtained and plotted in Figure 7 for the exact and estimated external forces and in Figure 8 for the measured and estimated displacements. From these figures we learned that reliable inverse solutions can still be obtained when the large measurement errors are considered.

6.2. NUMERICAL TEST CASE 2

The system parameters that are used in test case 2 are now taken as

$$M = 1{\cdot}0, \quad K(t) = 5{\cdot}0 - 0{\cdot}015t \quad \text{and} \quad C(t) = 2{\cdot}0 + 0{\cdot}01t.$$

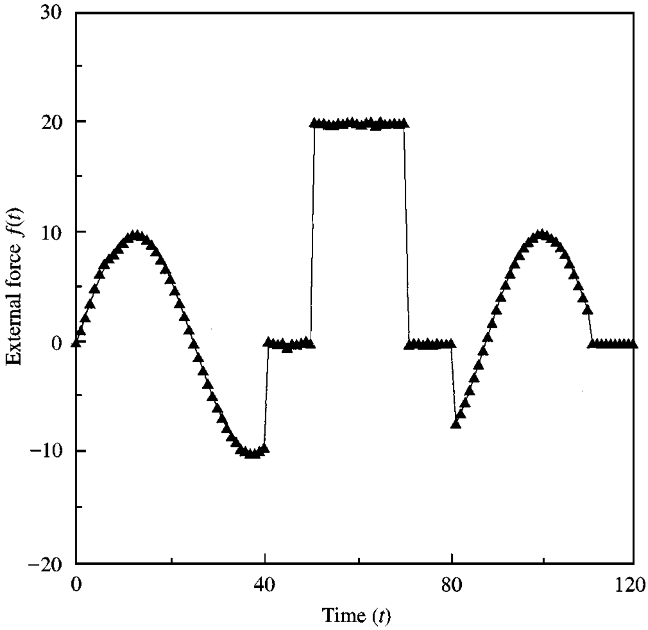

Figure 5. The exact (f) and estimated ($\hat{f}$) external forces using only displacement measurements with $\sigma_1 = \sigma_2 = 0{\cdot}0$ in numerical test case 1: ——, exact; ▲, estimated.

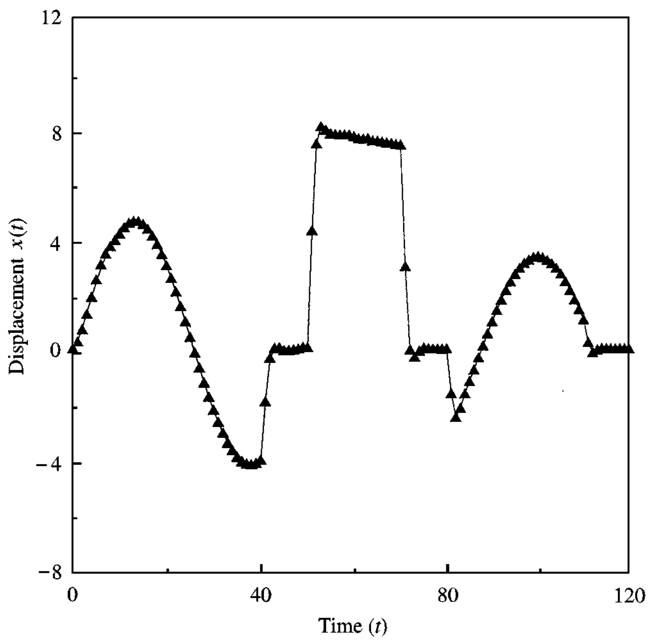

Figure 6. The measured (X) and estimated (x) displacement with $\sigma_1 = \sigma_2 = 0{\cdot}0$ in numerical test case 1: ——, measured $X(t)$; ▲, estimated $x(t)$.

The initial conditions are taken as $x(0) = 0$ and $y(0) = 0$. Time interval is also chosen as 120 and a time step $\Delta t = 1$ is used. Therefore, a total of 120 unknown discretized external forces are to be determined in the present study. The number of measured displacements is 120.

