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A COMPARISON OF SOME THIN SHELL THEORIES USED FOR THE DYNAMIC ANALYSIS OF STIFFENED CYLINDERS

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The aim of this article is to compare Donnell's, Love's, Sanders' and Flügge's thin shell theories in the evaluation of natural frequencies of cylinders stiffened with rings and stringers, whose effect is smeared over the entire surface of the cylinder. It is demonstrated that due to the large increase in bending stiffness related to rings, Donnell's theory provides highly inaccurate results with respect to the other three theories. Numerical results related to aluminum and composite stiffened cylinders and a comparison with results obtained with a finite element model of a stiffened cylinder complete the work.

1. INTRODUCTION

Cylinders stiffened by rings and stringers are extensively used in engineering practice in order both to increase stiffness and strength, and to reduce the weight of the structure to be designed; aircraft fuselages, missile bodies and shells of ships and submarines are examples of such structures. A number of researchers have dealt with this topic and a review of the available literature can be found in reference [1].

In the analysis of stiffened cylinders, the effects of rings and stringers can be considered by treating them as discrete elements or by averaging their properties over the surface of the shell. An example of the application of the first approach is shown in reference [2], where the analysis of free vibration of orthogonally stiffened cylindrical shells is addressed. In contrast, when the stiffeners are large in number, closely and equally spaced, the second approach can be applied.

The latter method, in which the eccentricity of stiffeners is taken into account, was originally derived for buckling analysis in reference [3], used in reference [4] to determine the static and dynamic behavior of stiffened plates and cylindrical shells, and extensively employed in the study of the general instability of cylindrical and conical shells in references [5, 6]. Moreover, this smeared approach was also taken into account during the Apollo program in the design of the Saturn V launch vehicle as described in reference [7].

More recently, several authors [8–11] addressed the problem of predicting interior noise levels in the fuselage of propeller driven aircraft and, in order to obtain a first estimate of the performance of both active and passive control methods in reducing the interior pressure level and vibrations of the fuselage sidewall, used cylinders stiffened by rings and stringers to represent the dynamic behavior of the fuselage, usually representing their effect by the smeared approach.

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It is well known that the averaging procedure introduces modifications into the extensional, coupling and bending submatrices $[A]$, $[B]$ and $[D]$ described in reference [12], such that from this viewpoint a stiffened shell can be analyzed with methods usually applied for composite laminates.

1.1. STUDIES ON LAMINATE AND ISOTROPIC SHELLS

Laminate thin elastic shells have been examined by several investigators. Usually, the problem of determining the static and dynamic response, as well as the dynamic behavior, or laminate and/or homogeneous isotropic shells is addressed by using some simplifying assumptions proposed initially by Love that led to the development of a sub-class of the theory of elasticity known as the theory of thin elastic shells. Love's First Approximation Theory (LFAT) for thin elastic shells is based upon the following postulates [13]: (1) the shell is thin; (2) the deflections of the shell are small; (3) the transverse normal stress is negligible; (4) normals to the middle surface of the shell remain normal to it and undergo no change in length during deformation.

Several theories have been developed according to these four postulates. As described in reference [14], the Donnell–Mushtari shallow shell theory is based on other approximations on terms present in curvature and twist of the shell. Moreover, this theory is inconsistent with respect to rigid-body motions; indeed, in reference [13] it is demonstrated that a rigid-body rotation gives a torsion if the shell is not a plate, a sphere or a symmetrically loaded shell of revolution. Sanders' theory [13] is aimed to remove this inconsistency, so that it is preferable to Donnell's theory in some cases.

By suspending one or more of Love's four postulates, Love's Second Approximation Theories (LSAT) are derived. A theory of elastic shells in which Love's first postulate is delayed was independently derived by Flügge, Lur'e and Byrne [13].

As described by Reddy [15], several other theories have been developed by delaying Love's fourth postulate. For thick laminates and laminates with high degree of anisotropy, the transverse deformation effects can be significant and theories which hold the fourth postulate are not able to determine accurately the response to static and dynamic loads. As a consequence, First order Shear Deformation Theories (FSDT) have been developed, which consider a constant transverse shear strain over the thickness of the laminate [15]. In contrast, second and higher order SDT use higher order polynomials to describe displacement components through the thickness of the laminate. A good survey of all these theories was written by Carrera [16].

In the analysis of static and dynamic response of homogeneous isotropic shells Novozhilov [14] underlined the importance of the thickness to curvature ratio (h/R) in order to make a distinction between thick and thin shells, i.e., cases in which theories holding Love's first postulate can or cannot be applied with success.

