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# NON-LINEAR FREE VIBRATIONS OF A ROTATING FLEXIBLE ARM

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The non-linear, moderately large amplitude flexural free vibrations of an arm clamped with a setting angle to a rigid rotating hub are studied. The shear deformation and rotary inertia effects are assumed to be negligible, but account is taken of axial inertia, non-linear curvature and the inextensibility condition. The Lagrangian approach in conjunction with the assumed modes method, assuming constant hub rotation speed, is used in a consistent manner to obtain the third order non-linear uni-modal temporal problem. Because of the strength of the non-linearities in the temporal problem, which includes elastic and inertial geometric stiffening as well as inertial softening terms, a time transformation method is employed to obtain an approximate solution to the frequency–amplitude relation of arm free oscillation. Results in non-dimensional form are presented graphically, for the effect of hub rotation speed, blade setting angle, and hub radius on the variation of the natural frequency with vibration amplitude.



## 1. INTRODUCTION

Flexural vibrations of rotating flexible beam elements are a major concern in many practical engineering applications such as turbine blades, helicopter rotors, satellite antennas and robotic arms. These vibrations have been the subject of numerous analytical and numerical investigations using a variety of mathematical models with various assumptions concerning the effects of beam geometric non-linearities; references [1–15] are examples of such studies. These studies, and others related, have shown that the axial displacement due to bending has a major effect on the stability and control of flexible rotating beam structures. It was shown that increasing the rotation speed tends significantly to stiffen the beam and thus increases its flexural natural frequencies. The effect of axial shortening due to bending in rotating flexible beams has been addressed in different ways in various available dynamic models leading to significantly different results depending on the way this effect of axial displacement is accounted for, in the formulation of the mathematical model [5]. In most cases, the available models accounted for the effect of axial shortening only partially, usually using the method of virtual work, in the formulation of either the stiffness or inertia forces. In order to obtain stable beam motions at high angular velocities, some models included the shortening effect in the formulation of both the inertia and elastic forces, but only partially, wherein some of the non-linear terms which may arise as a result of axial shortening were ignored.

In the present work, non-linear free vibrations of a rotating beam are studied by using a consistent formulated dynamic model which adopts the inextensibility condition to account for the axial shortening due to bending in the formulation of both the kinetic and potential energy. The shear deformation and rotary inertia effects are assumed to be negligible, but axial inertia and non-linear curvature are taken into account. The beam is assumed to be undergoing moderately large-amplitude planar flexural vibrations, and the hub rotation is assumed to constant. An assumed single mode Lagrangian approach is used to formulate directly an equivalent third order non-linear temporal problem which includes elastic and inertial geometric stiffening as well as inertial softening non-linear terms. For the range of amplitudes to be considered in this work, e.g., for amplitudes of tip vibration up to 0·5 of beam length for the first mode, the equivalent temporal equation of motion presents a strongly non-linear oscillator. Therefore, the time transformation method [16–18] which is suitable for such oscillators is used to obtain an approximate expression for the amplitude–frequency relation. A parametric study is carried out to show the effects of rotational speed, hub radius, hub mass and setting angle on the frequency–amplitude variation for each of the first three modes. The results are presented graphically in non-dimensional forms and their trends are compared with those available in the literature.

## 2. ASSUMPTIONS AND EQUATION OF MOTION

### 2.1. SYSTEM DESCRIPTION AND ASSUMPTIONS

The beam system under consideration is shown schematically in Figure 1, $X, Y, Z$ denotes the set of rectangular Cartesian co-ordinate axes fixed in space with origin $O$ at the center of the rotating hub. The $x$, $y$, $z$ is the system of orthogonal axes rotating with the hub with origin $o$ at the root of the beam (i.e., on the hub surface) and with the $x$-axis oriented along the neutral axis of the beam in the undeformed configuration, while the $x'$, $y'$, $z'$ are the principal rotating axes of the undeformed beam with common origin $o$ with the $xyz$ and with the $y'z'$ plane inclinded to the $yz$ plane at the angle $\psi$ called the setting angle. The hub has a radius $R_H$ and angular velocity $\dot{\theta}$ about the $Z$-axis. The beam is assumed to be homogeneous, initially straight along the $x'$-axis, cantilevered at the base, having a uniform cross-sectional area $A$, flexural stiffness $EI$ in the $x'y'$ plane, constant length $l$, and mass $\rho$ per unit volume. The beam thickness is assumed to be small compared with beam length so that the effects of rotary inertia and shear deformation can be ignored. The beam motion is assumed to be confined to the $x'y'$ plane (i.e. only planar flexural vibrations are possible),

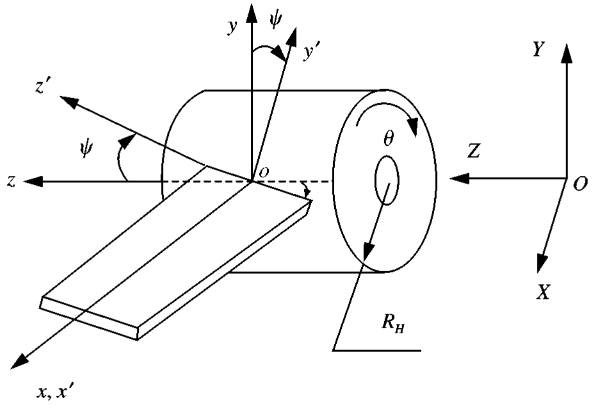


Figure 1. Disk–blade schematic diagram.

where this motion is purely in the $xy$ plane (lead–lag) when $\Psi = 0^\circ$ and it is purely in the $xz$ plane (flapping) when $\psi = 90^\circ$. Furthermore, it is assumed that the peak amplitude of this planar flexural motion may reach arbitrary, moderately large values (i.e., can be of the order of the beam length for the lower modes) but the slope of the elastica may not have a slope tangent to the neutral $x'$-axis, i.e., $\phi$ may not reach $\pm 90^\circ$; also the beam is assumed to be conservative. In the following section the governing temporal equation describing this motion is formulated using a combined Lagrangian-assumed mode method wherein the effect of axial displacement due to bending determined using the inextensibility condition [19] and its time derivative to eliminate the dependence of the beam Lagrangian on the axial displacement.