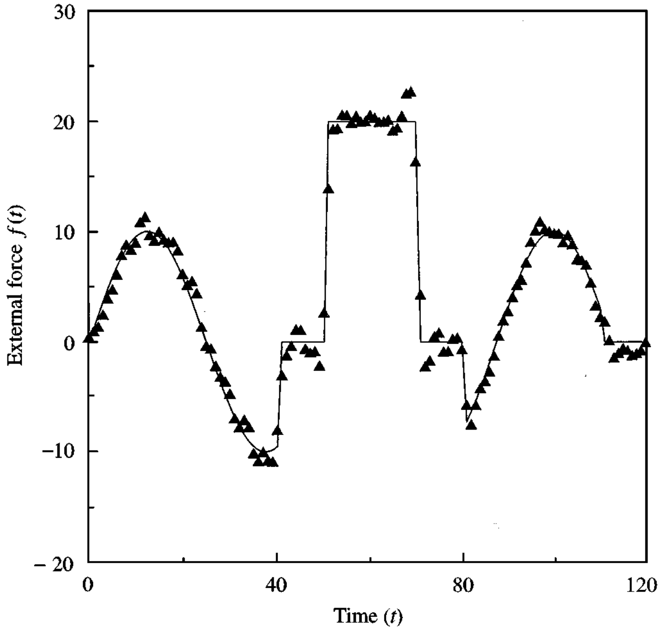

Figure 7. The exact (f) and estimated ($\hat{f}$) external forces using only displacement measurements with $\sigma_1 = 0{\cdot}2$ in numerical test case 1: ——, exact; ▲, estimated.

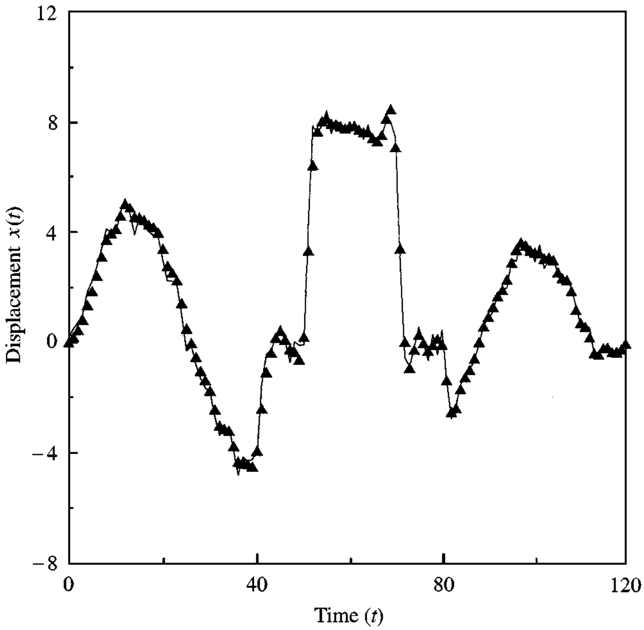

Figure 8. The measured (X) and estimated (x) displacement with $\sigma_1 = 0{\cdot}2$ in numerical test case 1: ——, measured $X(t)$; ▲, estimated $x(t)$.

The unknown transient external forces $f(t)$ are assumed as the impulse function, which means large external forces will be applied to the system for only certain short time and forces remain zero for the rest of the time. The impulse function can be expressed as

$$f(t) = \begin{cases} 200\delta(t-11) \\ 200\delta(t-12) \\ 500\delta(t-31) \\ 500\delta(t-32) \\ 400\delta(t-61) \\ 400\delta(t-62) \\ 300\delta(t-81) \\ 300\delta(t-82) \\ 80\delta(t-101) \\ 80\delta(t-102) \end{cases} \quad \text{for } 0 < t \leqslant 120, \tag{22}$$

where $\delta(\cdot)$ is the Dirac-delta function. The inverse analysis is first performed by using exact displacement measurements, i.e., assuming no measurement errors $\sigma_1 = 0{\cdot}0$. When the stopping criteria is set as $\varepsilon = 35$, after 19 iterations the inverse solutions are converged, J is calculated as 33·5 and CPU time is about 11 s. The exact and estimated external forces are shown in Figure 9. From the figure we know that for this strict test the estimated external forces are still very accurate. The measured and estimated displacements $X(t)$ and $x(t)$, are shown in Figure 10.

