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LETTERS TO THE EDITOR

BENDING VIBRATIONS OF BEAMS COUPLED BY A DOUBLE SPRING–MASS SYSTEM

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1. INTRODUCTION

There are large number of works in the literature on vibrations of combined dynamical systems consisting of beams or rods to which spring-mass secondary systems are attached. Some of them with recent dates are given references [1–8]. The common feature of the cited papers is that the corresponding continuous systems, i.e., primary systems, are made up of a single beam or rod.

Motivated by the interesting study by Kukla *et al.* [9] which was published on the problem of the natural longitudinal vibrations of two rods coupled by many translational springs, the study published by Gürgöze *et al.* [10] dealt with a similar system which was made up of two clamped–free longitudinally vibrating rods carrying tip masses to which a double spring–mass system was attached as a secondary system across the span. Then, an alternative formulation of the frequency equation of the same system without tip masses was given by Gürgöze [11]. Natural longitudinal vibrations of the same system with n secondary systems via the Green function method was investigated in the study by Inceoğlu *et al.* [12].

The present work is concerned essentially with a mechanical system, similar to that described in reference [10], but here natural bending vibrations of two Bernoulli–Euler beams attached with a double spring-mass system are investigated as a counterpart of that publication.

2. THEORY

The problem to be investigated in the present note is the natural vibration problem of the system shown in Figure 1. It consists of two clamped–free laterally vibrating Bernoulli–Euler beams carrying tip masses as the primary system (ps) to which a double spring-mass secondary system (ss) is attached across the span. The length, mass per unit length, location of the spring attachment point, bending rigidity and tip mass of the ith beam are L_i, m_i, $\eta_i L_i$, $E_i I_i$, and M_i, respectively ($i = 1, 2$). The secondary system consists of two springs of stiffness k_1, k_2 and the mass M. Let the bending displacements of the first and second beams to the left and right of the spring attachment points be denoted as $w_{11}(x, t), w_{12}(x, t)$ and $w_{21}(x, t), w_{22}(x, t)$, respectively, as depicted in Figure 1. $z(t)$ represents the displacement of the mass M.

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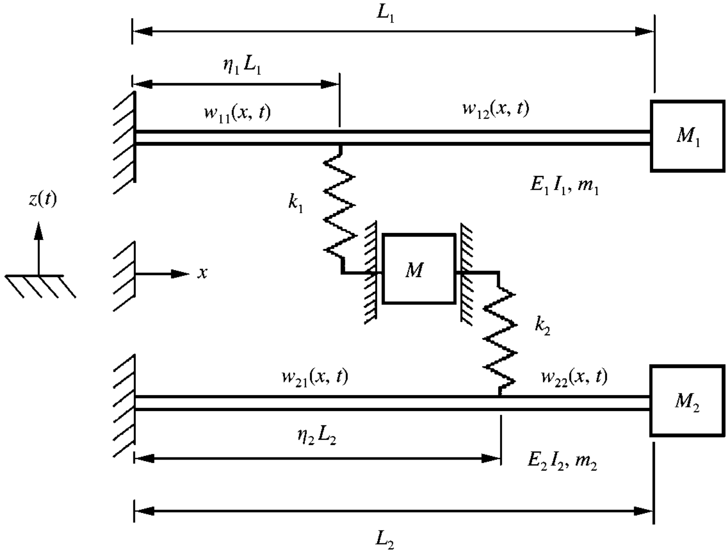

Figure 1. Two clamped–free beams with tip masses to which a double spring-mass system is attached in span.

The equations of bending motion of the four beam portions are governed by the partial differential equations.

$$E_i I_i \partial^4 w_{ij}(x, t)/\partial x^4 + m_i \partial^2 w_{ij}(x, t)/\partial t^2 = 0 \quad (i, j = 1, 2) \tag{1}$$