## 2.2. EQUATION OF MOTION

In the co-ordinate systems shown in Figures 1 and 2, the components of the inertial displacement vector $\mathbf{R}_P$ of the beam cross-sectional area centroid $P$ at an arbitrary point $s$ along the length of the beam after deformation are, at time $t$, given by

$$\begin{Bmatrix} R_{PX} \\ R_{PY} \\ R_{PZ} \end{Bmatrix} = R_H \begin{Bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{Bmatrix} + [A_\theta][A_\psi] \begin{Bmatrix} s+u \\ v \\ 0 \end{Bmatrix}, \tag{1}$$

where $R_H$ is the hub radius, $\theta$ is the angular position of the $x'$-axis in the inertial frame, $\psi$ is the setting angle, $[A_\psi]$ is the rotation transformation matrix from the $x'y'z'$ rotating co-ordinate system to the $xyz$ rotating co-ordinate system and is given by

$$[A_\psi] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}, \tag{2}$$

$[A_\theta]$ is the transformation matrix from the $xyz$ rotating co-ordinate system to the fixed $XYZ$ inertial co-ordinate system defined by

$$[A_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3}$$

$u = u(s, t)$ is the axial displacement (shortening, i.e. $u \leqslant 0$) of the deformed beam at the point $P$ along the rotating $x'$-axis, and $v = v(s, t)$ is the lateral displacement of the point $P$ along the rotating $y'$-axis.

By differentiating equation (1), one obtains the absolute velocity $\dot{\mathbf{R}}_P$ of the point $P$ as

$$\begin{Bmatrix} \dot{R}_{PX} \\ \dot{R}_{PY} \\ \dot{R}_{PZ} \end{Bmatrix} = \dot{\theta} R_H \begin{Bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{Bmatrix} + \dot{\theta}\left[\frac{\mathrm{d}A_\theta}{\mathrm{d}\theta}\right][A_\psi] \begin{Bmatrix} s+u \\ v \\ 0 \end{Bmatrix} + [A_\theta][A_\psi] \begin{Bmatrix} \dot{u} \\ \dot{v} \\ 0 \end{Bmatrix}. \tag{4}$$

Substituting equations (2) and (3) into equation (4) and carrying out the matrix multiplication in the resulting equation leads to the following expression for the inertial velocity vector $\dot{\mathbf{R}}_P$:

$$\dot{\mathbf{R}}_P = [\dot{u}\cos\theta - R_H\dot{\theta}\sin\theta - v\dot{\theta}\cos\theta\cos\psi - \dot{v}\sin\theta\cos\psi - (s+u)\dot{\theta}\sin\theta]\mathbf{I}$$
$$+ [R_H\dot{\theta}\cos\theta + \dot{u}\sin\theta - v\dot{\theta}\sin\theta\cos\psi + \dot{v}\cos\theta\cos\psi + (s+u)\dot{\theta}\cos\theta]\mathbf{J}$$
$$+ [\dot{v}\sin\psi]\mathbf{K}, \qquad (5)$$

where **I**, **J** and **K** are respectively unit vectors along the $X$, $Y$ and $Z$ inertial axes. The kinetic energy $KE$ of the homogenous uniform beam–hub system is given by

$$KE = \frac{\rho A}{2}\int_0^l \dot{\mathbf{R}}_P\cdot\dot{\mathbf{R}}_p\,\mathrm{d}s + \frac{1}{2}I_H\dot{\theta}^2, \qquad (6)$$

where $I_H = \frac{1}{2}m_H R_H^2$ is the mass moment of inertia of the hub about the inertial $Z$-axis, and $m_H$ is the mass of hub. Upon substituting equation (5) into equation (6), and noting that the angular speed $\dot{\theta}$ is independent of the spatial variable $s$, the above beam–hub system kinetic energy expression simplifies to

$$KE = \frac{m_b}{2\lambda^2}\left[C^2\left(1+\frac{1}{2}\mu\right)\right]\dot{\theta}^2 + \frac{m_b}{2}\int_0^1\left[\dot{v}^2+\dot{u}^2+(s+u)^2\dot{\theta}^2+2(s+u)\dot{v}\dot{\theta}\cos\psi+2\mathrm{R}_H\dot{v}\dot{\theta}\cos\psi\right.$$
$$\left. + 2R_H(s+u)\dot{\theta}^2 - \dot{u}v\dot{\theta}\cos\psi + v^2\dot{\theta}^2\cos^2\psi\right]\mathrm{d}\zeta, \qquad (7)$$

where $\lambda = 1/l$, $m_b$ is the beam total mass, $C = R_H/l$ and $\mu = m_H/m_b$ are dimensionless hub radius and hub mass ratios, and $\zeta = s/l$ is a dimensionless beam arc length variable. Note that the kinetic energy expression $KE$ in equation (7) is a function of the beam displacement and velocity variables $u, \dot{u}, v$ and $\dot{v}$ as well as of the hub angular speed $\dot{\theta}$. The axial displacement $u$ and axial velocity $\dot{u}$ can be eliminated from this equation by noting that for the present inextensible planar beam motion, the inextensibility condition dictates that [19]

$$(1+\lambda u')^2 + (\lambda v')^2 = 1, \qquad (8)$$

which may be rewritten as

$$1+\lambda u' = [1-(\lambda v')^2]^{1/2}, \qquad (9)$$

where a prime denotes a derivative with respect to the dimensionless arc length variable $\zeta$. Then, noting that $(\lambda v')^2 < 1$, expanding the right-hand side of equation (9) into a power series, retaining non-linear terms up to the desired (i.e., fourth) order, and integrating the result from 0 to an arbitrary value of $\zeta$ leads to the following expression of the axial displacement (shortening) $u$ due to the flexural bending $v$,

$$u = \frac{-1}{2}\int_0^\zeta\left(\lambda v'^2 + \frac{1}{4}\lambda^3 v'^4\right)\mathrm{d}\chi, \qquad (10)$$

which, upon differentiating with respect to time, yields

$$\dot{u} = \frac{-1}{2}\dot{\left[\int_0^\zeta\left(\lambda v'^2 + \lambda^3 v'^4\right)\mathrm{d}\chi\right]}. \qquad (11)$$