Next, when the measurement error for the displacements measured by sensors is taken as $\sigma_1 = 1{\cdot}2$ (about 10% of the average measured displacement), then the stopping criteria ε can be calculated from equation (19) where σ_2 should be equal to zero. The inverse solutions are converged after only seven iterations and CPU time is about 4 s. The exact and estimated

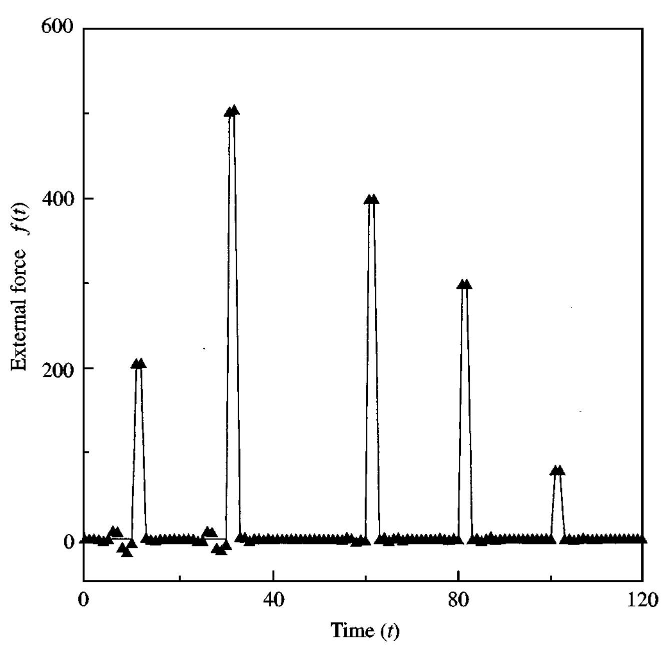

Figure 9. The exact (f) and estimated ($\hat{f}$) external forces using only displacement measurements with $\sigma_1 = 0{\cdot}0$ in numerical test case 2: ——, exact; ▲, estimated.

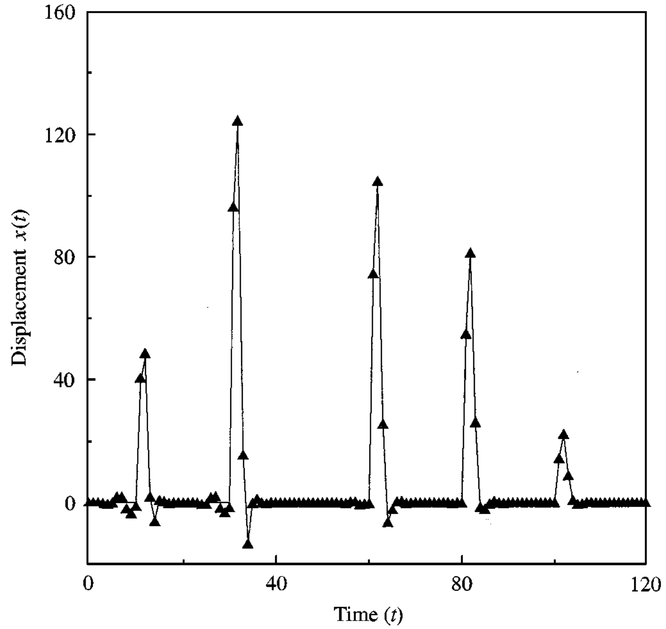

Figure 10. The measured (X) and estimated (x) displacement with $\sigma_1 = 0{\cdot}0$ in numerical test case 2: ——, measured $X(t)$; ▲, estimated $x(t)$.