The corresponding boundary and continuity conditions at the spring attachment points are as follows:

$$\begin{gathered}
w_{11}(0, t) = 0, \quad w_{21}(0, t) = 0, \quad w'_{11}(0, t) = 0, \quad w'_{21}(0, t) = 0, \\
w_{11}(\eta_1 L_1, t) = w_{12}(\eta_1 L_1, t), \qquad w_{21}(\eta_2 L_2, t) = w_{22}(\eta_2 L_2, t), \\
w'_{11}(\eta_1 L_1, t) = w'_{12}(\eta_1 L_1, t), \qquad w'_{21}(\eta_2 L_2, t) = w'_{22}(\eta_2 L_2, t), \\
w''_{11}(\eta_1 L_1, t) = w''_{12}(\eta_1 L_1, t), \qquad w''_{21}(\eta_2 L_2, t) = w''_{22}(\eta_2 L_2, t), \\
E_1 I_1 [w'''_{11}(\eta_1 L_1, t) - w'''_{12}(\eta_1 L_1, t)] - k_1 [w_{11}(\eta_1 L_1, t) - z(t)] = 0, \\
E_2 I_2 [w'''_{21}(\eta_2 L_2, t) - w'''_{22}(\eta_2 L_2, t)] + k_2 [z(t) - w_{22}(\eta_2 L_2, t)] = 0, \\
E_1 I_1 w'''_{12}(L_1, t) - M_1 \ddot{w}_{12}(L_1, t) = 0, \qquad E_2 I_2 w'''_{22}(L_2, t) - M_2 \ddot{w}_{22}(L_2, t) = 0, \\
E_1 I_1 w''_{12}(L_1, t) = 0, \qquad E_2 I_2 w''_{22}(L_2, t) = 0, \\
M \ddot{z}(t) - k_1 [w_{11}(\eta_1 L_1, t) - z(t)] + k_2 [z(t) - w_{22}(\eta_2 L_2, t)] = 0.
\end{gathered} \tag{2}$$

Here dots and primes denote partial derivatives with respect to time t and position co-ordinate x, respectively.

Using the standard method of separation of variables one assumes that

$$w_{ij}(x,t) = W_{ij}(x)\cos(\omega\ t) \quad (i,j = 1,2), \tag{3}$$

where $W_{ij}(x)$ are corresponding amplitude functions of the beam portions and ω is the unknown eigenfrequency of the combined system. Substituting these into the partial differential equation (1) results in the ordinary differential equations

$$W_{1j}''''(x) - \beta^4 W_{1j}(x) = 0, \qquad W_{2j}''''(x) - \mu^4\beta^4 W_{2j}(x) = 0 \quad (j = 1,2). \tag{4}$$

Here, the following abbreviations are introduced

$$\beta^4 = \omega^2 m_1/E_1I_1, \qquad \mu^4 = \alpha_{\mathrm{m}}/\chi, \qquad \alpha_{\mathrm{m}} = m_2/m_1, \qquad \chi = E_2I_2/E_1I_1, \tag{5}$$

where β denotes the eigenfrequency parameter. Assuming that

$$z(t) = Z\cos(\omega\ t) \tag{6}$$

and substituting equations (3) and (6) into equations (2) yields the corresponding boundary and matching conditions for amplitude functions $W_{ij}(x)$ and Z:

$$\begin{aligned}
&W_{11}(0) = 0, \qquad W_{21}(0) = 0, \qquad W'_{11}(0) = 0, \qquad W'_{21}(0) = 0,\\
&W_{11}(\eta_1L_1) = W_{12}(\eta_1L_1), \qquad W_{21}(\eta_2L_2) = W_{22}(\eta_2L_2),\\
&W'_{11}(\eta_1L_1) = W'_{12}(\eta_1L_1), \qquad W'_{21}(\eta_2L_2) = W'_{22}(\eta_2L_2),\\
&W''_{11}(\eta_1L_1) = W''_{12}(\eta_1L_1), \qquad W''_{21}(\eta_2L_2) = W''_{22}(\eta_2L_2),\\
&E_1I_1[W'''_{11}(\eta_1L_1) - W'''_{12}(\eta_1L_1)] - k_1[W_{11}(\eta_1L_1) - Z] = 0,\\
&E_2I_2[W'''_{21}(\eta_2L_2) - W'''_{22}(\eta_2L_2)] + k_2[Z - W_{22}(\eta_2L_2)] = 0,\\
&E_1I_1W'''_{12}(L_1) + \omega^2M_1W_{12}(L_1) = 0, \qquad E_2I_2W'''_{22}(L_2) + \omega^2M_2W_{22}(L_2) = 0,\\
&E_1I_1W''_{12}(L_1) = 0, \qquad E_2I_2W''_{22}(L_2) = 0,\\
&\omega^2MZ + k_1[W_{11}(\eta_1L_1) - Z] - k_2[Z - W_{22}(\eta_2L_2)] = 0.
\end{aligned} \tag{7}$$

