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A LOCAL MODAL/PERTURBATIONAL METHOD FOR ESTIMATING FREQUENCY RESPONSE STATISTICS OF BUILT-UP STRUCTURES WITH UNCERTAIN PROPERTIES

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A local modal/perturbational method is described which enables estimates to be made of the statistics of frequency response functions of a system whose properties are uncertain. The method is computationally very efficient, providing these estimates at little cost above that required to perform a single deterministic frequency response prediction. Such an approach is typically required when there is significant uncertainty or variability in the properties of the system under consideration. This would often be at higher frequencies, where the dynamic behaviour becomes increasingly sensitive to small changes in system properties, but not at frequencies high enough such that broad-brush statistical methods, such as statistical energy analysis, are appropriate.

In the approach the system is divided into subsystems. The frequency response of the baseline system is found using modal analysis. The global modes of the baseline system are found in terms of the subsystem modes using component mode synthesis. Uncertainty is then assumed to exist in the local modal properties of the subsystems. A perturbation is found which relates small changes in the local modal properties to those in the global modal properties, enabling the frequency response of the perturbed system to be estimated. Finally, a Monte Carlo simulation is used to estimate the frequency response function statistics. Numerical results are presented for a system comprising two spring-coupled rods.

1. INTRODUCTION

Different physical realizations of a manufactured product have physical and geometric properties which, while similar, inevitably differ in detail. A number of factors cause this variability. These include normal manufacturing variations in component dimensions, material properties, joint properties, etc., variations in environmental conditions such as ambient temperature, aging, wear, and variations in operating conditions.

The inherent variability in the properties of the system produces consequent variability in dynamic response, such as the frequency response function (FRF). The dynamic variability becomes larger at higher frequencies, especially in complex, built-up structures, because of

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the increasing sensitivity of the dynamic behaviour to even small variations in system properties. In a similar manner, numerical predictions are subject to uncertainty in the physical and geometric properties or the boundary conditions of a given system: these cannot be known exactly. Although model refinement, updating and parameter estimation may improve the agreement between predictions and a single, specific realization of the system, this is of little value if the next example of the product is not identical to the first.

Thus, at higher frequencies, in what might be termed the "mid-frequency" range, a single "exact", deterministic prediction (e.g., by way of finite element analysis (FEA)) is not able to describe accurately the behaviour of all possible physical realizations of a system which share the same nominal properties. Instead, the engineer who is trying to model or design for noise and vibration behaviour may need to consider this variability: a stochastic approach is required. He or she should consider the FRFs of an *ensemble* of systems, the variability of these FRFs and their statistics. These statistics include the baseline FRF (i.e., that of the baseline system, the engineer's "best guess"), the mean, minimum, maximum, variance and so on. While high frequency, stochastic approaches exist (e.g., statistical energy analysis (SEA) [1]), these are broad-brush, and lose all details of the response.

In this paper, a local modal/perturbational (LM/P) method for such "mid-frequency" vibration analysis is described. The aim is to predict the baseline FRF and the statistics of the FRFs of an ensemble of systems whose detailed properties vary randomly about the baseline values. The method follows that used in reference [2] to form energy flow models of systems.

The system is assumed to be constructed from a number of connected subsystems. The subsystems are modelled in terms of their local modes using component mode synthesis (CMS) [3]. A fixed interface CMS approach is adopted in this paper. The baseline response is found by assembling the subsystems and solving for the global modes of vibration. Ensemble statistics are estimated by assuming that the properties of the members of the ensemble vary randomly in some way. In particular, in the LM/P method which is described here, it is assumed that the local modal properties of each component subsystem vary randomly and a perturbation analysis is used to obtain the corresponding variations in the global modal properties. A Monte Carlo simulation is then used to estimate the response statistics. This is in contrast to more traditional stochastic finite element approaches [4], where elements in the stiffness and/or mass matrices of the system vary randomly.

One major consideration is that of computational cost. In the mid-frequency range the system will typically be large and complex so that it is costly to perform even the single FEA of the baseline system. It is not practical to repeat such an analysis many times over for different values of system properties in an attempt to estimate the response statistics. In this regard, the LM/P approach is very efficient and the response statistics can be estimated for very little cost above that required to perform the analysis of the baseline system.

In the next section the LM/P method for mid-frequency FRF estimation is described. The application to a system comprising two coupled rods is then considered as a simple numerical example.

2. MID-FREQUENCY VIBRATION ANALYSIS AND THE LM/P METHOD

The aim of a mid-frequency analysis is to predict the FRF statistics of an ensemble of systems whose properties vary randomly. The baseline response (i.e., the response of some nominal, baseline system) is found and so, too, are the statistics of the responses (e.g., mean response, maximum response, variance, percentiles, etc.) of the ensemble member systems.

A mid-frequency model therefore incorporates some variability in system properties and therefore differs from a low-frequency, deterministic analysis. However, the degree of variability is not as wide as that assumed in a high-frequency, statistical approach such as SEA.

In its simplest form, a mid-frequency FRF analysis follows the steps indicated in Figure 1(a). Given the physical properties of the structure, a global modal analysis is performed. This would typically involve a global FEA of the system. The desired FRF is then found by modal summation. If there is uncertainty in the system's properties then, in principle, a Monte Carlo simulation can be performed: the properties are allowed to vary randomly and the analysis process repeated many times. The statistics of the FRFs are then inferred from the statistics of the sample. This approach is impractical because of the large computational expense incurred in repeating the full analysis many times.

It is more efficient computationally, and quite natural in built-up structures, to divide the system into subsystems which are joined together at boundaries or interfaces. The steps in the analysis then become those indicated in Figure 1(b). Each subsystem is described by its local modes using CMS. Again the component modes will typically be found using FEA. The global modal properties can then be found by assembling the subsystem modal models. The computational effort is significantly reduced because the global eigenvalue problem is much smaller when written in terms of local modes: relatively few component modes are required to describe the subsystem behaviour. Thus, rather than solving one large eigenvalue problem, the solutions to a number of smaller eigenvalue problems are required. Another advantage is that large structures are often constructed from a number of component subsystems whose properties vary randomly but independently. In this case, the local modal analysis for a subsystem need only be repeated when the properties of that subsystem vary. However, despite the reduction in computational expense, the cost of a full Monte Carlo simulation is still generally prohibitive.

The most computationally expensive steps in the analysis are indicated in Figure 1(b) and are the local FEA of each subsystem, the eigensolution to find the component modes and assembling the subsystem models and solving for the global modes of vibration.

In the local modal/perturbational approach indicated in Figure 1(c) these steps are removed in the following ways.

- Variability is assumed to exist in the *local modal properties* of each subsystem rather than in their physical properties (i.e., material and geometric properties) directly. Of course, the two are related, since the physical properties uniquely define the local modal properties. However, including variability in this manner avoids the computational expense associated with recalculating the local modes.
- A *perturbational relationship* is found that relates the global and local modal properties. This avoids having to solve the global eigenvalue problem for every member of the sample in the Monte Carlo simulation. Here a linear perturbation is used (e.g., changes in the global natural frequencies and mode shapes are related to changes in the local natural frequencies using a Taylor series expansion) as described below.