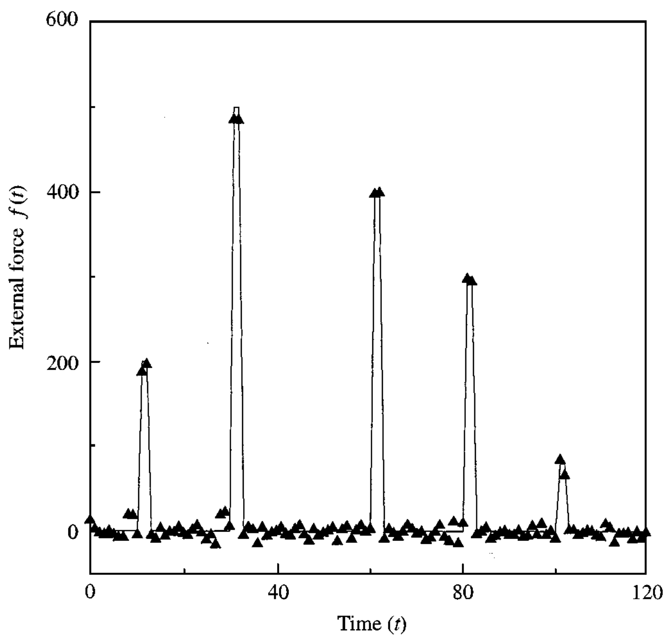

Figure 11. The exact (f) and estimated ($\hat{f}$) external forces using only displacement measurements with $\sigma_1 = 1{\cdot}2$ in numerical test case 2: ——, exact; ▲, estimated.

external forces are plotted in Figure 11, and Figure 12 shows the measured and estimated displacements. Again, from these figures we learned that the reliable inverse solutions can still be obtained when the large measurement errors are considered.

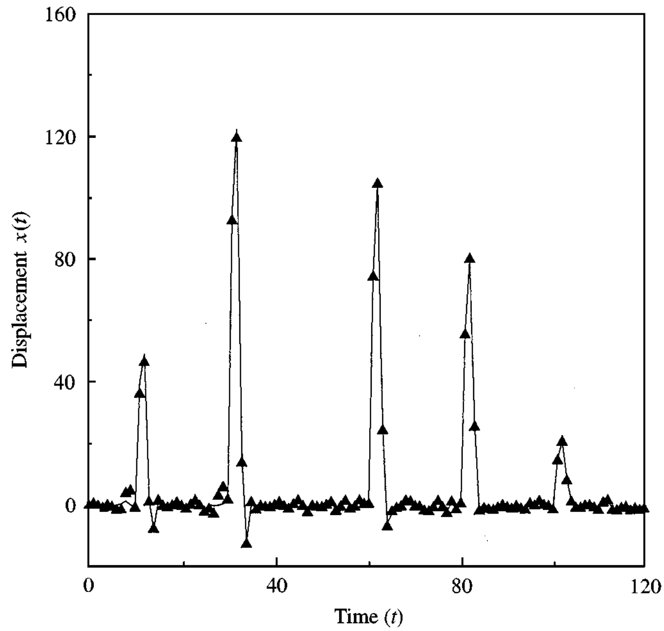

Figure 12. The measured (X) and estimated (x) displacement with $\sigma_1 = 1{\cdot}2$ in numerical test case 2: ——, measured $X(t)$; ▲, estimated $x(t)$.

From the above two test cases we learned that an inverse non-linear damped force vibration problem in estimating unknown external forces is now completed. Reliable estimations can be obtained when using either exact or error measurements.

Moreover, even though the algorithm developed in this paper is for only a SDOF problem, it can readily be extended to a MDOF problem as well as to a problem with displacement-dependent system parameters, i.e., $K(x)$ and $C(x)$. Those problems are now undertaken and the results of the first investigation seem quite good.

7. CONCLUSIONS

The CGM was successfully applied for the solution of the inverse non-linear force vibration problem to determine the unknown transient external forces by utilizing simulated displacement and velocity readings obtained from sensors. Several test cases involving different system parameters, measurement errors and external forces were considered. The results show that the inverse solutions obtained by CGM remain stable and regular as the measurement errors are large. Moreover, the CPU time needed in the inverse calculations is very short and the initial guesses for the external forces can be arbitrarily chosen as zero.