The general solutions of the ordinary differential equations (4) are simply

$$\begin{aligned}
&W_{1j}(x) = C_{1j}\sin(\beta x) + C_{2j}\cos(\beta x) + C_{3j}\sinh(\beta x) + C_{4j}\cosh(\beta x),\\
&W_{2j}(x) = C_{5j}\sin(\mu\beta x) + C_{6j}\cos(\mu\beta x) + C_{7j}\sinh(\mu\beta x) + C_{8j}\cosh(\mu\beta x) \quad (j = 1,2),
\end{aligned} \tag{8}$$

where C_{1j}–C_{8j} are 16 integration constants to be evaluated via conditions (7). The application of these boundary and matching conditions to the solutions (8) and the amplitude Z yields a set of 17 homogeneous equations for the 17 unknown constants $C_{1j} - C_{8j}\,(j = 1,2)$ and Z. A non-trivial solution of this set of equations is possible only if the characteristic determinant of the coefficients vanishes. Upon taking into account that

C_{31}, C_{41}, C_{71} and C_{81} vanish, the characteristic equation reduces to the form

$$R = \begin{vmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} & R_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} & R_{26} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} & R_{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} & R_{46} & 0 & 0 & 0 & 0 & 0 & 0 & R_{4,13} \\ R_{51} & 0 & R_{53} & 0 & 0 & 0 & 0 & R_{58} & 0 & R_{5,10} & R_{5,11} & R_{5,12} & R_{5,13} \\ 0 & 0 & 0 & 0 & 0 & 0 & R_{67} & R_{68} & R_{69} & R_{6,10} & R_{6,11} & R_{6,12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & R_{77} & R_{78} & R_{79} & R_{7,10} & R_{7,11} & R_{7,12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & R_{87} & R_{88} & R_{89} & R_{8,10} & R_{8,11} & R_{8,12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & R_{97} & R_{98} & R_{99} & R_{9,10} & R_{9,11} & R_{9,12} & R_{9,13} \\ 0 & R_{10,2} & 0 & R_{10,4} & R_{10,5} & R_{10,6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & R_{11,8} & 0 & R_{11,10} & R_{11,11} & R_{11,12} & 0 \\ 0 & R_{12,2} & 0 & R_{12,4} & R_{12,5} & R_{12,6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & R_{13,8} & 0 & R_{13,10} & R_{13,11} & R_{13,12} & 0 \end{vmatrix} = 0, \quad (9)$$

where the corresponding elements are given in Appendix A.

It is worth noting that $\bar{\beta}$, of which all elements are functions, represents the dimensionless frequency parameter of the combined system.

3. EXEMPLARY NUMERICAL RESULTS

The complicated frequency equation (9) was solved by using MATLAB version 5.1 on a PC Pentium III. The first 10 non-dimensional frequency parameters are given in Table 1 to allow comparison with other studies related to the same system, that may be made in the future. This table is based on the numerical values $\eta_1 = \eta_2 = 0{\cdot}5$, $\alpha_{M1} = \alpha_{M2} = 2$, $\alpha_M = 1$, $\alpha_{k1} = \alpha_{k2} = 1000$ with all other dimensionless parameters set to one.

However, there are many system parameters which can be varied, so it is meaningless to tabulate the values obtained for various combinations of these many parameters. Instead, results for three examples are given in a graphical form.

TABLE 1

The first 10 non-dimensional frequency parameters of the system in Figure 1; $\eta_1 = \eta_2 = 0{\cdot}5$, $\alpha_{M1} = \alpha_{M2} = 2$, $\alpha_M = 1$, $\alpha_{k1} = \alpha_{k2} = 1000$

From equation (9)
1·07026286
1·55878983
3·30959152
6·33425479
6·81983036
7·46938324
7·92595594
10·66673582
10·74697127
13·40231846

In all numerical evaluations presented here, the values of some of the non-dimensional physical parameters are as follows: $\alpha_k = 1$, $\alpha_L = 1$, $\alpha_m = 1$, $\chi = 1$, and $\mu = \delta = 1$.