As a result, the FRF for each member of the Monte Carlo simulation (apart from the baseline response) can be found at a trivial cost and the ensemble statistics can be found in a small fraction of the time it takes to calculate the baseline response.

In summary, the steps are as follows:

- The system is divided into subsystems joined together at interfaces.
- The subsystems are described in terms of their local modes of vibration. Here, the modes are taken to be those obtained when the interfaces are clamped, although other approaches are possible.

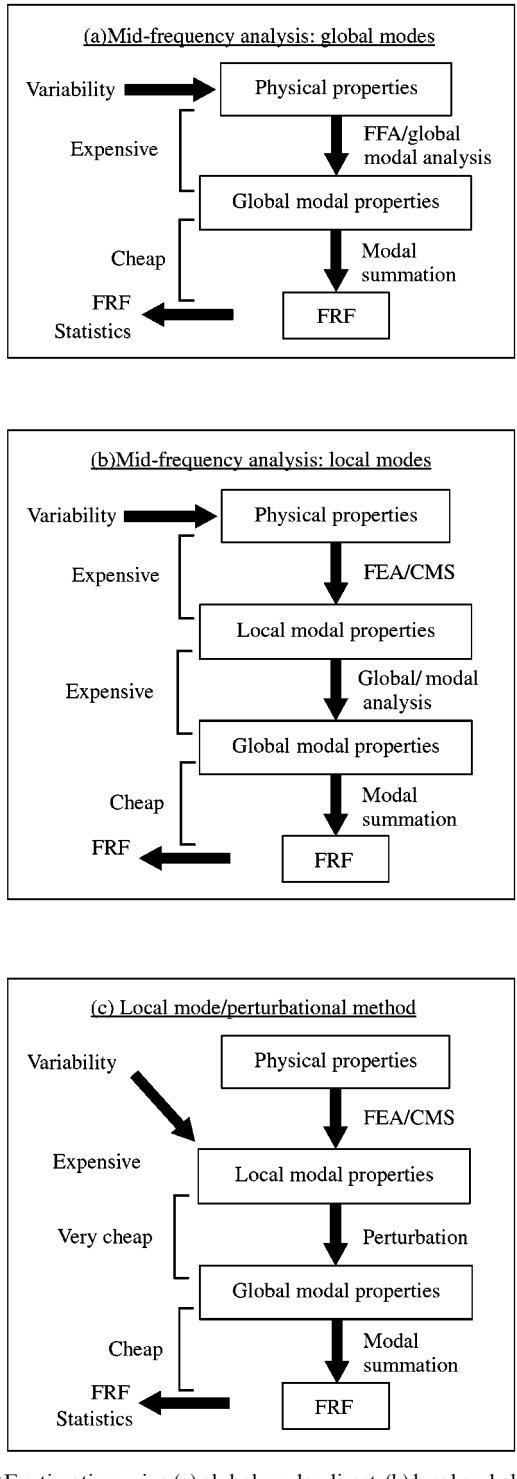

Figure 1. Mid-frequency FRF estimation using (a) global modes direct, (b) local and global modes and (c) LM/P method.

- The global modes of vibration are found in terms of the local modes using component mode synthesis.
- The response of the baseline system is found by summing contributions from each of the global modes of vibration.
- The modal properties of the subsystems are assumed to be random with known statistical distributions.
- A perturbation is found which relates (small) changes in the subsystem component modal properties to consequent changes in the global modal properties. This avoids the need to solve the global eigenvalue problem for each member of the sample considered.
- A Monte Carlo simulation is then performed: a (typically) large sample of systems is chosen, the properties of the members of this sample varying at random; the global modal properties and hence the response of each member of the sample is calculated.
- The statistics of the response are then inferred from the statistics of the responses of the chosen sample.

In the following, these steps are described in more detail. The emphasis is placed on the application of the LM/P method to discrete models, although comments are made regarding continuous models. An example application is presented in the next section.

2.1. THE SYSTEM, SUBSYSTEMS AND DEGREES OF FREEDOM

The system is divided into a number of subsystems joined together at boundaries or interfaces.

2.1.1. *Discrete models*

In a discrete model the response is described by a vector of physical degrees of freedom $\mathbf{w}$, which gives the displacements and/or rotations at the various response points. In a finite element model these are typically the locations of the nodes. Those degrees of freedom lying in the ith subsystem are denoted by $\mathbf{w}^{(i)}$. In the absence of damping, the equation of motion of the ith subsystem can then be written as

$$\mathbf{M}_w^{(i)}\ddot{\mathbf{w}}^{(i)} + \mathbf{K}_w^{(i)}\mathbf{w}^{(i)} = \mathbf{f}, \tag{1}$$

where $\mathbf{M}_w^{(i)}$ and $\mathbf{K}_w^{(i)}$ are the mass and stiffness matrices in physical co-ordinates and $\mathbf{f}$ is a vector of applied forces (a list of symbols is given in Appendix A).

2.1.2. *Continuous models*

In a continuous model the displacement $w^{(i)}(x_i)$ is a continuous function of position x_i in the ith subsystem. In the absence of damping, the equation of motion now takes the form

$$\rho^{(i)}(x_i)\ddot{w}^{(i)} + L^{(i)}(x_i)w^{(i)} = f(x_i), \tag{2}$$

where $\rho^{(i)}$ is the mass density, $L^{(i)}$ a differential (stiffness) operator and f the applied force.

2.2. SUBSYSTEM MODES: COMPONENT MODE SYNTHESIS

Each subsystem is modelled in turn using component mode synthesis (CMS) [3]. Here, and in [2], the subsystem response is decomposed into modes of the following types, although other approaches are possible. The *fixed interface component modes* are the modes

of vibration when the component interfaces are clamped. They are given by the eigensolutions to equations (1) or (2) assuming free vibration and time-harmonic behaviour and with all the interface degrees of freedom set to zero. These modes are assumed to be mass normalized. In the subsequent analysis, a discrete model is then used: the physical response is approximated by assuming that only a finite number of component modes gives significant contributions to the total response in the frequency range of interest. This will usually lead to a substantial reduction in the number of degrees of freedom of a subsystem, especially when a (possibly large) finite element model is used to calculate the component modes. The *constraint modes* are the deformed shapes obtained when each boundary degree of freedom is given a unit displacement (or rotation), with the other boundary degrees of freedom remaining fixed. These constraint modes enforce compatibility of the motions of the subsystems at their interfaces. Other functions can also be included in the subsystem description [3].

The response of the ith subsystem is then given in terms of vectors $\mathbf{q}^{(i)}$ of component modal degrees of freedom (i.e., both fixed interface component modes and constraint modes), which is partitioned into fixed interface and constraint degrees of freedom as

$$\mathbf{q}^{(i)} = \begin{Bmatrix} \mathbf{q}_f^{(i)} \\ \cdots\cdots \\ \mathbf{q}_c^{(i)} \end{Bmatrix}. \tag{3}$$

One advantage of the component mode approach is that a subsystem can be modelled using fewer degrees of freedom than would exist in a full finite element model, thus reducing computation expense. A second advantage for statistical vibration analysis is that it fits neatly within the SEA philosophy, in which the vibrational behaviour of a structure arises from the interaction of modal behaviours in the various subsystems. The third advantage arises because it gives an intuitive and computationally efficient way of including uncertainty, as discussed below.