During the analysis of laminate shells also the difference in mechanical properties due to layer orientation must be considered. This aspect was addressed by Soldatos [17], who performed an interesting work comparing results obtained with Donnell's, Sanders', Love's and Flügge's theories in the evaluation of natural frequencies of cross-ply laminated panels, a case in which mechanical properties are very different along the 0 and 90° directions. In reference [17], the importance of such a difference in mechanical properties over the accuracy of natural frequencies evaluation was pointed out: it was demonstrated that for a cylinder with $h/R = 0{\cdot}01$ and length over radius ratio (L/R) of 10, Donnell's theory can provide results with an error of about 10% with respect to results obtained through the other three theories (the corresponding error for a homogeneous isotropic cylinder is much lower).

1.2. PROJECT SCOPE

As previously stated, the averaging approach permits one to deal with stiffened cylinders with the same tools used with composite laminate shells. As a result, by recalling Soldatos' work [17] and the influence of rings and stringers on the extensional, coupling and bending sub-matrices, it should be expected that similar errors may occur in the analysis of stiffened shells.

In several previous works, different thin shell theories were used to determine the static, dynamic or vibroacoustic behavior of stiffened cylinders (e.g., in references [9–11] Donnell's theory was used, in reference [8] Flügge's theory, while in reference [18] Cole III used the Love–Timoshenko theory). Furthermore, Rinehart and Wang [19] compared Donnell's and Flügge's theories for cylindrical shells with longitudinal stiffeners, concluding that "Donnell's approximate theory gives excellent results for stiffened shells," whereas Rosen and Singer [6] compared Donnell's and Flügge's theories for vibration of axially loaded stiffened cylindrical shells with experiments and obtained good agreement. However, it appears that a study permitting one to understand which is the best theory to deal with cylinders stiffened by both rings and stringers and what is the extent of the error related to each different theory, is not available.

As a consequence, the aim of this article is to compare results provided by Donnell's, Love's, Sanders' and Flügge's theories. By following a technique introduced in works related to the field of composite laminates [16, 17], some traces are used such that the problem of determining the vibration behavior of stiffened cylinders is written in a form which includes all these four theories as particular cases.

In the following section, the four considered theories are briefly reviewed, while section 3 presents results related to stiffened cylinders. In section 4, numerical results are discussed to highlight inaccuracies related to Donnell's theory.

2. SHELL THEORIES

Figure 1 shows a differential element of the shell under analysis with the curvilinear orthogonal co-ordinates α and β. Each layer of the laminated shell is considered to behave macroscopically as a homogeneous, orthotropic, linearly elastic material and all layers are assumed to be perfectly bonded together. By recalling Love's fourth postulate, related to the preservation of the normal during the deformation, it is assumed that displacements are linearly distributed through the thickness of the shell,

$$\begin{aligned} u(\alpha,\beta,z) &= u^0(\alpha,\beta) + z u_{,\alpha}(\alpha,\beta), \\ v(\alpha,\beta,z) &= v^0(\alpha,\beta) + z v_{,\beta}(\alpha,\beta), \\ w(\alpha,\beta,z) &= w(\alpha,\beta), \end{aligned} \tag{1}$$

where u^0 and v^0 represent the displacement of the middle surface of the shell along α and β directions respectively.

The kth layer is considered to be in a state of plane stress governed by the following relation:

$$\begin{Bmatrix} \sigma_\alpha^{(k)} \\ \sigma_\beta^{(k)} \\ \sigma_{\alpha\beta}^{(k)} \end{Bmatrix} = \begin{bmatrix} Q_{11}^{(k)} & Q_{12}^{(k)} & 0 \\ Q_{21}^{(k)} & Q_{22}^{(k)} & 0 \\ 0 & 0 & Q_{66}^{(k)} \end{bmatrix} \begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_{\alpha\beta} \end{Bmatrix}. \tag{2}$$

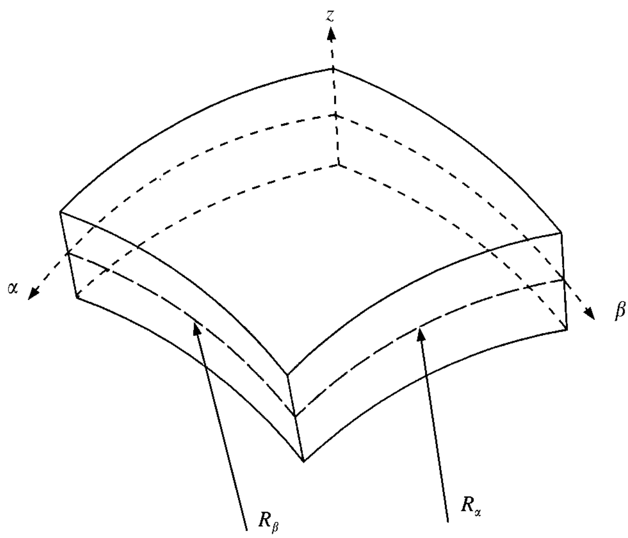

Figure 1. Differential element of a shell.