The first example aims to explore the effect of the variation of the location of the spring attachment points on the natural frequencies of the combined system where the mass ratio $\alpha_{M1}(= \alpha_{M2})$ is taken as a parameter. At first, the attachment point to the second beam is fixed at $\eta_2 = 0{\cdot}5$ and the attachment point to the first beam is varied along the beam and then *vice versa*. Tip mass ratio and spring stiffnesses are chosen as $\alpha_M = 1$ and $\alpha_{k1} = \alpha_{k2} = 1000$, respectively. The large value of $\alpha_{k1} = \alpha_{k2}$ implies that the investigated system has a very rigid secondary system. As expected from the symmetry of the system, the same curves are obtained in both situations, but for the sake of briefness only one of the figures is placed here. The results are shown in Figure 2.

As is shown in the figure, all the dimensionless frequency parameters diminish as $\alpha_{M1}(= \alpha_{M2})$ parameter becomes larger. It seems that the reductions in the first two frequency parameters are move obvious.

However, the fundamental frequency parameter decreases highly for small values of $\alpha_{M1}(= \alpha_{M2})$ while η_1 gets larger. It seems that the fundamental frequency parameter is not so sensitive to the variation of η_1.

The next example deals with the effect of the variation of the mass M of the secondary system on the eigenfrequencies of the combined system. The spring attachment points $\eta_2 = \eta_1 = 0{\cdot}5$ are chosen. The mass parameter α_M is varied in the range 0–20. The results are shown in Figure 3. The fundamental and the third non-dimensional eigenfrequencies diminish continuously as α_M gets larger, whereas the other two frequencies remain constant.

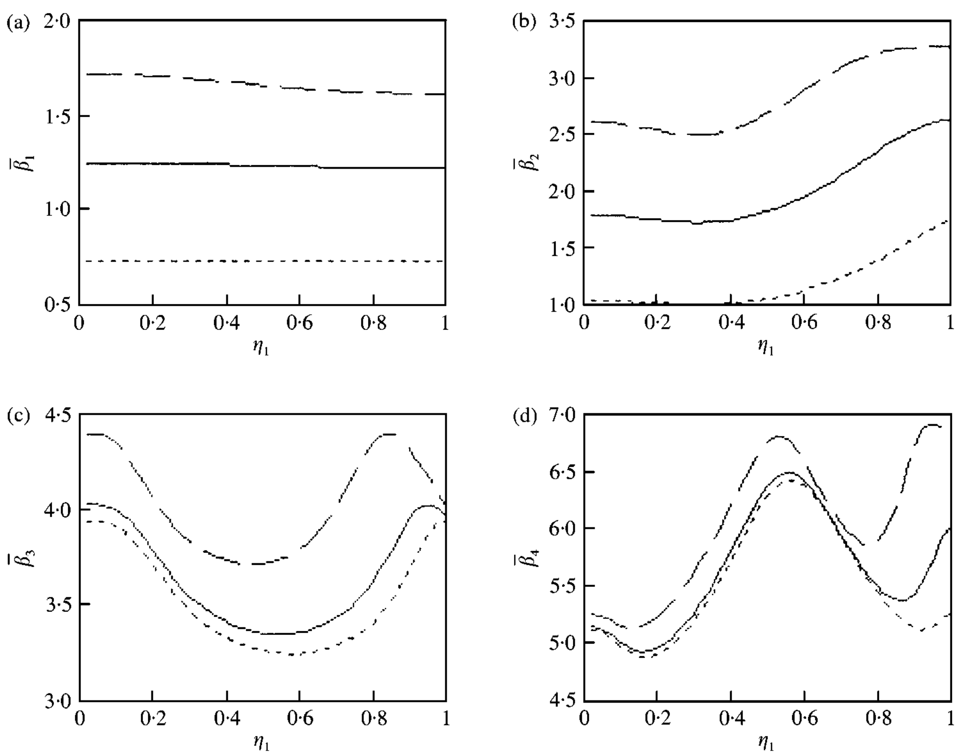

Figure 2. The first four dimensionless frequency parameters of the system shown in Figure 1 as a function of η_1, ($\eta_2 = 0{\cdot}5, \alpha_M = 1, \alpha_k = 1, \alpha_{k1} = \alpha_{k2} = 1000, \alpha_{M1} = \alpha_{M2} = 0{\cdot}1$ - - - -, $\alpha_{M1} = \alpha_{M2} = 1$; ——, $\alpha_{M1} = \alpha_{M2} = 10$), (a) $\bar{\beta}_1$; (b) $\bar{\beta}_2$; (c) $\bar{\beta}_3$; (d) $\bar{\beta}_4$.