Substituting equations (10) and (11) into equation (7), integrating terms which do not include the spatial-dependent variable $v$ or its derivatives, one obtains the beam–hub kinetic energy $KE$ as

$$KE = \frac{m_b}{2\lambda^2} C_o \dot{\theta}^2 + \frac{m_b}{2} \int_0^1 \left\{ \frac{1}{4} \left( \int_0^\zeta \lambda v'^2 \, \mathrm{d}\chi \right)^2 \dot{\theta}^2 - \frac{\zeta}{\lambda} \int_0^\zeta \left( \lambda v'^2 + \frac{1}{4} \lambda^3 v'^4 \right) \mathrm{d}\chi \, \dot{\theta}^2 \right.$$

$$- R_H \int_0^\zeta \left( \lambda v'^2 + \frac{1}{4} \lambda^3 v'^4 \right) \mathrm{d}\chi \, \dot{\theta}^2 + \frac{1}{4} \left[ \left( \int_0^\zeta \lambda v'^2 \, \mathrm{d}\chi \right) \right]^2 + \dot{v}^2 + v^2 \dot{\theta}^2 \cos^2 \psi$$

$$\left. + 2 R_H \dot{v} \dot{\theta} \cos \psi + 2 \zeta \dot{v} \dot{\theta} \cos \psi - \int_0^\zeta \lambda v'^2 \, \mathrm{d}\chi \, \dot{v} \dot{\theta} \cos \psi + \left( \int_0^\zeta \lambda v'^2 \, \mathrm{d}\chi \right) v \dot{\theta} \cos \psi \right\} \mathrm{d}\zeta, \quad (12)$$

where

$$C_0 = C^2(1 + \tfrac{1}{2}\mu) + \tfrac{1}{3} + C, \quad (13)$$

with $C$ and $\mu$ the dimensionless parameters defined from equation (7).

Next, to formulate the system Lagrangian, the potential energy $PE$ of the beam is obtained. For the present beam system, the potential energy $PE$ due to the assumed inextensional planar bending motion is given by

$$PE = \frac{EI}{2\lambda} \int_0^1 K^2(\zeta, t) \, \mathrm{d}\zeta, \quad (14)$$

where $EI$ is the principal flexural stiffness about the $z'$-axis, and $K^{-1}(\zeta, t)$ is the radius of curvature at a position $\zeta$. Following reference [20], it can be seen from Figure 2 that

$$K(\zeta, t) = \lambda \phi'(\zeta, t), \quad (15)$$

where

$$\sin \phi = \lambda v', \quad (16)$$

$\phi = \phi(\zeta, t)$ is the slope of the elastica at a position $\zeta$, and a prime, as before, denotes a derivative with respect to the dimensionless arc variable $\zeta$.

Differentiating equation (16) with response to $\zeta$, and noting that $\cos \phi = \sqrt{(1 - \sin^2 \phi)}$ lead to

$$\phi' = \lambda v'' [1 - (\lambda v')^2]^{-1/2}. \quad (17)$$

Substituting equation (17) into equation (15), noting that $(\lambda v')^2 < 1$, expanding the bracketed term on the right-hand side of this equation into power series, retaining terms up to fourth order and substituting the results into equation (14), one obtains the bending potential energy $PE$ of the beam as

$$PE = (EI\lambda^3/2) \int_0^1 [v''^2 + (\lambda v' v'')^2] \, \mathrm{d}\zeta, \quad (18)$$

which is not a function of the axial displacement variable $u$ nor any of its derivatives. It is to be noted that the same potential expression $PE$ as given in equation (18) was also obtained

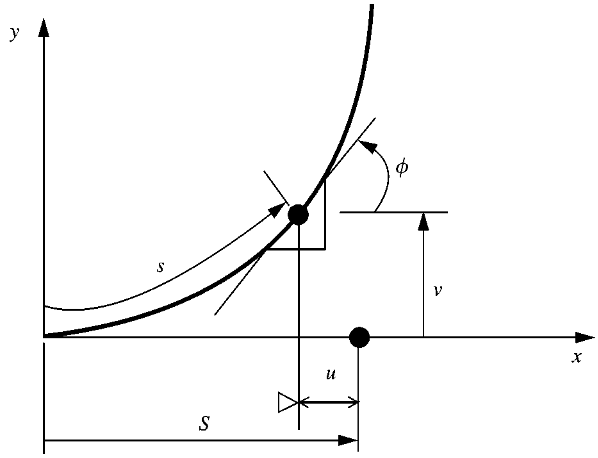


Figure 2. Deflected configuration of the blade.

by using, instead of equation (16), the kinematic relation $\tan\phi = \lambda v'/(1+\lambda u')$, defined in reference [20], along with the inextensibility condition in equation (9) to calculate the curvature $K$ in equation (15). Substituting the calculated $K$ into equation (14) and retaining non-linear terms up to fourth order was found to lead to the same potential energy expression $PE$ as given in equation (18). This indicates that, except for the restriction of constant beam length, the flexural potential energy of the planar inextensible beam is not affected by the inextensibility constraint.

The beam–hub system Lagrangian $L$ is defined as

$$L = KE - PE, \tag{19}$$

where $KE$ and $PE$ are given by equations (13) and (18) respectively. The continuous system in equation (19), like most other non-linear continuous systems, does not admit a closed form solution. However, the interest in this work is in the case where the beam motion is dominated by a single active mode (i.e., the modal subspaces are invariant [21] and are assumed to be individually active); therefore, an assumed single-mode approach may be used to discretize this continuous Lagrangian. Accordingly, one assumes

$$v(\zeta, t) = \Phi_i(\zeta)g(t), \tag{20}$$

where $\Phi_i(\zeta)$ is a normalized, self-similar (i.e., independent of motion amplitude) assumed mode shape deflection of the beam and $g(t)$ is an unknown time modulation of the assumed deflection mode $\Phi_i(\zeta)$. In the present work, the beam deflection shape $\Phi_i(\zeta)$ is assumed to be that of the associated non-rotating linear cantilever beam which can be written in the form

$$\Phi_i(\zeta) = \left(\frac{1}{r_i}\right)[\cosh p_i\zeta - \cos p_i\zeta - \gamma_i(\sinh p_i\zeta - \sin p_i\zeta)], \tag{21}$$

where $r_i = |\Phi_i(\zeta)|_{\max}$ is a scaling factor, $p_i = m_b\omega_{ni}l^3/(EI)$ ($\omega_{ni}$ is the $i$th mode natural frequency of the non-rotating linear cantilever beam) is the $i$th dimensionless frequency parameter found from the solution of the transcendental frequency equation

$$\cos p_i \cosh p_i + 1 = 0 \tag{22}$$

and $\gamma_i$ is a weighting constant associated with each mode, defined as

$$\gamma_i = \frac{\sinh p_i - \sin p_i}{\cosh p_i + \cos p_i}. \tag{23}$$

Upon substituting equations (13), (18) and (20) into equation (19), the following expression for system Lagrangian is obtained:

$$L = \frac{m_b l^2}{2}[C_o\dot{\theta}^2 + \beta_2^i q^2\dot{\theta}^2 + \beta_3^i q^4\dot{\theta}^2 + \beta_4^i \dot{q}\dot{\theta} + \beta_5^i q^2\dot{q}\dot{\theta} + \beta_1^i \dot{q}^2 + \beta_6^i \dot{q}^2 q^2$$

$$- \beta^2\beta_7^i q^2 - \beta^2\beta_8^i q^4], \tag{24}$$

where $\beta = (EI/m_b l^3)^{1/2}$ is a frequency parameter, $q = g/l$ is the dimensionless displacement of the beam at the point of maximum deflection, and $\beta_j^i$, $j = 1, \ldots, 8$, are dimensionless coefficients associated with each of the assumed $i$th mode, defined as follows:

$$\beta_1^i = \int_0^1 \Phi_i^2(\zeta)\,\mathrm{d}\zeta, \quad \beta_2^i = \int_0^1 \left[\Phi_i^2(\zeta)\cos^2\psi - (\zeta + C)\int_0^\zeta \Phi_i'^2(\chi)\,\mathrm{d}\chi\right]\mathrm{d}\zeta,$$

$$\beta_3^i = \frac{1}{4}\int_0^1\left[\left(\int_0^\zeta \Phi_i'^2(\chi)\,\mathrm{d}\chi\right)^2 - (\zeta + C)\int_0^\zeta \Phi_i'^4(\chi)\,\mathrm{d}\chi\right]\mathrm{d}\chi, \quad \beta_4^i = 2\cos\psi\int_0^1 (\zeta + C)\Phi_i(\zeta)\,\mathrm{d}\zeta,$$

$$\beta_5^i = \cos\psi\int_0^1\left[\int_0^\zeta \Phi_i'^2(\chi)\,\mathrm{d}\chi\right]\Phi_i(\zeta)\,\mathrm{d}\zeta, \quad \beta_6^i = \int_0^1\left[\int_0^\zeta \Phi_i'^2(\chi)\,\mathrm{d}\chi\right]^2\mathrm{d}\zeta,$$

$$\beta_7^i = \int_0^1 \Phi_i''^2(\zeta)\,\mathrm{d}\zeta, \quad \beta_8^i = \int_0^1 \Phi_i'^2(\zeta)\Phi_i''^2(\zeta)\,\mathrm{d}\zeta. \tag{25}$$

Upon assuming the hub rotation speed $\dot{\theta}$ to be a constant, and applying the Euler–Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0, \tag{26}$$

the following non-linear non-dimensional uni-modal equation of motion is obtained:

$$\ddot{w} + w - \varepsilon_1\Omega^2 w - \varepsilon_2\Omega^2 w^3 + \varepsilon_3(w^2\ddot{w} + w\dot{w}^2) + \varepsilon_4 w^3 = 0. \tag{27}$$

Here a dot now denotes a derivative with respect to the non-dimensional time, $t^* = (\beta^2\beta_7^i/\beta_1^i)^{1/2}t$, $w = p_i q = p_i g/l$ is a dimensionless tip displacement, $\Omega = \dot{\theta}/\omega_{ni}$ is a dimensionless hub speed ratio, $\omega_{ni}$ is, as defined before, the $i$th mode natural frequency of the associated non-rotating linear beam, and

$$\varepsilon_1 = \beta_2^i\Omega^2/\beta_1^i, \quad \varepsilon_2 = 2\beta_3^i\Omega^2/(\beta_1^i p_i^2), \quad \varepsilon_3 = \beta_6^i/(\beta_1^i p_i^2), \quad \varepsilon_4 = 2\beta_8^i/(\beta_7^i p_i^2) \tag{28}$$

are dimensionless coefficients.

Equation (27) describes the non-linear, planar, flexural free motion of the inextensible beam which is rotating at a constant speed around its hub center. In this equation, the terms $\varepsilon_3 w^2\ddot{w}$ and $\varepsilon_3 w\dot{w}^2$ are inertial non-linearities due to kinetic energy of axial motion which arise as a result of using the inextensibility condition. The first of these non-linear terms has a softening effect (i.e., leads to a decreasing frequency with increasing amplitude), while the second has a hardening effect (i.e., leads to a increasing frequency with increasing amplitude). The non-linear term $\varepsilon_4 w^3$ in equation (27) is a hardening static type due to potential energy stored in bending and arises as a result of using non-linear curvature, while the non-linear centrifugal term $\varepsilon_2 \Omega^2 w^3$ is an inertial hardening (i.e., numerical results, samples of which are shown later, indicate that $\varepsilon_2 < 0$ for all modes and selected system parameters) which arises as a result of using the inextensibility condition and always has a stabilizing effect (i.e., since $\varepsilon_2 < 0$ leads to an increasing non-linear natural frequency with increasing hub rotation speed $\Omega$). On the other hand, the linear centrifugal term $\varepsilon_1 \Omega^2 w$, which when using linear beam theory always has a destabilizing effect for any mode, is found (as will be shown later), as a result of using the inextensibility, to be always a stabilizing factor (i.e., $\varepsilon_1 < 0$) for the second and higher modes, and above some critical value of beam parameters it also becomes a stabilizing factor for the first mode. An approximate analytic solution of the non-linear equation (27) is presented in the next section.

## 3. METHOD OF SOLUTION

The calculations of the coefficients and examination of the various terms in equation (27) indicate that the non-linear oscillator described by this equation is in general strongly non-linear. Samples of the results of these calculations are shown in Table 1. These sample results and others not shown indicate that for the range of motion amplitudes to be considered in the present work (i.e., for values of vibration amplitude $g = lq = lw/p_i$ up to $0{\cdot}5l$ for the first mode), the non-linear terms in equation (27) are not small compared to the linear ones. Therefore, a first-order approximation to the frequency–amplitude relation of this oscillator, which includes static and inertia non-linearities, obtained by using perturbation methods or a single-mode harmonic balance method is not expected to yield fairly accurate results when the vibration amplitude is not relatively small compared to unity. Instead, the time transformation method, described in detail in references [16–18], which is not restricted to weakly non-linear oscillators is used to obtain an approximation to the frequency–amplitude relation of the strongly non-linear conservative oscillator in equation (27). According to this method, a single-valued transformation $T(t^*)$ (recall that the derivatives in equation (27) are with respect to the dimensionless time $t^*$), is sought between the time $t^*$ and a new time $T$, such that in the new time domain $T$ the solution of equation (27) is simple harmonic with period equal to $2\pi$: i.e., one assumes $w(T) = A\cos T$, where $A$ is the amplitude of motion and $T(0) = 0$. Transforming equation (27) to the new time domain $T$, defining $F = \mathrm{d}T/\mathrm{d}t^*$ and substituting for $w(T) = A\cos T$ in the result,

