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LETTERS TO THE EDITOR

AN ADDITIONAL CONTRIBUTION ON THE VIBRATIONS OF TWO ELASTIC BODIES IN ROLLING CONTACT

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1. INTRODUCTION

The problem of the vibrations of a rigid roller in rolling contact with a flexible roller has been studied by Nayak in reference [1]. One of the cylinders has a smooth surface, while the other cylinder has a wavy surface. Assuming that the rolling contact is Hertzian, the equation of contact vibration can be written as [1]

$$\ddot{x} + 2\zeta\dot{x} + (\tfrac{2}{3})[H(x)x^{3/2} - 1] = -\xi_0\Omega[\Omega\cos\Omega t + 2\zeta\sin\Omega t], \tag{1}$$

where $H(x)$ is the Heaviside unit step function, ζ is the damping factor and ξ_0 represents the amplitude of the surface waviness which is assumed harmonic with frequency Ω.

In a very interesting paper [2], Narayanan and Sekar introduced a frequency-domain-based numeric–analytical method for non-linear dynamical systems combined with a stability analysis and a path-following algorithm. The efficiency of the continuation technique in conjunction with the stability study based on the Floquet theory was illustrated with the example of the vibrations of the two rollers described by equation (1). Period-1, -2, -4 and -3 orbits were obtained for $0 \leqslant \Omega \leqslant 3$ as mentioned in references [2–4].

In addition to the orbits reported in references [2–4], the author found [5, 6] period-6 motion and a corresponding period doubling cascade near to $\Omega = 1{\cdot}87$ with $\zeta = 0{\cdot}05$ and $\xi_0 = 0{\cdot}5$. The main characteristics of the period-6 orbit and its bifurcations such as the phase plane plot, the Poincaré section points, the response amplitude diagram and the basins of attraction of coexisting periodic solutions have been reported in reference [5, 6].

Recently, the author discovered period-8 motion and remarkably another period doubling cascade $8\mathrm{P} \to 16\mathrm{P} \to 32\mathrm{P} \cdots$ in the vicinity of $\Omega = 1{\cdot}85$ with the same values for ζ and ξ_0. The aim of this letter is to report the features of this additional cascade with special emphasis on the metamorphoses of the domains of attraction.

2. CHARACTERISTICS OF THE PERIOD-8 MOTION

It has been pointed out that the period-8 orbit is created at a value of the frequency very near to $\Omega = 1{\cdot}865$ with $\zeta = 0{\cdot}05$ and $\xi_0 = 0{\cdot}5$. At this value of Ω, stable period-3, 4, 6 and 8 orbits coexist. By direct numerical integration of equation (1) starting from the initial point $x = -2$, $\dot{x} = 4$ in the phase plane, the period-8 orbit has been discovered. Figure 1 shows the limit cycle for the period-8 motion at $\Omega = 1{\cdot}865$. The Poincaré section points at

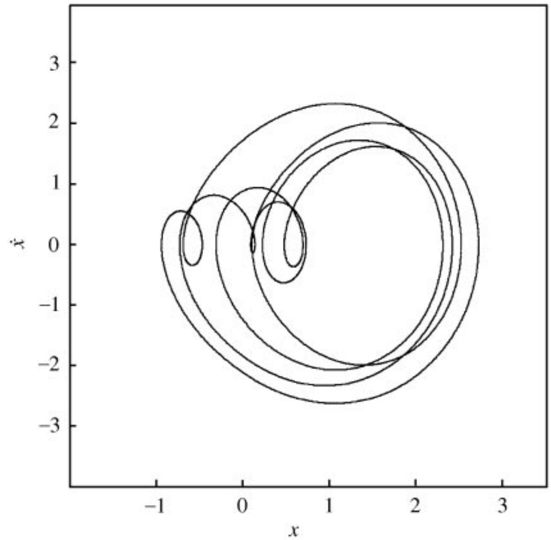

Figure 1. The period-8 orbit in the phase plane with $\zeta = 0{\cdot}05$, $\xi_0 = 0{\cdot}5$ and $\Omega = 1{\cdot}865$.