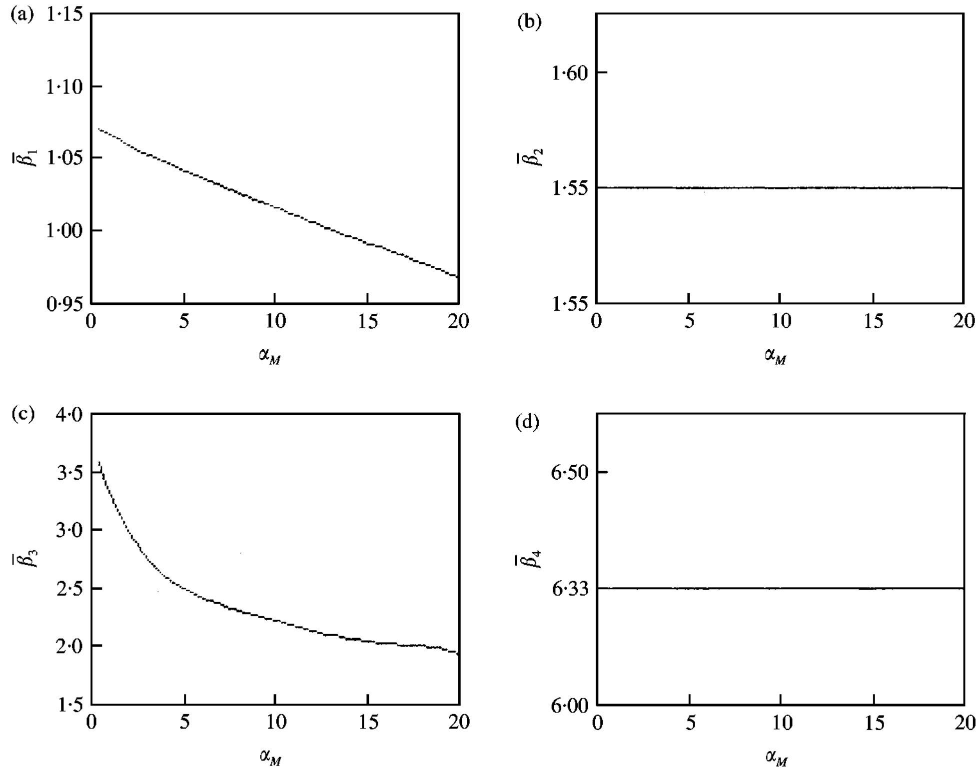

Figure 3. Effect of the variation of the dimensionless mass α_M of the secondary system on the dimensionless frequency parameters of the system shown in Figure 1 ($\eta_1 = \eta_2 = 0{\cdot}5$, $\alpha_k = 1$, $\alpha_{k1} = \alpha_{k2} = 1000$, $\alpha_{M1} = \alpha_{M2} = 2$). (a) $\overline{\beta_1}$; (b) $\overline{\beta_2}$; (c) $\overline{\beta_3}$; (d) $\overline{\beta_4}$.

It is also interesting to note that the fundamental frequency changes linearly while the secondary system mass becomes larger. The behavior of the curves can be explained as follows. The physical system corresponding to the selected parameter values is a symmetric system. For this reason, the mass M in between remains stationary in the second and fourth modes. As a result, eigenfrequencies are not affected during variations of this mass. On the other hand, in the first and third modes the mass M participates in the vibration, and naturally its increase decreases the related frequency values.