Table 1

*Samples of calculated values of parameters in temporal equation* (27)

| Mode | $C$ | $\psi$ (deg) | $\Omega$ | $\Omega^2\varepsilon_1$ | $\Omega^2\varepsilon_2$ | $\varepsilon_3$ | $\varepsilon_4$ |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 30 | 1 | – 0·131133 | – 3·202327 | 0·029306 | 0·930404 |
| 1 | 1 | 0 | 1 | 0·521665 | – 1·477399 | 0·029306 | 0·930404 |
| 2 | 1 | 15 | 0·7 | – 12·24061 | – 6·274303 | 2·410494 | 154231 |
| 3 | 0·4 | 0 | 0·5 | – 33·25740 | – 3·732080 | 12·46556 | 1·125653 |

leads to

$$(1-\varepsilon_1 \Omega^2 - F^2)\cos T - \varepsilon_3 A^2 FF' - \varepsilon_2 \Omega^2 A^2 \cos^3 T + \varepsilon_3 A^2 \cos^2 T(-FF'\sin T - F^2 \cos T) + \varepsilon_3 A^2 F^2 \cos T \sin^2 T + \varepsilon_4 A^2 \cos^3 T = 0, \tag{29}$$

where a prime denotes a derivative with respect to time $T$. Next, one solves equation (29) for $F^2$ by noting that, for the present oscillator which does not involve even non-linearities, a series solution of period $2\pi$ may be assumed in the form [16]

$$F^2 = \sum_{n=0,2}^{\infty} G_n \cos nT. \tag{30}$$

Substituting equation (30) into equation (29), using appropriate trigonometric identities to simplify some of the trigonometric terms, ignoring harmonics greater than the third, and equating to zero the coefficient of each harmonic in the resulting equation, one obtains the following set of two independent, simultaneous linear algebraic equations for the coefficients $G_0$ and $G_2$:

$$a_{11}G_0 + a_{12}G_2 = b_1, \quad a_{21}G_0 + a_{22}G_2 = b_2. \tag{31, 32}$$

Here

$$a_{11} = 1 + \tfrac{1}{2}\varepsilon_3 A^2, \quad a_{12} = \tfrac{1}{4}\varepsilon_3 A^2, \quad a_{21} = \tfrac{1}{2}\varepsilon_3 A^2, \quad a_{22} = 1 + \tfrac{3}{8}\varepsilon_3 A^2,$$
$$b_1 = 1 - \varepsilon_1 \Omega^2 - \tfrac{3}{4}\varepsilon_2 \Omega^2 A^2 + \tfrac{3}{4}\varepsilon_4 A^2, \quad b_2 = \tfrac{1}{4}A^2(\varepsilon_2 \Omega^2 - \varepsilon_4). \tag{33}$$

Solving equations (31) and (32) for $G_0$ and $G_2$, substituting the result into equation (30), using the relation $F = \mathrm{d}T/\mathrm{d}t^*$, integrating the resulting equation from 0 to $2\pi$ in $T$ and noting that the period in the time $T$ domain is $2\pi$ leads to the non-linear frequency–amplitude relation in the dimensionless $t^*$:

$$\omega = G_0^{1/2}\left[1 + \frac{3}{16}H^2 + \frac{105}{1024}H^4 + \cdots\right]^{-1}. \tag{34}$$

Here

$$H = \frac{G_0}{G_2}, \quad G_0 = \frac{b_1 a_{22} - b_2 a_{12}}{\Delta}, \quad G_0 = \Delta = a_{11}a_{12} - a_{12}a_{21}, \quad G_2 = \frac{b_2 a_{11} - b_1 a_{21}}{\Delta}. \tag{35}$$

Note that $t^* = (\beta^2 \beta_7^i/\beta_1^i)^{1/2}\, t = \omega_{ni} t$, where $\omega_{ni}$ is the $i$th mode natural frequency (in time $t$) of the associated non-rotating linear beam; thus the dimensionless non-linear frequency parameter $\omega$ calculated from equation (34) is the ratio of the frequency in time $t$ of the non-linear rotating beam to the natural frequency $\omega_{ni}$ of the associated non-rotating linear beam. Also note that the leading term in equation (34) is the result one obtains by using the single-mode harmonic, i.e., this term represents the effect of the fundamental harmonic of the system non-linearities, while the bracketed term in this equation represents a measure of the relative importance of the higher harmonics introduced by the system non-linearities. Furthermore, in arriving at equation (34), a truncated power series expansion in terms of the parameter $H$, assuming $|H| < 1$, was used. Thus, equation (33) is expected to yield reasonably accurate results for the range of amplitude $A$ for which $|H| < 1$ which is found to be the satisfied in the amplitude range considered in the present work. Examples of the results obtained by using equation (34) for selected values of the rotating beam system parameters are presented and discussed in the next section.

## 4. RESULTS AND DISCUSSION

By using equations (25), (28), (33) and (35), the non-linear free vibration frequency parameter $\omega$ of each of the first three modes of free vibration of the rotating beam shown in Figure 1 was calculated, for selected values of the hub radius ratio $C$, speed ratio $\Omega$, setting angle $\psi$ and range of motion amplitude $A$, by using equation (34). All of the calculations were programmed on a digital computer and a symbolic manipulator and the Gauss-quadrature 16-point integration scheme were used to evaluate the system coefficients defined in equation (28). Examples of the results of these calculations are shown in Figures 3–6 in which the variation of the non-linear frequency parameter $\omega$ ($\omega$ is the ratio of