2.2.1. *Discrete models*

The physical degrees of freedom can be related to the component modal degrees of freedom by

$$\mathbf{w}^{(i)} = \mathbf{S}^{(i)}\mathbf{q}^{(i)}, \tag{4}$$

where $\mathbf{q}^{(i)}$ is a vector of retained component modal degrees of freedom and where $\mathbf{S}^{(i)}$, a transformation matrix, is the component modal matrix. The equations of motion for the subsystem can thus be transformed from physical degrees of freedom $\mathbf{w}^{(i)}$ to component modal degrees of freedom $\mathbf{q}^{(i)}$ using the transformation matrix $\mathbf{S}^{(i)}$. The mass and stiffness matrices $\mathbf{M}_w^{(i)}$ and $\mathbf{K}_w^{(i)}$ of equation (1) written in terms of physical degrees of freedom can similarly be transformed, so that the mass and stiffness matrices for the subsystem in local modal co-ordinates become

$$\mathbf{M}^{(i)} = S^{(i)\mathrm{T}}\mathbf{M}_w^{(i)}\mathbf{S}^{(i)}, \quad \mathbf{K}^{(i)} = \mathbf{S}^{(i)\mathrm{T}}\mathbf{K}_w^{(i)}\mathbf{S}^{(i)}, \tag{5}$$

where T denotes the transpose. The matrices have special structures so that

$$\mathbf{M}^{(i)} = \begin{bmatrix} \mathbf{I} & \mathbf{m}_{fc}^{(i)} \\ \mathbf{m}_{fc}^{(i)\mathrm{T}} & \mathbf{m}_{cc}^{(i)} \end{bmatrix}, \quad \mathbf{K}^{(i)} = \begin{bmatrix} diag(\lambda_j^{(i)}) & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{k}_{cc}^{(i)} \end{bmatrix}, \tag{6}$$

where $\mathbf{I}$ is the identity matrix (since the local modes are mass normalized), $\mathbf{m}_{fc}$ is a coupling mass matrix, $\mathbf{m}_{cc}$ and $\mathbf{k}_{cc}$ are constraint mass and stiffness matrices, *diag(.)* represents a diagonal matrix and where $\lambda_j^{(i)}$ are the component modal eigenvalues. More details can be found in references [3, 5], for example.

2.2.2. *Continuous models*

For a continuous model the displacement is a continuous function of position so that

$$w(x) = \mathbf{S}(x)\mathbf{q}, \tag{7}$$

where $\mathbf{S}$ is a row vector of mode shape amplitudes. The mass and stiffness matrices are now given by

$$\mathbf{M}^{(i)} = \int \rho(x)\mathbf{S}^{\mathrm{T}}(x)\,\mathbf{S}(x)\,\mathrm{d}x, \quad \mathbf{K}^{(i)} = \int \mathbf{S}^{\mathrm{T}}(x)L(x)\mathbf{S}(x)\,\mathrm{d}x \tag{8}$$

and once again have the form of equation (6). The stiffness matrix is often found instead from potential energy considerations.

2.3. GLOBAL MODAL ANALYSIS OF THE BASELINE SYSTEM

The global modes are found by assembling the discrete local modal models and solving the resulting eigenproblem. A vector $\mathbf{q}$ of component modal degrees of freedom is defined. This contains the fixed interface and constraint modes $\mathbf{q}^{(i)}$ for all subsystems. Here the components of $\mathbf{q}$ are ordered so that $\mathbf{q}$ can be partitioned as

$$\mathbf{q} = \begin{Bmatrix} \mathbf{q}_f \\ \cdots\cdots \\ \mathbf{q}_c \end{Bmatrix}, \tag{9}$$

where $\mathbf{q}_f$ are the fixed interface modal degrees of freedom and $\mathbf{q}_c$ are the constraint modes. The equations of free vibration are of the form

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0},$$
$$\mathbf{M} = \begin{bmatrix} \mathbf{I} & \mathbf{m}_{fc} \\ \mathbf{m}_{fc}^{\mathrm{T}} & \mathbf{m}_{cc} \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} diag(\lambda_j^{(i)}) & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{k}_{cc} \end{bmatrix}, \tag{10}$$

where $\mathbf{m}_{fc}$ is a coupling mass matrix, $\mathbf{m}_{cc}$ and $\mathbf{k}_{cc}$ are constraint mass and stiffness matrices and where $\lambda_j^{(i)}$ are the component modal eigenvalues ordered in an appropriate way. These matrices are found by assembling the subsystem component mode mass and stiffness matrices given above.

The global modal properties can be found from this eigenvalue problem. This results in the global eigenvalues λ_j (i.e., the squares of the global natural frequencies) and the global mode shapes. These global mode shapes relate the component modal degrees of freedom $\mathbf{q}$ to the global modal degrees of freedom $\mathbf{y}$ by

$$\mathbf{q} = \mathbf{T}\mathbf{y}, \tag{11}$$

where the columns of **T** give the mode shapes, expressed in terms of the subsystem component modes. The jth column of **T** contains the jth global mode shape ϕ_j in terms of component modal degrees of freedom. These mode shapes are assumed to be mass normalized.

2.3.1. *Discrete models*

The transformation matrices $\mathbf{S}^{(i)}$ are assembled to find the matrix **S** such that

$$\mathbf{w} = \mathbf{S}\mathbf{q} \tag{12}$$

is a transformation relating the global physical degrees of freedom **w** to the component modal degrees of freedom **q**.

In summary, three sets of global co-ordinates are used. These are

$$\begin{gathered} \textit{physical degrees of freedom}: \quad \mathbf{w}, \\ \textit{component modal degrees of freedom}: \quad \mathbf{q}, \\ \textit{global modal degrees of freedom}: \quad \mathbf{y}. \end{gathered} \tag{13}$$

Three matrices are used to transform from one set to another, namely

$$\mathbf{w} = \mathbf{S}\mathbf{q}, \qquad \mathbf{q} = \mathbf{T}\mathbf{y}, \qquad \mathbf{w} = \mathbf{P}\mathbf{y}. \tag{14}$$

These matrices are found by solving the local eigenvalue problems (**S**) and the global eigenvalue problem in terms of local modes (**T**). The global modal matrix $\mathbf{P} = \mathbf{S}\mathbf{T}$ follows from them.

In a continuous model, similar equations can be written except that w, **S** and **P** become continuous functions of position.

2.4. RESPONSE PREDICTION: SUMMATION OF GLOBAL MODAL CONTRIBUTIONS

The physical response of the system is described in terms of the finite number of retained global modes of vibration. These global modes are found (1) for the baseline system from a global modal analysis using component modes as described above and (2) for the other systems in the sample by perturbing the modal properties of the baseline system as described in the next section.