By introducing relations (1) into the exact expressions of strain and recalling Love's postulates, the following equations are derived [13]:

$$\varepsilon_\alpha = (\varepsilon_\alpha^0 + zk_\alpha)/I_\alpha,$$

$$\varepsilon_\beta = (\varepsilon_\beta^0 + zk_\beta)/I_\beta,$$

$$\varepsilon_{\alpha\beta} = (I_\beta v^0_{,\alpha} + I_\alpha u^0_{,\beta} + z(I_\beta v^0_{,\alpha}/R_\beta + I_\alpha u^0_{,\beta}/R_\alpha - w_{,\alpha\beta}(I_\alpha + I_\beta)))/I_\alpha I_\beta. \tag{3}$$

Here

$$\varepsilon_\alpha^0 = u^0_{,\alpha} + w/R_\alpha, \qquad \varepsilon_\beta^0 = v^0_{,\beta} + w/R_\beta,$$

$$k_\alpha = u^0_{,\alpha}/R_\alpha - w_{,\alpha\alpha}, \qquad k_\beta = v^0_{,\beta}/R_\beta - w_{,\beta\beta}, \tag{4}$$

represent linear deformation and curvature of the middle surface of the shell, while $I_\alpha = 1 + z/R_\alpha$ and $I_\beta = 1 + z/R_\beta$.

To introduce Flügge's theory and simplify the derivation of the equation of motion, Kraus [13] rewrote $\varepsilon_{\alpha\beta}$ as

$$\varepsilon_{\alpha\beta} = \frac{1}{I_\alpha I_\beta}\left(\varepsilon^0_{\alpha\beta}\left(1 - \frac{z^2}{R_\alpha R_\beta}\right) + zk^0_{\alpha\beta}\left(1 + \frac{z}{2}\left(\frac{1}{R_\alpha} + \frac{1}{R_\beta}\right)\right)\right), \tag{5}$$

in order to divide terms depending on the z co-ordinate from constant terms, the latter representing strains of the middle surface:

$$\varepsilon^0_{\alpha\beta} = u^0_{,\beta} + v^0_{,\alpha},$$

$$k^0_{\alpha\beta} = 2(v^0_{,\alpha}/R_\beta + u^0_{,\beta}/R_\alpha - w_{,\alpha\beta}). \tag{6}$$

Strains ε_α and ε_β are written according to relations (3) and (4).

Donnell's, Love's and Sanders' theories take into account all four of Love's approximations. As a result, they neglect the term z/R with respect to one, i.e., $I_\alpha = I_\beta \approx 1$ in equation (3). Furthermore, the main differences between these theories are due to different expressions for curvatures. In particular, the expression for strains used by Love is derived by introducing Love's first assumption into equation (3), while in Donnell's theory it is also necessary to neglect all the tangential displacements and related derivatives in the curvatures. Sanders' theory utilizes a different expression for curvatures in order to assure that any rigid-body motion cannot introduce strain into the shell [13].

Expressions for strains according to these four shell theories can be written by using some tracers δ. In particular,

$$\varepsilon_\alpha = \frac{\varepsilon_\alpha^0 + zk_\alpha}{1 + \delta_F z/R_\alpha}, \qquad \varepsilon_\beta = \frac{\varepsilon_\beta^0 + zk_\beta}{1 + \delta_F z/R_\beta},$$

$$\varepsilon_{\alpha\beta} = \frac{\varepsilon_{\alpha\beta}^0(1 - \delta_F z^2/R_\alpha R_\beta) + zk_{\alpha\beta}^0(1 + \delta_F z/2(1/R_\alpha + 1/R_\beta))}{(1 + \delta_F z/R_\alpha)(1 + \delta_F z/R_\beta)}, \tag{7}$$

where ε_α^0 and ε_β^0 are given by equation (4) and

$$k_\alpha = (\delta_L + \delta_S + \delta_F)u_{,\alpha}^0/R_\alpha - w_{,\alpha\alpha},$$

$$k_\beta = (\delta_L + \delta_S + \delta_F)v_{,\beta}^0/R_\beta - w_{,\beta\beta}, \qquad \varepsilon_{\alpha\beta}^0 = v_{,\alpha}^0 + u_{,\beta}^0,$$

$$k_{\alpha\beta}^0 = -2w_{,\alpha\beta} + (\delta_L + \delta_S + 2\delta_F)(v_{,\alpha}^0/R_\beta + u_{,\beta}^0/R_\alpha) + \delta_S(1/R_\beta - 1/R_\alpha)(v_{,\alpha}^0 - u_{,\beta}^0)/2, \tag{8}$$

where tracers δ_S, δ_L and δ_F provide strain expressions related to Sanders', Love's and Flügge's theories, respectively, according to Donnell, $\delta_S = \delta_L = \delta_F = 0$; Sanders, $\delta_S = 1$, $\delta_L = \delta_F = 0$; Love, $\delta_L = 1$, $\delta_S = \delta_F = 0$; Flügge, $\delta_F = 1$, $\delta_L = \delta_S = 0$.