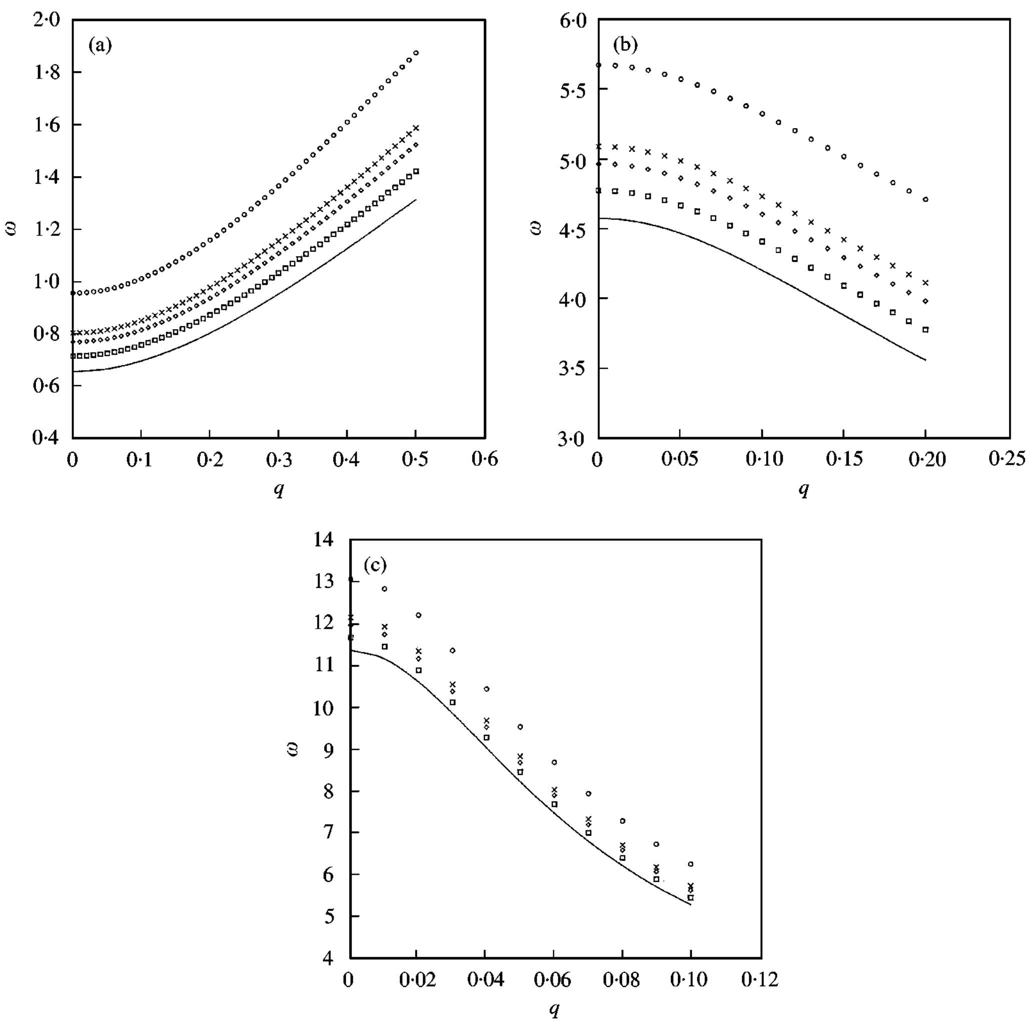


Figure 3. (a) Effect of hub radius $C$ on the variation of the frequency parameter $\omega$ with amplitude $q$ for the first mode; $\psi = 10°$ and $\Omega = 1$; ——, $C = 0{\cdot}2$; □, $C = 0{\cdot}5$; ◇, $C = 0{\cdot}8$; $x$, $C = 1{\cdot}0$; ○, $C = 2{\cdot}0$. (b) As (a) but for the second mode. (c) As (a) but for the third mode.

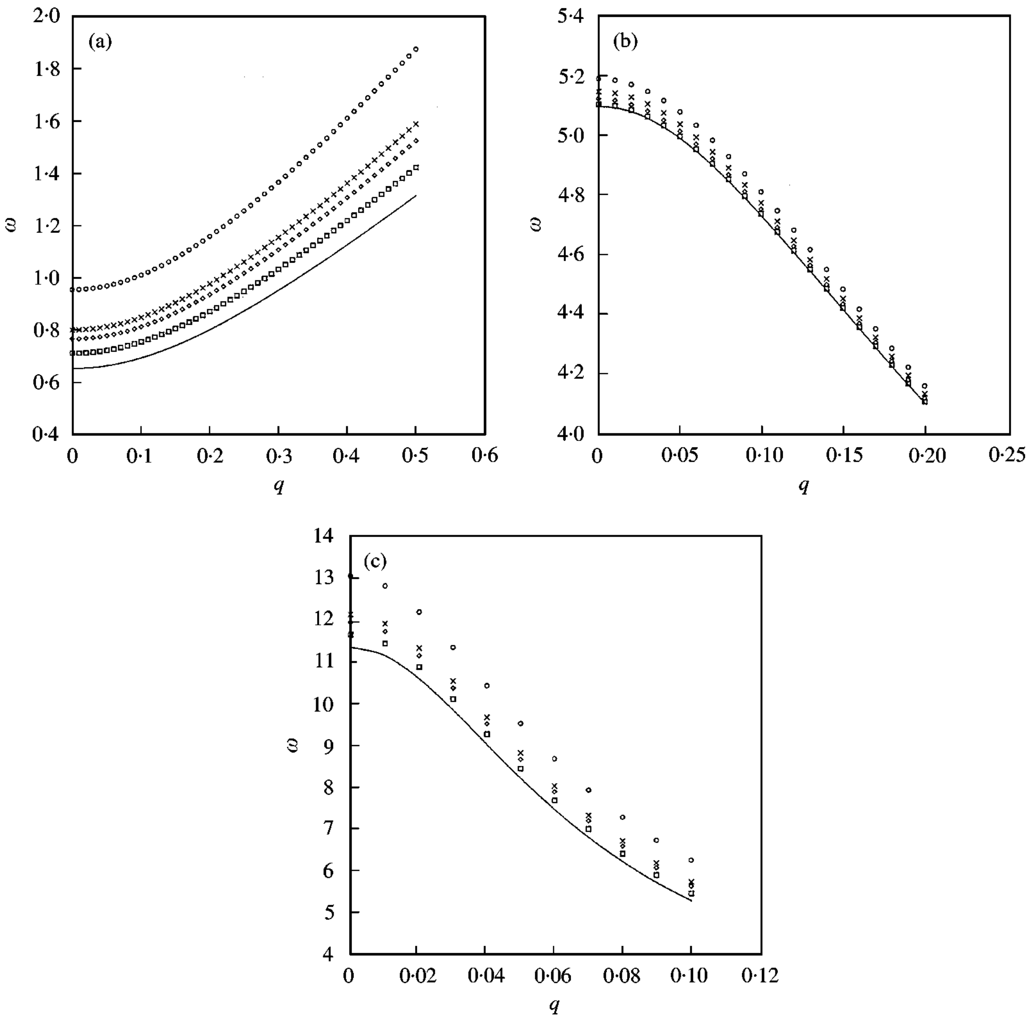


Figure 4. (a) Effect of the setting angle $\psi$ on the variation of the frequency parameter $\omega$ with amplitude $q$ for the first mode; $C = 1$; $\Omega = 1$; ——, $\psi = 0°$; □, $\psi = 15°$; ◇, $\psi = 30°$; $x$, $\psi = 45°$; ○, $\psi = 75°$. (b) As (a) but for the second mode. (c) As (a) but the third mode.