In a discrete model the physical degrees of freedom **w** are related to the global modal degrees of freedom by

$$\mathbf{w} = \mathbf{P}\mathbf{y}, \tag{15}$$

where **P** is a transformation matrix, the global modal matrix. The jth column of **P** contains the jth global mode shape in terms of physical degrees of freedom.

It is now assumed that time-harmonic forces $\mathbf{F}\exp(\mathrm{i}\omega t)$ act on the system at a frequency ω. Here, it is assumed that the applied excitation is correlated. If, instead, uncorrelated, random forces act on the system then the mean square response is found by summing the mean square responses to each force acting alone. The resulting computations can be

reordered to reduce much of the computation. This is the situation in reference [2], where distributed, "rain-on-the-roof" excitation is applied to a whole subsystem. In practice, the excitation will often be applied to only a small region of one subsystem, so that many elements of $\mathbf{F}$ will be zero.

Suppose that there is (light) proportional damping. Under these circumstances the response $\mathbf{w}$ can be found using a conventional global modal decomposition. The modal forces become $\mathbf{P}^{\mathrm{T}}\mathbf{F}$. The modal amplitudes are

$$\mathbf{Y} = diag(\beta_j)\mathbf{P}^{\mathrm{T}}\mathbf{F}, \tag{16}$$

where

$$\beta_j = \frac{1}{(\omega_j^2(1 + \mathrm{i}\eta_j) - \omega^2)} \tag{17}$$

is the modal receptance of the jth global mode of vibration. This mode has a natural frequency ω_j and a loss factor η_j.

The physical response is thus $\mathbf{w} = \mathbf{W}\exp(\mathrm{i}\omega t)$ where

$$\mathbf{W} = \alpha\mathbf{F}, \quad \alpha = \mathbf{P}\, diag(\beta_j)\mathbf{P}^{\mathrm{T}}, \tag{18}$$

α being a matrix of receptances. Often the main interest will centre on the responses of only a few degrees of freedom. Finally, if the excitation is broadband, the frequency average mean square response of the kth degree of freedom W_k can be found by averaging $\frac{1}{2}|W_k|^2$ over frequency. Again, similar equations can be written for a continuous model.

2.5. VARIABILITY AND PERTURBATIONS: GLOBAL MODAL PROPERTIES

In general, the global modal analysis described in sections 2.1–2.3 would be computationally expensive, even if the CMS approach were adopted. Thus, it would be impractical to estimate the response statistics of the ensemble by choosing a sample of systems whose properties are random and repeating this global analysis for each member of the chosen sample. In this section, a first order perturbation is described which relates small changes in the local modal eigenvalues of each subsystem to changes in the global modal properties (i.e., the global natural frequencies and mode shapes). This enables the global modal properties of each member of the sample to be found at a trivial computational cost.

2.5.1. *Variability in subsystem properties*

The properties in each subsystem are assumed to vary randomly. In general, this would require information about the statistical properties (e.g., joint probability density functions) of the subsystem properties. Here certain simplifying assumptions are made for convenience.

First, it is assumed that the variations in the properties of one subsystem are independent of those of the other subsystems. This is reasonable since subsystems are typically manufactured by different processes and then later assembled.

In principle, each ensemble-member subsystem could be defined in terms of its primitive variables, i.e., its material and geometric properties. These would vary with some assumed statistical distributions. The variability could be specified by defining the distributions, together with the relevant mean values, variances, covariances and so on. Equally, however,

the ensemble could be defined in terms of the component modal properties, and particularly the elements of the mass and stiffness matrices in the component modal equations of motion (equation (10)). This follows because the material and geometric properties implicitly define the component modal properties. Here the approach in which variability is defined in terms of subsystem modal properties is adopted.

For simplicity, it is assumed here that the properties of the interfaces between subsystems are deterministic, and hence do not vary across the ensemble (i.e., $\mathbf{m}_{cc}$ and $\mathbf{k}_{cc}$ are deterministic) although this is not necessary for implementing the LM/P method. It is also assumed that the subsystem coupling mass matrices $\mathbf{m}_{fc}$ are deterministic. The subsystem component modal natural frequencies, however, are assumed to vary randomly. Consequently, the mass matrix $\mathbf{M}^{(i)}$ is deterministic and so, too, are all the elements of $\mathbf{K}^{(i)}$ except for those diagonal entries which correspond to the fixed interface eigenvalues $\lambda_j^{(i)}$. Each member subsystem in the ensemble is therefore defined by its fixed interface eigenvalues $\lambda_j^{(i)}$, whose statistics (distributions, covariances, etc.) are assumed known. It is also assumed that, in any single realization, the perturbation $\delta\lambda_j^{(i)}$ of $\lambda_j^{(i)}$ from the nominal value that is assumed for the baseline system (i.e., $\overline{\lambda_j^{(i)}}$) is small.

2.5.2. *Perturbational relationship between local and global modal properties*

The global eigenvalue problem in terms of component modal degrees of freedom $\mathbf{q}$ is given in equation (10). Suppose there is a variation in some parameter μ, on which $\mathbf{M}$ and $\mathbf{K}$ depend. There is therefore a consequent variation in the global eigenvalues and eigenvectors. In reference [6] it is shown that, to first order, the variation in the kth global eigenvalue is

$$\delta\lambda_k = \phi_k^{\mathrm{T}}\left[\frac{\partial \mathbf{K}}{\partial \mu} - \lambda_k \frac{\partial \mathbf{M}}{\partial \mu}\right]\phi_k \delta\mu, \tag{19}$$

where ϕ_k is the kth eigenvector, that is, the kth column of $\mathbf{T}$.

Under the assumptions of the previous subsection, the mass matrix $\mathbf{M}$ is deterministic and is hence constant from one member of the sample to another, while those diagonal elements of the stiffness matrix $\mathbf{K}$ which correspond to the component modal eigenvalues vary from one member to another. Thus, a variation $\delta\lambda_j^{(i)}$ in the jth component modal eigenvalue produces a variation

$$\delta\lambda_k = (\phi_k)_j^2 \delta\lambda_j^{(i)} \tag{20}$$

in the kth global eigenvalue, where $(\phi_k)_j = T_{jk}$ is the jth element of the kth global eigenvector. Thus, a perturbation in a component modal eigenvalue gives a large (or small) perturbation in a global eigenvalue if the corresponding component of the eigenvector is large (or small). If all component modal eigenvalues are perturbed, the total perturbation in the kth global eigenvalue is given by

$$\delta\lambda_k = \sum_j (\phi_k)_j^2 \delta\lambda_j^{(i)}. \tag{21}$$

Expressions for the perturbations in the global eigenvectors are also given in reference [6]. For a variation $\delta\lambda_j^{(i)}$ in the jth component modal eigenvalue, the first order variation in the kth global eigenvector becomes

$$\delta\phi_k = \left(\sum_{r \neq k} \frac{(\phi_k)_j (\phi_r)_j}{\lambda_k - \lambda_r}\phi_r\right)\delta\lambda_j^{(i)}. \tag{22}$$

The total first order perturbation in ϕ_k is obtained by summing the contributions from perturbations in each component mode.