$t = 0$ of the coexisting periodic orbits have the following co-ordinates in the phase plane:

period-8: (2·5124, 0·2455), (0·7251, 0·0674), (2·2864, − 0·3968), (0·6431, 0·3676), (2·3325, − 0·7693), (0·0681, 0·3708), (2·6461, − 0·7809), (− 0·4839, 0·1219);
period-3: (− 1·2030, 1·4990), (3·0940, − 1·8206), (− 2·9615, − 0·5711);
period-4: (2·6162, − 0·5070), (− 0·0594, 0·2061), (2·6311, − 0·3801), (0·0650, 0·2126);
period-6: (0·1894, 0·3330), (2·5882, − 0·7561), (− 0·3454, 0·1748), (2·5967, − 0·0100), (0·4742, 0·1811), (2·4368, − 0·5648).

Let us investigate the continuation of the period-8 orbit. Therefore, equation (1) is rewritten as

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -2\zeta x_2 - (\tfrac{2}{3})[H(x_1)x_1^{3/2} - 1] - \xi_0 \Omega[\Omega \cos \Omega t + 2\zeta \sin \Omega t]. \qquad (2)$$

In the continuation technique described in references [7–9], which is based on the shooting method, we consider in addition to system (2), the system of the first variational equations expressed with respect to the periodic solution under consideration with the relevant period denoted by T. This yields the following equations:

$$\dot{x}_3 = x_4,$$
$$\dot{x}_4 = -x_1^{1/2}x_3 - 2\zeta x_4 \ \text{ if } x_1 > 0 \quad \text{or} \quad \dot{x}_4 = -2\zeta x_4 \ \text{ if } x_1 \leqslant 0,$$
$$\dot{x}_5 = x_6,$$
$$\dot{x}_6 = -x_1^{1/2}x_5 - 2\zeta x_6 \ \text{ if } x_1 > 0 \quad \text{or} \quad \dot{x}_6 = -2\zeta x_6 \ \text{ if } x_1 \leqslant 0. \qquad (3)$$

The numerical integration of the first variational equation (3) is performed simultaneously with that of system (2). Thus, one has to integrate a sixth order system with the initial conditions at $t = 0$:

$$x_1 = x_{10}, \quad x_2 = x_{20}, \quad x_3 = 1, \quad x_4 = 0, \quad x_5 = 0, \quad x_6 = 1. \qquad (4)$$

The correction vector $\Delta\mathbf{x}$ with components Δx_1 and Δx_2, i.e. the corrections of x_1 and x_2, has to satisfy the system of the linear equations:

$$[\mathbf{I} - \mathbf{A}(T)]\,\Delta\mathbf{x} = \mathbf{e}, \tag{5}$$

where $\mathbf{I}$ is the identity matrix. The matrix $\mathbf{A}(T)$ is the fundamental matrix of the system of the first variational equations evaluated at time $t = T$, $\mathbf{e}$ in equation (5) is the error vector in the phase plane x_1x_2 at $t = T$ of the numerical integration of system (2). Equation (5) defines an iterative algorithm in which we have to solve the linear system for the corrections. This is repeated until the iterative method converges numerically. The stability of the periodic solutions is investigated by computing the eigenvalues of $\mathbf{A}(T)$. Stable periodic solutions correspond to eigenvalues lying inside the unit circle. The solutions become unstable when one of the eigenvalues of $\mathbf{A}(T)$ leaves the unit circle along the real axis at the value -1. Repeated use of this procedure where one alternatively doubles the period, allows us to compute each transition in the period doubling cascade: $8\mathrm{P} \to 16\mathrm{P} \to 32\mathrm{P} \cdots$.

Following the continuation procedure by taking small changes of the frequency parameter Ω, the period-8 orbit created near $\Omega = 1{\cdot}865$ becomes unstable at $\Omega = 1{\cdot}85058$. At this transition value, a bifurcation to a stable period-16 orbit takes place. The continuation procedure allows us to determine three further transitions ($16\mathrm{P} \to 32\mathrm{P}$, $32\mathrm{P} \to 64\mathrm{P}$, $64\mathrm{P} \to 128\mathrm{P}$) which occur at $\Omega = 1{\cdot}84359$, $1{\cdot}84197$ and $1{\cdot}84162$ respectively.