the $i$th mode non-linear natural frequency of the rotating beam to the $i$th mode natural frequency $\omega_{ni}$ of the associated linear non-rotating beam) is displayed, with the dimensionless beam displacement amplitude $q = g/l = A/p_i$ at the point of maximum beam deflection for different modes and various selected values of $C$, $\psi$ and $\Omega$.

The results in these figures, and others not shown, indicate that: (1) the $\omega$–$q$ curves of the non-linear inextensible rotating beam are, for given $C$, $\psi$ and $\Omega$, of the hardening type (i.e., the frequency $\omega$ increases with increasing motion amplitude $q$) for the first mode and are of the softening type for the second and higher modes (i.e., for given $C$, $\Psi$ and $\Omega$, $\omega$ decreases with increasing $q$); (2) at a given amplitude $q$, and for all modes, the effect of increasing any of the parameters $C$, $\psi$ or $\Omega$ leads to increasing $\omega$, where this effect becomes significantly larger as the mode number is increased; (3) at a given amplitude $q$, and for all modes, the change in $\omega$ is more significant when $\Omega$ is changed and less significant when $\psi$ is changed. Furthermore, note that the first-mode results shown in Figure 5(a) are for a case for which

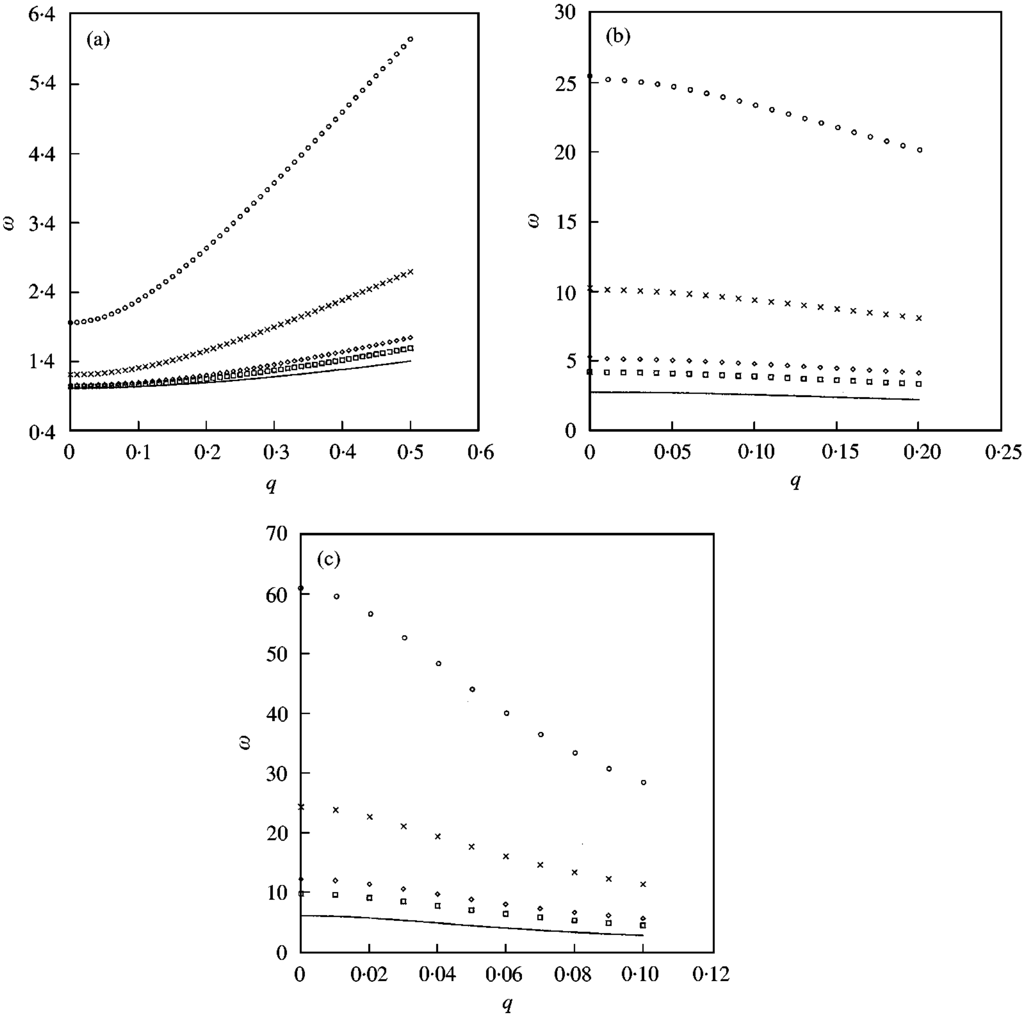


Figure 5. (a) Effect of the speed ratio $\Omega$ on the variation of the frequency parameter $\omega$ with amplitude $q$ for the first mode; $\psi = 45°$; and $C = 1$; ——, $\Omega = 0{\cdot}5$; □, $\Omega = 0{\cdot}8$; ◇, $\Omega = 1$; $x$, $\Omega = 2$; ○, $\Omega = 5$. (b) As (a) but for the second mode. (c) As (a) but for the third mode.