The last example deals with the effect of the variation of the tip masses on the eigenfrequencies of the combined system, where $\alpha_{M1} = \alpha_{M2}$ is taken for the sake of simplicity, and, $\alpha_M = 1$ and $\eta_2 = \eta_1 = 0{\cdot}5$ are chosen. The results are shown in Figure 4. The curves indicate clearly that the eigenfrequencies of the system decrease as the tip of masses become larger, as expected.

4. CONCLUSIONS

The subject of this note is the lateral vibration problem of a combined system consisting of two clamped–free Bernoulli–Euler beams, coupled by a double spring-mass system attached to them in-span. After formulating the complicated frequency equation of the combined system, the effects of the variation of some parameters upon the natural frequencies were investigated through numerical examples.

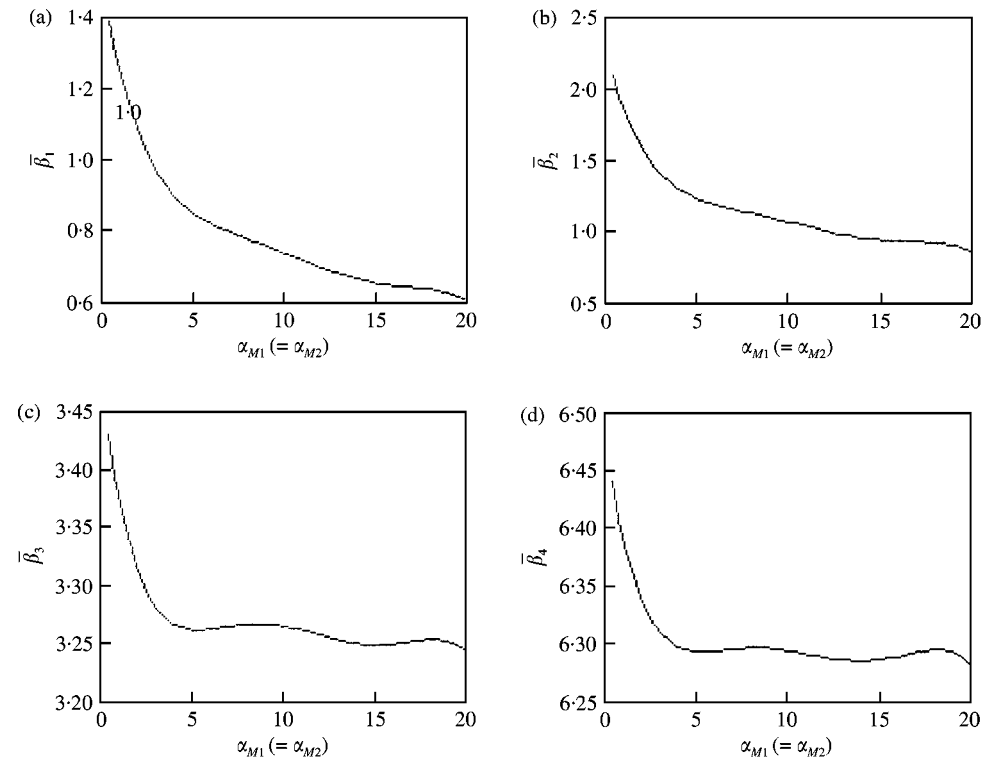

Figure 4. Effect of the variation of the dimensionless tip masses α_{M_1} and α_{M_2} of the secondary system on the dimensionless frequency parameters of the system shown in Figure 1 ($\eta_1 = \eta_2 = 0{\cdot}5$, $\alpha_k = 1$, $\alpha_{k1} = \alpha_{k2} = 1000$, $\alpha_M = 1$). (a) $\overline{\beta_1}$; (b) $\overline{\beta_2}$; (c) $\overline{\beta_3}$; (d) $\overline{\beta_4}$.