Recall from reference [5, 6] that the period-doubling cascade starting with period-6 motion first appears at $\Omega = 1{\cdot}8933$ and that the subsequent transitions $6\mathrm{P} \to 12\mathrm{P}$, $12\mathrm{P} \to 24\mathrm{P}$, $24\mathrm{P} \to 48\mathrm{P}$, $48\mathrm{P} \to 96\mathrm{P}$ take place at $\Omega = 1{\cdot}86156$, $1{\cdot}85078$, $1{\cdot}84845$ and $1{\cdot}84795$ respectively.

At the limit of both sequences of the transition values starting with period-8 and -6 motion, the behavior of the system becomes chaotic. Chaos sets in at $\Omega = 1{\cdot}8415$ for the period-8 cascade and at $\Omega = 1{\cdot}8478$ for the period-6 cascade. The predictions of these limit values from Universality theory and Feigenbaum's relation [10] have been confirmed by numerical experiments.

3. METAMORPHOSES OF THE BASINS OF ATTRACTION

The occurrence of the period-doubling cascades starting with period-8 and 6 motion, in addition to period-1 to -4 motions reported in references [2, 3], suggests a very rich pattern of metamorphoses for the domains of attraction in the problem of the rolling contact of two elastic bodies. The basins of attraction are constructed by considering a grid of initial conditions in the phase plane. By integrating system (2) for each set of initial conditions, the periodic and chaotic attractors to which the orbit converges, are detected. Depending on the different attractor that is reached, each initial condition is assigned a distinct color. The domains of attraction for the cases with $\zeta = 0{\cdot}05$ and $\xi_0 = 0{\cdot}5$ have been constructed taking a 400×400 grid of pixels in the region of the phase plane defined by $-4 \leqslant x \leqslant 4$ and $-4 \leqslant \dot{x} \leqslant 4$.

A detailed investigation based on extensive numerical experiments reveals that the most interesting metamorphoses occur when the forcing frequency varies between $\Omega = 1{\cdot}83$ and $1{\cdot}895$. This is precisely the range where the period-8 and -6 cascades appear. Table 1 for this range of Ω illustrates the highly interesting pattern of coexisting periodic and chaotic attractors. Very complex pattern is found for $1{\cdot}845 \leqslant \Omega \leqslant 1{\cdot}865$ where four different attractors coexist. In addition to the period-3 and -4 attractors, the two attractors resulting from the period-8 and -6 cascades occur. In the last column of Table 1 the percentage of

TABLE 1

The coexisting periodic (with period nP) and chaotic attractors, and their corresponding percentage of pixels with 400×400 *resolution for* $1{\cdot}83 \leqslant \Omega \leqslant 1{\cdot}895$

Ω	Coexisting attractors				Pixels percentage			
1·830	3P	4P			15	85		
1·835	3P	4P			15	85		
1·840	3P	4P		Chaos	15	64		21
1·845	3P	4P	Chaos	16P	16	36	31	17
1·850	3P	4P	24P	16P	16	37	30	17
1·855	3P	4P	12P	8P	16	39	33	12
1·860	3P	4P	12P	8P	16	40	30	14
1·865	3P	4P	6P	8P	16	37	31	16
1·870	3P	2P	6P		17	45	38	
1·875	3P	2P	6P		17	43	40	
1·880	3P	2P	6P		17	44	39	
1·885	3P	2P	6P		17	44	39	
1·890	3P	2P	6P		17	42	41	
1·895	3P	2P			17	83		

pixels for the basins of each attractor is mentioned. These results for the basins of attraction have been obtained by the use of Nusse and Yorke's package DYNAMICS [11]. Note the nearly constant percentage of pixels (16%) for the period-3 basin. The period-8 (or bifurcated) basin has a percentage between 12 and 21 while the period-6 (or bifurcated) basin is more extended having a percentage between 30 and 41 for Ω varying in the mentioned range. The percentage of pixels for the period-2 (or period-4) basin is slightly higher than that for the period-6 basin in case of coexistence.