2.5.3. *Monte Carlo simulation*

A Monte Carlo simulation is finally used to estimate the statistics of the responses of the ensemble by choosing at random a sample of systems. For each member in the sample the local modal eigenvalues are chosen at random and the perturbations in them thus found. First order perturbations in the global eigenvalues and eigenvectors are then estimated from equations (21) and (22). The global response is then found from these global modes.

The overall result of the local modal/perturbational approach is that the global eigenproblem only needs to be solved once for the baseline system, the global modal properties of other members of the ensemble being estimated at a trivial computational cost.

3. APPLICATION TO TWO COUPLED RODS

3.1. THE SYSTEM

In this section, the LM/P method is applied to the system comprising two, end-coupled rods shown in Figure 2. Each rod forms one subsystem and undergoes axial vibration which is excited by a time-harmonic force applied at various points in rod 1. The vibrational velocity at a given location in rod 2 is found, with results for the transfer mobility being given. Local co-ordinate axes x_i are defined in each rod ($i = 1, 2$). The density, cross-sectional area, length and elastic modulus of each rod are ρ_i, A_i, l_i and E_i respectively. The interface is where the end $x_1 = l_1$ in subsystem 1 is joined to the end $x_2 = 0$ in subsystem 2. A spring of stiffness K_c is attached to this junction.

In this particular case, the response $w(x)$, the axial displacement of the rod, is a continuous function. The equation of motion of the ith rod is

$$\rho_i A_i \frac{\partial^2 w_i}{\partial t^2} - E_i A_i \frac{\partial^2 w_i}{\partial x_i^2} = f(x_i, t), \tag{23}$$

where f is the applied force per unit length.

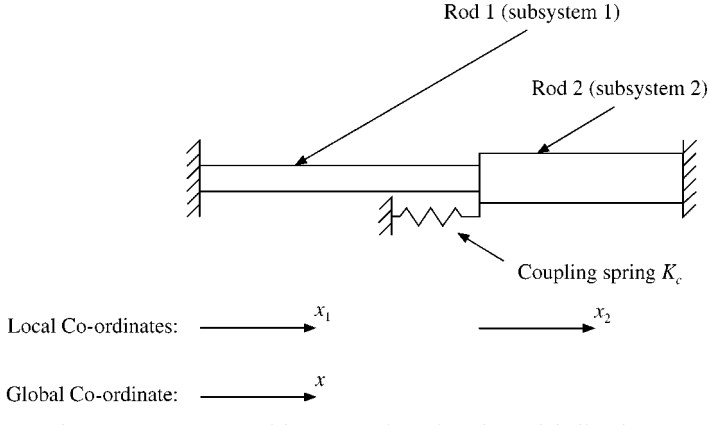

Figure 2. System comprising two rods undergoing axial vibrations.

3.2. SUBSYSTEM MODES

The jth fixed interface (mass normalized) mode shape and natural frequency are found from equation (23) with $f = 0$ and subject to the boundary equations $w_i(0) = w_i(l_i) = 0$. They are given by

$$\phi_j^{(i)}(x_i) = \sqrt{\frac{2}{\rho_i A_i l_i}} \sin \frac{\mathrm{j}\pi x_i}{l_i}, \quad \omega_j^{(i)} = \sqrt{\frac{E_i}{\rho_i}} \left(\frac{\mathrm{j}\pi}{l_i}\right), \quad i = 1, 2. \tag{24}$$

The constraint mode shapes are

$$\phi_{c,0}^{(i)}(x_i) = 1 - \frac{x_i}{l_i}, \quad \phi_{c,l}^{(i)}(x_i) = \frac{x_i}{l_i}. \tag{25}$$

These result from prescribing a unit displacement at the ends $x_i = 0$ and l_i respectively.

The submatrices of the component modal mass and stiffness matrices (equation (6)) of the ith subsystem are found by evaluating the integrals in equation (8). The mass submatrices are given by

$$\begin{aligned}
\mathbf{m}_{cc}^{(i)} &= \int_0^{l_i} \rho_i A_i \begin{Bmatrix} 1 - \dfrac{x}{l_i} \\ \dfrac{x}{l_i} \end{Bmatrix} \begin{bmatrix} 1 - \dfrac{x}{l_i} & \dfrac{x}{l_i} \end{bmatrix} \mathrm{d}x = \left(\frac{\rho_i A_i l_i}{6}\right) \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \\
\mathbf{m}_{fc}^{(i)^{\mathrm{T}}} &= \int_0^{l_i} \rho_i A_i \sqrt{\frac{2}{\rho_i A_i l_i}} \begin{Bmatrix} 1 - \dfrac{x}{l_i} \\ \dfrac{x}{l_i} \end{Bmatrix} \begin{bmatrix} \sin \dfrac{\pi x}{l_i} & \sin \dfrac{2\pi x}{l_i} & \cdots\cdots \end{bmatrix} \mathrm{d}x \\
&= \left(\frac{\sqrt{2\rho_i A_i l_i}}{\pi}\right) \begin{bmatrix} \dfrac{1}{1} & \dfrac{1}{2} & \cdots & \dfrac{1}{j} & \cdots \\ \dfrac{1}{1} & -\dfrac{1}{2} & \cdots & \dfrac{-(-1)^j}{j} & \cdots \end{bmatrix},
\end{aligned} \tag{26}$$

while the coupling stiffness matrix is

$$\mathbf{k}_{cc}^{(i)} = \int_0^{l_i} E_i A_i \begin{Bmatrix} \dfrac{\partial \phi_{c,0}^{(i)}}{\partial x} \\ \dfrac{\partial \phi_{c,1}^{(i)}}{\partial x} \end{Bmatrix} \begin{bmatrix} \dfrac{\partial \phi_{c,0}^{(i)}}{\partial x} & \dfrac{\partial \phi_{c,1}^{(i)}}{\partial x} \end{bmatrix} \mathrm{d}x = \left(\frac{E_i A_i}{l_i}\right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{27}$$

3.3. GLOBAL MODAL ANALYSIS

For this system, the vector of component modal degrees of freedom $\mathbf{q}$ (equation (9)) is defined by

$$\mathbf{q} = \begin{Bmatrix} \mathbf{q}_f^{(1)} \\ \mathbf{q}_f^{(2)} \\ \mathbf{q}_c \end{Bmatrix}, \tag{28}$$

where $\mathbf{q}_f^{(1)}$ and $\mathbf{q}_f^{(2)}$ are the amplitudes of the fixed interface modes (equation (24)) in rods 1 and 2, respectively, only a finite number of these modes being retained. Also q_c is the displacement of the interface where the two rods are connected.