$\varepsilon_1 = -\,0{\cdot}11285 < 0$, and those in Figure 6(a) are for a case for which $\varepsilon_1 = 0{\cdot}38746 > 0$. From the results in these two figures, and other similar ones not shown, one can see that when $\varepsilon_1$ corresponding to the first mode is positive the non-linear free vibration at this mode becomes unstable (i.e., no real solution for $\omega$ is found) at low vibration amplitude when the hub rotation parameter $\Omega$ is increased above a certain critical value, where the width of this instability region increases as $\Omega$ is increased further above the critical value. Furthermore, these results show that when $\varepsilon_1 > 0$, increasing the hub rotation speed parameter $\Omega$ tends, with vibration amplitude $q$ kept constant, to decrease the first-mode frequency parameter $\omega$ (i.e., tends to destabilize the beam free vibration) in the low vibration amplitude region, while in the high vibration amplitude region, increasing $\Omega$ tends to increase $\omega$, i.e., tends to stiffen the beam. On the other hand, as can be seen from Figure 5(a), the instability region of the first-mode free vibration at low vibration amplitude, indicated above, is absent when $\varepsilon_1$ corresponding to this mode is negative. Note that, as was indicated

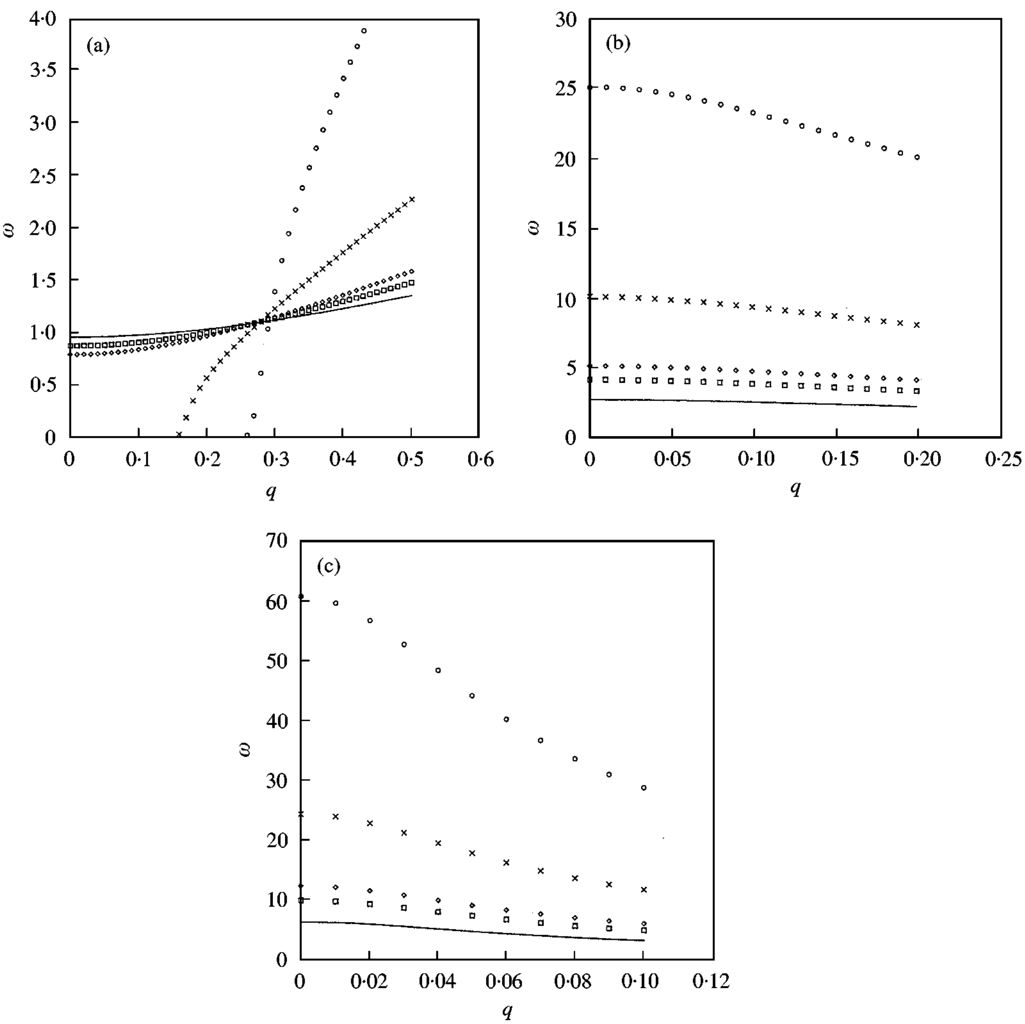


Figure 6. (a), (b), (c) as Figure 5(a), (b), (c) but for $\Psi = 0°$.

at the end of section 2, $\varepsilon_1$ is always negative for the second and higher modes; thus this behavior of the first-mode free vibration is not shared with those of the second and higher modes. Also, using the definitions given in equations (21), (25) and (28) and noting that $\beta_1^1$ is always positive, one finds that the condition $\varepsilon_1 < 0$ for the first mode leads to $\cos^2 \Psi - 0{\cdot}34413 - 0{\cdot}26841C < 0$ which is satisfied for all values of $C$, $(C > 0)$ when $\Psi > 54°$. Thus, according to this result, the above indicated destabilization effect of the high rotation speed $\Omega$, which may occur for the first mode only in the low vibration amplitude region, is totally removed provided that the setting angle $\Psi > 54°$.

## 5. CONCLUSIONS

The dynamic characteristics of a rotating inextensible blade attached with a setting angle to a rigid hub, rotating at constant speed, are studied analytically using a consistent formulated model which takes into account the beam axial inertia and non-linear curvature.

The model utilized the multibody dynamic approach though using two different co-ordinate transformations. One transformation is from the blade principal co-ordinate system to the disk body co-ordinate system using the blade setting angle as the transformation angle. The second transformation is from the disk co-ordinate system to the inertial reference frame using the rigid body rotation degree-of-freedom as the transformation angle. This approach resulted in a geometrically valid model for any possible setting angle. The effect of axial displacement (shortening effect) due to bending is accounted for in the description of the deflected material point position vector. By using the inextensibility condition, the effect of axial displacement and its associated axial inertia due to bending were incorporated into the system Lagrangian consistently, which leads to non-linear effects that in general are not accounted for in other available dynamic models. The results of numerical simulation demonstrated that the developed model solution remains stable (i.e., does not break down) at high values of angular velocity of the rotating beam. These results indicated that the rotating beam non-linear free vibration frequency exhibits a hardening behavior for the first mode and softening behavior for the second and higher modes. They also indicated that increasing the rotation speed, hub radius or setting angle leads to a significant increase in the natural frequency corresponding to any of the modes. It has also been found that, for some values of system parameters, the rotating beam free vibration exhibits interesting behavior, especially at the first vibration mode which requires further investigation.

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