Figure 2 shows the metamorphoses of the basins of attraction for several variations of Ω. The basins of the attractors have been colored as follows: magenta (period-3 basin), cyan (period-2 or 4), blue (period-8 or bifurcated) and yellow (period-6 or bifurcated). Figure 2(a) with $\Omega = 1{\cdot}83$ illustrates the coexistence of only two attractors having the periods 3 and 4. In Figure 2(b) with $\Omega = 1{\cdot}84$ we note the appearance of a chaotic attractor in addition to the periodic attractors having the periods 3 and 4 of the previous case. This chaotic attractor lying in the coherent part of the basin of the period-8 cascade is represented in white and consists of eight small clusters of points. In Figure 2(c) with $\Omega = 1{\cdot}845$ one has the coexistence of four attractors, three of which are periodic (period-3, -4 and -16 from the cascade generated by period-8). The six clusters of points in brown in this figure represent the chaotic attractor generated by the period-6 cascade. This attractor lies in the coherent part of its basin colored in yellow. Figure 2(d) with $\Omega = 1{\cdot}86$ shows the basins of four coexisting periodic attractors (period-3, 4, 12 from the cascade generated by the period-6 motion, and period-8). In Figure 2(e) with $\Omega = 1{\cdot}88$ the period-8 attractor has disappeared, yielding three basins of attractors corresponding to period-3, -2 (dedoubled from the previous period-4) and -6. Finally, for $\Omega = 1{\cdot}895$ in Figure 2(f) we remain with the basins of merely two periodic attractors having the periods 3 and 2.

With regard to the Figures 2(b) and (c) where one notes the appearance of the chaotic attractors, one of the most reliable criteria for discerning whether the motion is chaotic is to compute the Liapounov exponents λ_i and the Liapounov dimension d_L. In the case with $\Omega = 1{\cdot}84$, i.e., the case for the attractor resulting from the cascade starting with period-8, we find by the use of Wolf's method [12] for the computation of the Liapounov