The two rods each contribute to the mass and stiffness matrices of equation (10), the result being found by assembling the contributions from each. Since in this system the far ends of the rods remain fixed, only the constraint modes $\phi_{c,1}^{(1)}(x_1)$ and $\phi_{c,0}^{(2)}(x_2)$ are relevant. Furthermore, only the second column of $\mathbf{m}_{fc}^{(1)}$ (equation (26)) and the first column of $\mathbf{m}_{fc}^{(2)}$ then contribute to $\mathbf{m}_{fc}$. The stiffness K_c associated with the spring at the coupling also contributes to $\mathbf{K}$. The submatrices of $\mathbf{M}$ and $\mathbf{K}$ are then

$$\mathbf{m}_{cc} = \frac{\rho_1 A_1 l_1 + \rho_2 A_2 l_2}{3}, \quad \mathbf{k}_{cc} = \frac{E_1 A_1}{l_1} + \frac{E_2 A_2}{l_2} + K_c,$$

$$diag(\lambda_j^{(i)}) = diag\left(diag\left(\frac{E_1}{\rho_1}\left(\frac{\mathrm{j}\pi}{l}\right)^2\right), \quad diag\left(\frac{E_2}{\rho_2}\left(\frac{\mathrm{j}\pi}{l}\right)^2\right)\right),$$

$$m_{fc}^{\mathrm{T}} = \left[\frac{\sqrt{2\rho_1 A_1 l_1}}{\pi}\left(\left[\frac{1}{1} \quad -\frac{1}{2} \quad \cdots \quad \frac{-(-1)^j}{j} \quad \cdots\right]\right)\right.$$

$$\left.\frac{\sqrt{2\rho_2 A_2 l_2}}{\pi}\left(\left[\frac{1}{1} \quad \frac{1}{2} \quad \cdots \quad \frac{1}{j} \quad \cdots\right]\right)\right]. \tag{29}$$

The global modal properties are found numerically from an eigenanalysis of equation (10).

3.4. NUMERICAL EXAMPLES

3.4.1. *System properties*

The properties of the baseline system for which numerical results are presented are given in Table 1. The two rods are identical except for their lengths. The lengths are chosen so that the ratio l_1/l_2 is irrational and the global modal density is 1 (i.e., on average N global modes have natural frequencies at or below $\omega = N$). The spring stiffness is such that the impedance of the coupling equals the characteristic impedance of the rods at a frequency $\omega = 20$. The excitation and response points are such that all the local modes whose natural frequencies lie in the frequency range under consideration are excited or respond (although some very weakly).

TABLE 1

Physical and geometric properties of the baseline system

$E_1A_1 = E_2A_2$	1
$\rho_1A_1 = \rho_2A_2$	1
l_1	$\pi\sqrt{26}/8$
l_2	$\pi(1 - \sqrt{26}/8)$
η	0·01
Excitation point: x_1	$0{\cdot}245l_1$
Response point: x_2	$0{\cdot}745l_2$
K_c	40

Following the component mode synthesis, 72 and 42 fixed interface subsystem modes were retained for rods 1 and 2 respectively. Thus, there are 115 elements in the global component modal vector $\mathbf{q}$ of equation (9). The eigenvalue problem of equation (10) was solved using Matlab and 60 global modes of vibration retained for later calculation of the response using equation (18). The numbers of modes are such that negligible errors are introduced by truncating the modal sums.

In this example, the ensemble is defined by assuming that the jth eigenvalue in the ith subsystem is given by

$$\lambda_j^{(i)} = \overline{\lambda_j^{(i)}}(1 + \varepsilon^{(i)})(1 + \varepsilon_j^{(i)}), \tag{30}$$

where $\overline{\lambda_j^{(i)}}$ is the nominal value that is assumed for the baseline system and where $\varepsilon^{(i)}$ and $\varepsilon_j^{(i)}$ are small, random variables with zero mean, Gaussian distribution and with specified variances. The perturbation in this eigenvalue is then

$$\delta\lambda_j^{(i)} = \overline{\lambda_j^{(i)}}(\varepsilon^{(i)} + \varepsilon_j^{(i)} + \varepsilon^{(i)}\varepsilon_j^{(i)}). \tag{31}$$

The perturbation in $\lambda_j^{(i)}$ arises from two sources. First, there are fully correlated variations in all the fixed interface eigenvalues for subsystem i as defined by $\varepsilon^{(i)}$. Physically this might arise, for example, because of uncertainties in the elastic modulus of the material from which the subsystem is constructed, the result of which is that all the eigenvalues of that subsystem change by the same proportion. Secondly, there are uncorrelated variations in the individual eigenvalues as defined by the j random variables $\varepsilon_j^{(i)}$. These might arise from localized variations in geometric or material properties within a subsystem, for example. The data use to define the ensemble is given in Table 2. The values for the standard deviations of the parameters ε give coefficients of variation (i.e., the standard deviation divided by the mean) of about 2% for the global natural frequencies.

3.4.2. *The response of the baseline system*

The FRF (transfer mobility μ) of the baseline system is shown in Figure 3. Clear resonances can be seen. The asymptotic modal density of the system is $n(\omega) = 1$, while the bandwidth of the jth mode is $\Delta_j = \omega_i\eta \approx \mathrm{j}\eta$. Thus, the modal overlap $M = n\Delta = 0{\cdot}01\omega$ is smaller than 1 in the frequency range shown. The general level of the response tends to be lower in certain frequency bands. First, these may be where the excitation point is close to an integer multiple of half a wavelength from the fixed end at $x_1 = 0$: in this case the excitation point tends to lie close to nodal points of the modes which are resonant in these frequency ranges. These frequency ranges occur around integer multiples of $\omega = 6{\cdot}5$. Secondly, similar mode shape coherence effects occur when the response point is close to an

TABLE 2

Statistical properties of the ensemble

Parameter	Standard deviation
$\varepsilon^{(1)}$	0·02
$\varepsilon^{(1)}$	0·02
$\varepsilon_j^{(1)}$	0·02
$\varepsilon_j^{(2)}$	0·02

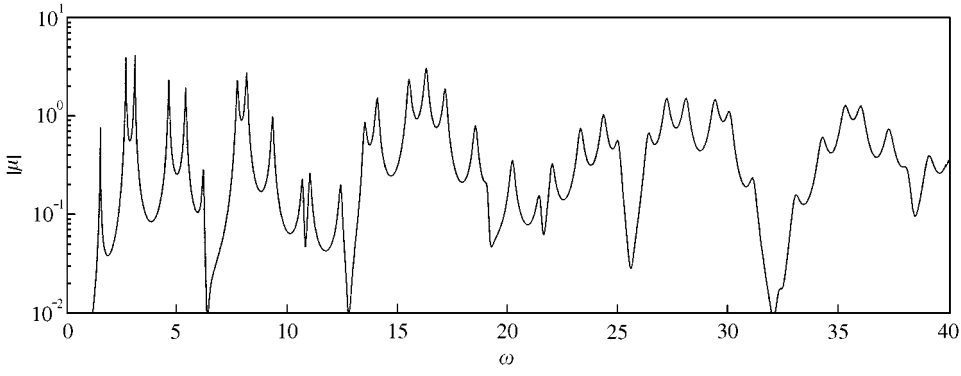

Figure 3. FRF (transfer mobility) of baseline system.