ACKNOWLEDGMENTS

This work was supported in part through the National Science Council, R.O.C., Grant number NSC-89-2611-E-006-058.

REFERENCES

1. C. H. Huang and H. M. Chen 1999 *Numerical Heat Transfer*, *Part A* **35**, 435–450. An inverse geometry problem of identifying growth of boundary shapes in a multiple region domain.
2. C. H. Huang, D. M. Wang and H. M. Chen 1999 *Inverse Problems in Engineering* **7**, 119–141. Prediction of local thermal contact conductance in plate finned-tube heat exchangers.
3. C. H. Huang, C. C. Chiang and H. M. Chen 1998 *AIAA*, *Journal of Thermophysics and Heat Transfer* **12**, 270–277. A shape identification problem in estimating the geometry of multiple cavities.
4. C. H. Huang and W. Y. Shih 1999 *International Journal of Numerical Methods in Engineering* **45**, 1547–1576. An inverse problem in estimating interfacial cracks in bimaterials by boundary element technique.
5. G.M.L. Gladwell 1986 *Inverse Problem in Vibration*. The Netherlands: Kluwer Academic Publishers.
6. L. Starek and D. J. Inman 1991 *Transactions of the American Society of Mechanical Engineers*, *Journal of Applied Mechanics* **58**, 1101–1104. On the inverse vibration problem with rigid-body modes.
7. L. Starek and D. J. Inman and A. Kress 1992 *Transactions of the American Society of Mechanical Engineers*, *Journal of Vibration and Acoustics* **114**, 565–568. A symmetric inverse vibration problem.
8. L. Starek and D. J. Inman 1995 *Journal of Sound and Vibration* **181**, 893–903. A symmetric inverse vibration problem with overdamped modes.
9. K. K. Stevens 1987 *Proceedings of the SEM Spring Conference on Experimental Mechanics*, *Houston*, *TX*, 14–19 June 838–844, Force identification problems—an overview.
10. G. Desanghere and R. Snoeys 1985 *Proceedings of the* 1985 *International Modal Analysis Conference*, *Orlando*, *FL*, 685–690. Indirect identification of excitation force by modal coordinate transformation.
11. V. I. Bateman, T. G. Carne, D. L. Gregory, S. W. Attaway and H. R. Yoshimura 1991 *Transactions of the American Society of Mechanical Engineers*, *Journal of Vibration and Acoustics* **113**, 192–200. Force reconstruction for impact tests.
12. J. E. Michaels and Y. H. Pao 1985 *Journal of Acoustical Society of American* **77**, 2005–2011. The inverse source problem for an oblique force on an elastic plate.
13. C. K. Ma, P. C. Tuan, D. C. Lin and C. S. Liu 1998 *International Journal of System Science* **29**, 663–672. A study of an inverse method for impulsive loads estimation.
14. O. M. Alifanov 1974 *Journal of Engineering Physics* **26**, 471–476. Solution of an inverse problem of heat conduction by iteration methods.
15. L. S. Lasdon, S. K. Mitter and A. D. Warren 1967 *IEEE Transactions on Automatic Control* **AC-12**, 132–138. The conjugate gradient method for optimal control problem.
16. IMSL Library Edition 10.0, 1987 *User's Manual*: *Math Library Version* 1.0, Houston, TX IMSL.

APPENDIX A: NOMENCLATURE

$C(t)$	damping coefficient
$f(t)$	external force
J	functional defined by equation (3)
J'	gradient of functional defined by equation (16)
$K(t)$	spring constant
M	system mass
P	direction of descent defined by equation (5)
t	time
x	estimated displacement
X	measured displacement
$\mathbf{y}$	estimated velocity
$\mathbf{Y}$	measured velocity

Greek letters

β	search step size defined by equation (10)
γ	conjugate coefficient defined by equation (6)

λ_1, λ_2	adjoint functions defined by equation (13)
Δx, Δy	sensitivity functions defined by equation (7)
ε	convergence criteria
ω	random number
σ	standard deviation of measurement errors

Superscript

$\hat{}$	estimated values