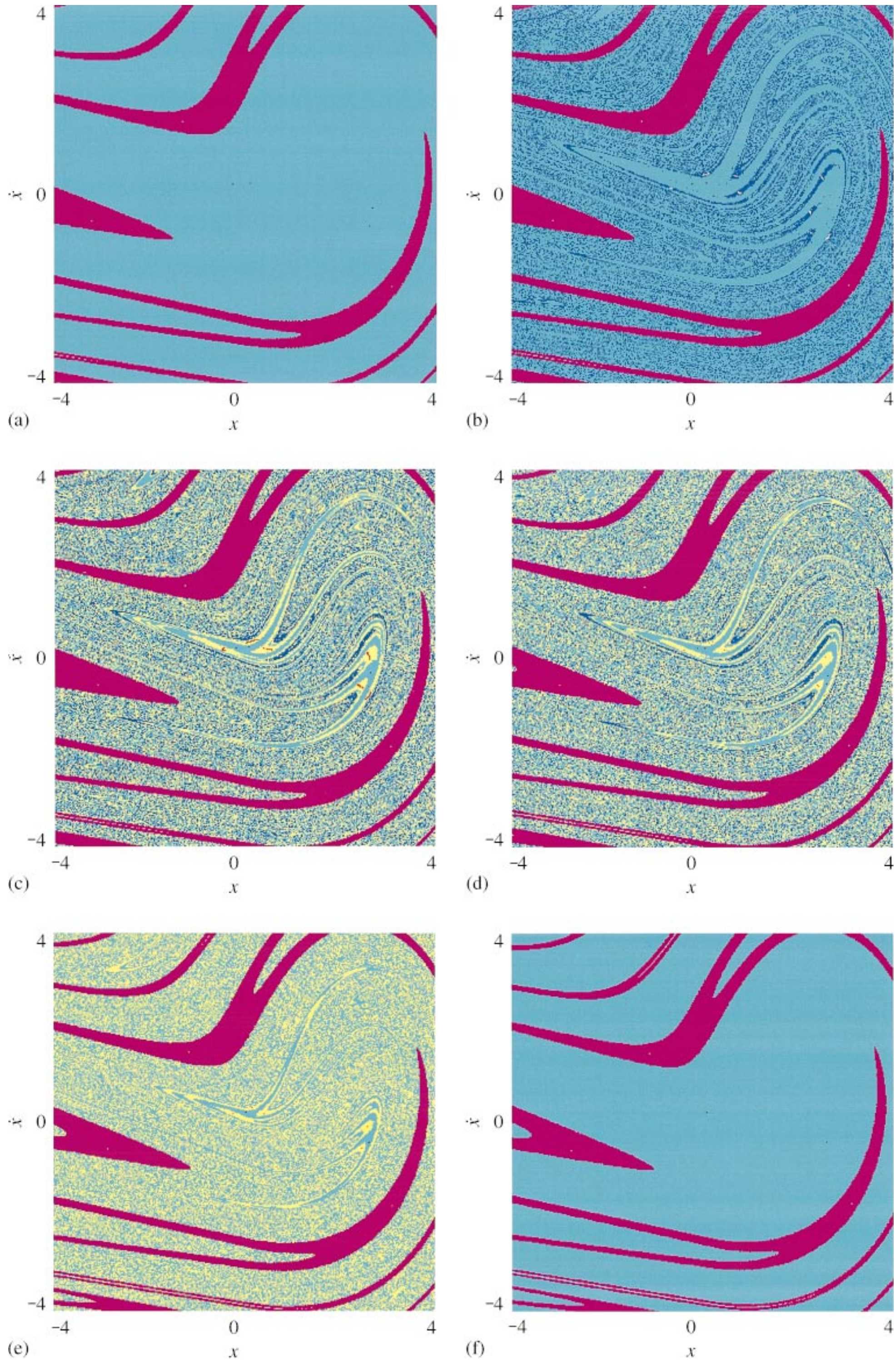

Figure 2. The basins of attraction with $\zeta = 0{\cdot}05$ and $\xi_0 = 0{\cdot}5$ for several variations of Ω: (a) $\Omega = 1{\cdot}83$ (3P, 4P); (b) $\Omega = 1{\cdot}84$ (3P, 4P, chaos); (c) $\Omega = 1{\cdot}845$ (3P, 4P, 16P, chaos); (d) $\Omega = 1{\cdot}86$ (3P, 4P, 12P, 8P); (e) $\Omega = 1{\cdot}88$ (3P, 2P, 6P); (f) $\Omega = 1{\cdot}895$ (3P, 2P). Used colors for basins: magenta (3P), cyan (2P or 4P), blue (8P or bifurcated) and yellow (6P or bifurcated).

exponents: $\lambda_1 = 0{\cdot}010$, $\lambda_2 = -0{\cdot}110$, $\lambda_3 = 0$. For the attractor with $\Omega = 1{\cdot}845$ in the period-6 cascade we have $\lambda_1 = 0{\cdot}019$, $\lambda_2 = -0{\cdot}119$, $\lambda_3 = 0$. In both cases, λ_1 is positive and hence the motion is chaotic. According to the Kaplan–Yorke relation [13] the Liapounov dimension, which is a measure of the fractal nature of the attractor, is defined by

$$d_L = 1 - \lambda_1/\lambda_2. \tag{6}$$

In the case with $\Omega = 1{\cdot}84$, we obtain $d_L = 1{\cdot}09$ and for the attractor with $\Omega = 1{\cdot}845$ we find $d_L = 1{\cdot}16$.

In numerical experiments, the chaotic attractor from the cascade generated by the period-8 motion has been found for Ω varying from $\Omega = 1{\cdot}838$ to $1{\cdot}841$. Hereby, the Liapounov dimension decreases from $d_L = 1{\cdot}13$ to $1{\cdot}06$. Chaotic motion generated from the period-6 motion has been obtained when Ω varies between $\Omega = 1{\cdot}843$ and $1{\cdot}847$ with the corresponding Liapounov dimension decreasing from $d_L = 1{\cdot}17$ to $1{\cdot}08$. The two chaotic attractors do not coexist at a single value of Ω.

4. CONCLUSIONS

New light has been thrown on the coexistence of the periodic and the chaotic attractors in the problem of the vibrations of two elastic bodies in rolling contact, one having a smooth surface, the other a wavy surface. The complex behavior of the system has been further unraveled with special emphasis on the metamorphoses of the domains of attraction. These metamorphoses are related to the period-doubling cascade of period-8 occurring near $\Omega = 1{\cdot}85$ and the cascade of period-6 in the vicinity of $\Omega = 1{\cdot}87$. The most complex pattern is obtained in the frequency range defined by $1{\cdot}843 \leqslant \Omega \leqslant 1{\cdot}865$ where four distinct attractors coexist.

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