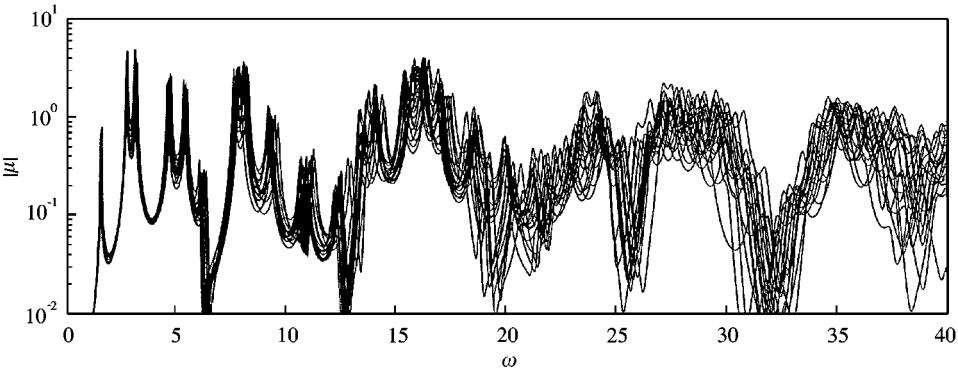

Figure 4. FRFs of 20 different realizations of the system.

integer multiple of half a wavelength from the fixed end at $x_2 = l_2$. These frequency ranges are around integral multiples of $\omega = 11$.

3.4.3. *FRF statistics*

A Monte Carlo simulation was performed using a sample of 200 systems. For each member in the sample the parameters ε in equation (31) were chosen at random and the perturbations in the local modal eigenvalues thus found. The FRFs of each member of the sample were estimated from a full global modal analysis and also using the LM/P method. The global modal analyses provide an accurate estimate of the ensemble statistics against which the approximate estimates of the LM/P method can be compared.

The discrete frequency responses of 20 members of the sample are shown in Figure 4. For the parameters used, the standard deviation of the uncertainty $\delta\omega_j$ in the jth global natural frequency ω_j is approximately $\delta\omega_j \approx 0{\cdot}02\omega_j$. This amount of variability has been deliberately chosen to demonstrate the effects of variability in system properties and to show the transition from low to high-frequency behaviour.

At low frequencies the responses of all systems are almost the same: this is the "low frequency" region where a deterministic analysis of the baseline system provides acceptably accurate estimates of the responses of all systems in the sample. At higher frequencies individual FRFs become more spread out, so that the ensemble responses as a whole do not

show distinct resonant behaviour, even though every single ensemble member does so. Nevertheless, there are still broad trends in the response at higher frequencies.

An "acoustic limit" or "uncertainty horizon" can be identified as the frequency where the "statistical overlap" equals 1; that is, where the typical uncertainty in a specific natural frequency (say, plus or minus one standard deviation) becomes equal to the average modal spacing. For this system, this occurs at around the 25th natural frequency. Here, for example, the 25th natural frequency of one member system may occur at a higher frequency than the 26th natural frequency of another member of the ensemble. Beyond this uncertainty horizon the amount of variability in the ensemble is large enough (in a dynamic sense) for there to be no real value in making deterministic predictions.

Figures 5–7 show estimates of various statistics of $|\mu|^2$obtained from full global modal analyses and using the LM/P method. The variability in FRFs increase with increasing frequency because the dynamics become increasingly sensitive to uncertainty in the system's parameters. This variability is particularly noticeable here because the system is resonant and has low modal overlap. At higher frequencies the resonant frequencies of individual systems become more spread out, so that the ensemble statistics tend not to show distinct resonant peaks, even though every single ensemble member does so. Finally, it is perhaps worth noting that different members will have the minimum (or maximum) responses in different frequency bands: one member will typically have a low response in one band and a high response in another.

Because the damping is light enough and the modal overlap small, the mean response is significantly greater than the median, the mean at any particular frequency being strongly

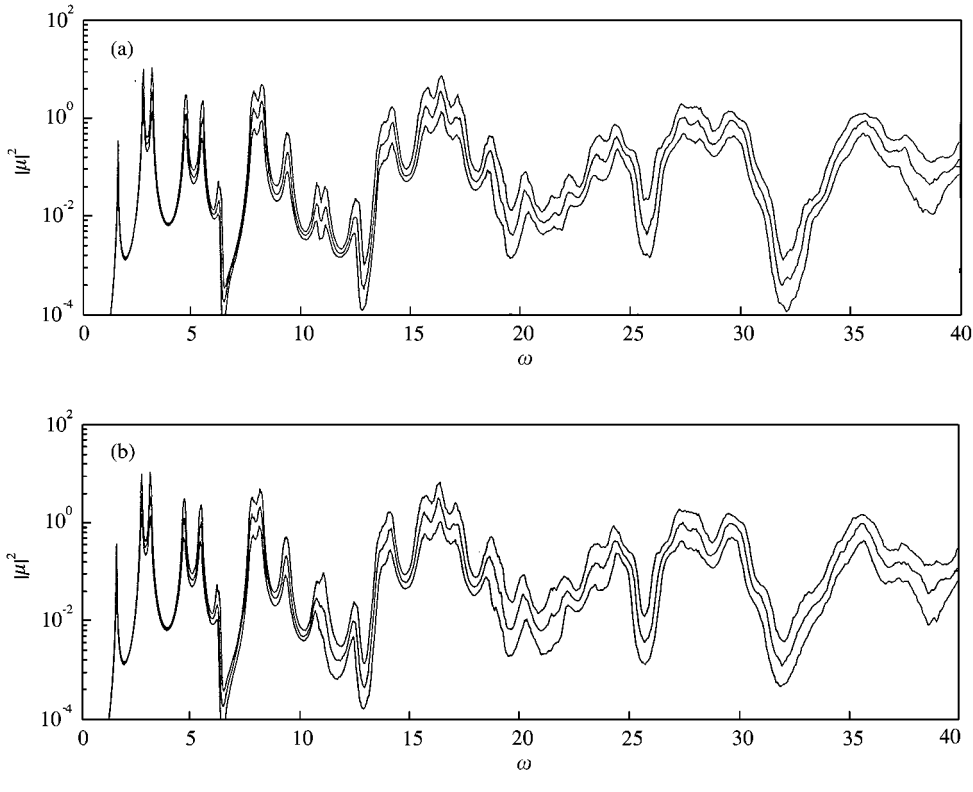

Figure 5. Ensemble FRF statistics using (a) full modal analysis and (b) LM/P method: 25th percentile, median, 75th percentile.