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APPENDIX A

The non-zero elements of the frequency determinant in equation (9) are as follows:

$$R_{11} = \sin(\bar{\beta}\eta_1) - \sinh(\bar{\beta}\eta_1), \qquad R_{12} = -\sin(\bar{\beta}\eta_1)$$

$$R_{13} = \cos(\bar{\beta}\eta_1) - \cosh(\bar{\beta}\eta_1), \qquad R_{14} = -\cos(\bar{\beta}\eta_1)$$

$$R_{15} = -\sinh(\bar{\beta}\eta_1), \qquad R_{16} = -\cosh(\bar{\beta}\eta_1)$$

$$R_{21} = \cos(\bar{\beta}\eta_1) - \cosh(\bar{\beta}\eta_1), \qquad R_{22} = -\cos(\bar{\beta}\eta_1)$$

$$R_{23} = -\sin(\bar{\beta}\eta_1) - \sinh(\bar{\beta}\eta_1), \qquad R_{24} = \sin(\bar{\beta}\eta_1)$$

$$R_{25} = -\cos(\bar{\beta}\eta_1), \qquad R_{26} = -\sinh(\bar{\beta}\eta_1)$$

$$R_{31} = \sin(\bar{\beta}\eta_1) + \sinh(\bar{\beta}\eta_1), \qquad R_{32} = -\sin(\bar{\beta}\eta_1)$$

$$R_{33} = \cos(\bar{\beta}\eta_1) + \cosh(\bar{\beta}\eta_1), \qquad R_{34} = -\cos(\bar{\beta}\eta_1)$$

$$R_{35} = \sinh(\bar{\beta}\eta_1), \qquad R_{36} = \cosh(\bar{\beta}\eta_1)$$

$$R_{41} = \cos(\bar{\beta}\eta_1) + \cosh(\bar{\beta}\eta_1) + (\alpha_{k_1}/(\bar{\beta}^3)(\sin(\bar{\beta}\eta_1) - \sinh(\bar{\beta}\eta_1)), \qquad R_{42} = -(\cos(\bar{\beta}\eta_1),$$

$$R_{43} = -\sin(\bar{\beta}\eta_1) + \sinh(\bar{\beta}\eta_1) + (\alpha_{k_1}/(\bar{\beta}^3)(\cos(\bar{\beta}\eta_1) - \cosh(\bar{\beta}\eta_1)), \qquad R_{44} = \sin(\bar{\beta}\eta_1),$$

$$R_{45} = \cosh(\bar{\beta}\eta_1), \qquad R_{46} = \sinh(\bar{\beta}\eta_1), \qquad R_{43} = -\alpha_{k_1}/\bar{\beta}^3,$$

$$R_{51} = (\alpha_{k_1}/\alpha_M\bar{\beta}^4)(\sin(\bar{\beta}\eta_1) - \sinh(\bar{\beta}\eta_1)), \qquad R_{53} = (\alpha_{k_1}/\alpha_M\bar{\beta}^4)(\cos(\bar{\beta}\eta_1) - \cosh(\bar{\beta}\eta_1)),$$

$$R_{58} = (\alpha_{k_1}\alpha_k/\alpha_M\bar{\beta}^4)\sin(\psi\bar{\beta}), \qquad R_{5,10} = (\alpha_{k_1}\alpha_k/\alpha_M\bar{\beta}^4)\cos(\psi\bar{\beta}),$$

$$R_{5,11} = (\alpha_{k_1}\alpha_k/\alpha_M\bar{\beta}^4)\sinh(\psi\bar{\beta}), \qquad R_{5,12} = (\alpha_{k_1}\alpha_k/\alpha_M\bar{\beta}^4)\cosh(\psi\bar{\beta}),$$

$$R_{5,13} = 1 - \alpha_{k_1}(\alpha_k + 1)/\alpha_M\bar{\beta}^4,$$

$$R_{67} = \sin(\psi\bar{\beta}) - \sinh(\psi\bar{\beta}), \qquad R_{68} = -\sin(\psi\bar{\beta}),$$

$$R_{6,9} = \cos(\psi\bar{\beta}) - \cosh(\psi\bar{\beta}), \qquad R_{6,10} = -\cos(\psi\bar{\beta}),$$

$$R_{6,11} = -\sinh(\psi\bar{\beta}), \qquad R_{6,12} = -\cosh(\psi\bar{\beta}),$$

$$R_{77} = \cos(\psi\bar{\beta}) - \cosh(\psi\bar{\beta}), \qquad R_{78} = -\cos(\psi\bar{\beta}),$$

$$R_{79} = -\sin(\psi\bar{\beta}) - \sinh(\psi\bar{\beta}), \qquad R_{7,10} = \sin(\psi\bar{\beta}),$$