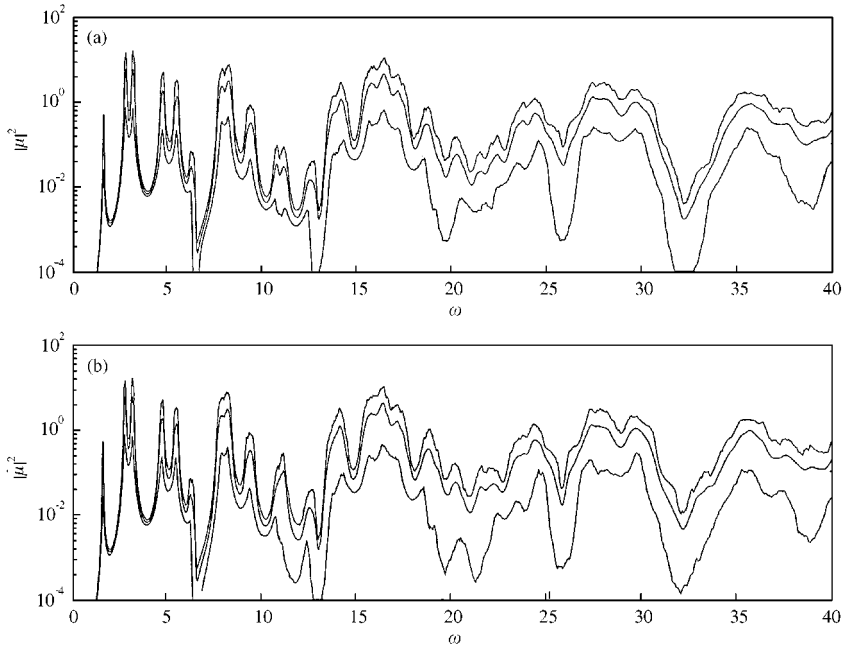

Figure 6. Ensemble FRF statistics using (a) full modal analysis and (b) LM/P method: 10th percentile, mean, 90th percentile.

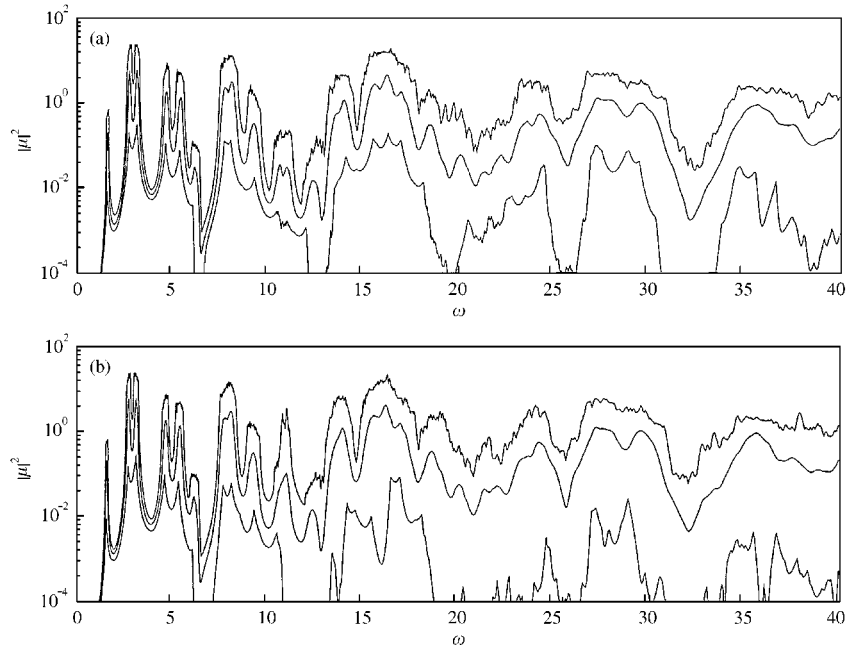

Figure 7. Ensemble FRF statistics using (a) full modal analysis and (b) LM/P method: minimum, mean and maximum.

weighted by the relatively few ensemble member systems which happen to be resonant at that frequency.

The LM/P estimates agree very closely with those based on 200 full global modal analyses but are obtained for very little cost. The mean and quartiles in particular show very close agreement. The main differences arise in estimating the "tails" of the FRF distribution, for example in the 10th and 90th percentiles and in the minimum and maximum values of the FRFs. This is primarily due to the difficulty in capturing the outlying resonant and antiresonant responses equally with a sample of only moderate size (200 members), since these responses occur rarely. This also accounts for the rather "ragged" nature of the estimates of the minimum and maximum FRFs.

4. CONCLUDING REMARKS

This paper concerned an approach to "mid-frequency" vibration analysis which used a local modal/perturbational approach. It provides a method of estimating FRF statistics in structures with uncertain parameters. The estimates are provided at very little computational cost above that required for the local and global modal analyses of the baseline system. The LM/P approach is applicable when levels of uncertainty and/or variability are such that neither low frequency, deterministic, "exact" predictions nor high frequency, statistical, "broad-brush" estimates are entirely appropriate.

The method requires data concerning subsystem variability. Here, variability is defined in terms of subsystem natural frequencies rather than primitive variables (density, thickness, etc.). The two are of course uniquely related. Defining variability in terms of subsystem natural frequencies may have advantages, however. First, the primitive variables are continuous random fields defined over the subsystem and their spatial correlation and cross-correlation would often be extremely difficult to quantify. However, only a finite number of subsystem natural frequencies are important in a given frequency band, and the measurement of their statistics may be more tractable. Secondly, the computational cost becomes very small, since the perturbational approach may be used. Finally, if the statistics of the primitive variables are known then those of the subsystem natural frequencies can easily be found using conventional stochastic finite element methods [4].

In the numerical example some quite sweeping assumptions were made regarding the statistics of the local modes. It is not known to what extent these assumptions define a realistic ensemble, since little experimental data are available regarding typical statistics. Equally, however, little data exist on the statistics of variability of the material and geometric properties themselves, especially their spatial dependencies and covariances.

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APPENDIX A: NOMENCLATURE

A	rod cross-sectional area
E	elastic modulus
$\mathbf{f}, f$	applied force, discrete and continuous models
$\mathbf{F}$	vector of amplitudes of applied forces
i, j	subsystem number
$\mathbf{k}$	stiffness sub-matrix
$\mathbf{K}$	stiffness matrix
K	stiffness of coupling spring between rods
l	rod length
L	differential operator
$\mathbf{m}$	mass sub-matrix
M	modal overlap
$\mathbf{M}$	mass matrix
n	asymptotic modal density
$\mathbf{P}$	global modal matrix ($\mathbf{w} = \mathbf{P}\mathbf{y}$)
$\mathbf{q}$	vector of component modal degrees of freedom
$\mathbf{S}$	component modal matrix ($\mathbf{w} = \mathbf{S}\mathbf{q}$)
$\mathbf{T}$	local/global modal transformation matrix ($\mathbf{q} = \mathbf{T}\mathbf{y}$)
$\mathbf{w}, w$	physical response, discrete and continuous models
x	position
$\mathbf{y}$	global modal d.o.f.s
α	receptance
β	modal receptance
Δ	modal bandwidth
ε	random perturbation
η	damping loss factor
λ	eigenvalue
$\overline{\lambda_j^{(i)}}$	mean value of jth component modal eigenvalue of the Ith subsystem
μ	perturbation parameter
ρ	density
ϕ	mode shape vector
ω	frequency
ω_j	jth natural frequency

Superscripts

i	ith subsystem

Subscripts

w	physical (response) co-ordinate system
f	fixed interface modes
c	constraint modes