$$R_{7,11} = -\cosh(\psi\bar{\beta}), \qquad R_{7,12} = -\sinh(\psi\bar{\beta}),$$

$$R_{87} = \sin(\psi\bar{\beta}) + \sinh(\psi\bar{\beta}), \qquad R_{88} = -\sin(\psi\bar{\beta}),$$

$$R_{89} = \cos(\psi\bar{\beta}) + \cosh(\psi\bar{\beta}), \qquad R_{8,10} = -\cos(\psi\bar{\beta})$$

$$R_{8,11} = \sinh(\psi\bar{\beta}), \qquad R_{8,12} = \cosh(\psi\bar{\beta}),$$

$$R_{97} = \cos(\psi\bar{\beta}) + \cosh(\psi\bar{\beta}), \qquad R_{98} = -\cos(\psi\bar{\beta}) + (\alpha_{k_2}/\delta^3\bar{\beta}^3)\sin(\psi\bar{\beta}),$$

$$R_{99} = -\sin(\psi\bar{\beta}) + \sinh(\psi\bar{\beta}), \qquad R_{9,10} = \sin(\psi\bar{\beta}) + (\alpha_{k_2}/\delta^3\bar{\beta}^3)\cos(\psi\bar{\beta}),$$

$$R_{9,11} = \cosh(\psi\bar{\beta}) + (\alpha_{k_2}/\delta^3\bar{\beta}^3)\sinh(\psi\bar{\beta}), \qquad R_{9,12} = \sinh(\psi\bar{\beta}) + (\alpha_{k_2}/\delta^3\bar{\beta}^3)\cosh(\psi\bar{\beta}),$$

$$R_{9,13} = -\alpha_{k_2}/\delta^3\bar{\beta}^3$$

$$R_{10,2} = \sin(\bar{\beta}), \qquad R_{10,4} = \cos(\bar{\beta}), \qquad R_{10,5} = -\sinh(\bar{\beta}), \qquad R_{10,6} = -\cosh(\bar{\beta}),$$

$$R_{11,8} = \sin(\delta\bar{\beta}), \qquad R_{11,10} = \cos(\delta\bar{\beta}), \qquad R_{11,11} = -\sinh(\delta\bar{\beta}), \qquad R_{11,12} = -\cosh(\delta\bar{\beta}),$$

$$R_{12,2} = -\cos(\bar{\beta}) + \alpha_{M_1}\bar{\beta}\sin(\bar{\beta}), \qquad R_{12,4} = \sin(\bar{\beta}) + \alpha_{M_1}\bar{\beta}\cos(\bar{\beta}),$$

$$R_{12,5} = \cosh(\bar{\beta}) + \alpha_{M_1}\bar{\beta}\sinh(\bar{\beta}), \qquad R_{12,6} = \sinh(\bar{\beta}) + \alpha_{M_1}\bar{\beta}\cosh(\bar{\beta}),$$

$$R_{13,8} = -\cos(\delta\bar{\beta}) + \alpha_{M_2}\delta\bar{\beta}\sin(\delta\bar{\beta}), \qquad R_{13,10} = \sin(\delta\bar{\beta}) + \alpha_{M_2}\delta\bar{\beta}\cos(\delta\bar{\beta}),$$

$$R_{13,11} = \cosh(\delta\bar{\beta}) + \alpha_{M_2}\delta\bar{\beta}\sinh(\delta\bar{\beta}), \qquad R_{13,12} = \sinh(\delta\bar{\beta}) + \alpha_{M_2}\delta\bar{\beta}\cosh(\delta\bar{\beta}).$$

Here, in addition to those given in equation (5), the following definitions are introduced:

$$\beta L_1 = \bar{\beta}, \qquad \alpha_M = M/m_1L_1, \qquad \alpha_{M_1} = M_1/m_1L_1,$$

$$\alpha_{M_2} = M_2/m_2L_2, \qquad \alpha_k = k_2/k_1, \qquad \alpha_{k_1} = k_1L_1^3/E_1I_1,$$

$$\alpha_{k_2} = k_2L_2^3/E_2I_2, \qquad \delta = \mu\alpha_L, \qquad \psi = \mu\alpha_L\eta_2, \qquad \alpha_L = L_2/L.$$