The stiffness matrix of the shell in the four theories considered can be derived by recalling the expression of the potential energy of deformation [14],

$$U = 1/2 \int_A \int_{-h/2}^{h/2} (\sigma_\alpha \varepsilon_\alpha + \sigma_\beta \varepsilon_\beta + \sigma_{\alpha\beta}\varepsilon_{\alpha\beta}) I_\alpha I_\beta \, \mathrm{d}\alpha \, \mathrm{d}\beta \, \mathrm{d}z, \tag{9}$$

where Love's postulates have been applied and the integral extends over the surface A of the shell and its thickness h.

By evaluating the integral firstly along the z direction, expression (9) can be rewritten as

$$U = 1/2 \int_A (N_\alpha \varepsilon_\alpha^0 + M_\alpha k_\alpha + N_\beta \varepsilon_\beta^0 + M_\beta k_\beta + N_{\alpha\beta}\varepsilon_{\alpha\beta}^0 + M_{\alpha\beta}k_{\alpha\beta}^0) \, \mathrm{d}\alpha \, \mathrm{d}\beta, \tag{10}$$

where resultants of stresses N_{ij} and of couples M_{ij} are evaluated such that expressions (9) and (10) are equal.

When Flügge's theory is considered, to simplify the integration along the z direction one can apply the approximation

$$(1 + z/R)^{-1} \approx 1 - z/R + (z/R)^2, \tag{11}$$

so that by introducing the vectors

$$\bar{N} = [N_\alpha N_\beta N_{\alpha\beta}], \qquad M = [M_\alpha M_\beta M_{\alpha\beta}],$$

$$\bar{\varepsilon} = [\varepsilon_\alpha^0 \varepsilon_\beta^0 \varepsilon_{\alpha\beta}^0], \qquad \bar{k} = [k_\alpha k_\beta k_{\alpha\beta}^0], \tag{12}$$

the stress and moment resultants can be related to strains of the middle surface through

$$\begin{Bmatrix} \bar{N} \\ \bar{M} \end{Bmatrix} = \begin{bmatrix} \bar{A}' & \bar{B}' \\ \bar{B}' & \bar{D}' \end{bmatrix} \begin{Bmatrix} \bar{\varepsilon} \\ \bar{k} \end{Bmatrix}, \tag{13}$$

where these matrices are functions of the well-known extensional, coupling and bending sub-matrices [12],

$$[A_{ij},\ B_{ij},\ D_{ij}] = \int_{-h/2}^{h/2} Q_{ij}[1,\ z,\ z^2]\,\mathrm{d}z, \quad i,j = 1,2,6 \tag{14}$$

as shown in Appendix A.

3. NUMERICAL RESULTS

In this section, several comparisons are shown, aimed to illustrate results obtained by applying Donnell's, Love's, Sanders' and Flügge's thin shell theories to stiffened cylinders.

In the following, it is assumed that $\alpha = x$, the axial co-ordinate of the cylinder, while $\beta = R\theta$; as a consequence, $R_\alpha \to \infty$ and $R_\beta = R$.

In every case, the cylinder is assumed to be simply supported at both ends, a boundary condition which is satisfied by the following functions for axial, circumferential and radial displacements [17]:

$$u^0(\alpha, \beta) = U_{mn} \cos(m\pi x/L) \sin(n\pi\beta/(2\pi R)),$$

$$v^0(\alpha, \beta) = V_{mn} \sin(m\pi x/L) \cos(n\pi\beta/(2\pi R)),$$

$$w(\alpha, \beta) = W_{mn} \sin(m\pi x/L) \sin(n\pi\beta/(2\pi R)), \tag{15}$$

where U_{mn}, V_{mn} and W_{mn} are unknown constant coefficients, while m and n are the number of half-waves along the axial and circumferential directions respectively.

The stiffness matrix obtained by using displacements (15) and the four thin shell theories described in section 2 is written in Appendix B. For stiffened cylinders the mass matrix has been evaluated according to the averaging approach, i.e., the mass of both rings and stringers has been smeared over the surface of the shell, and rotatory inertia has not been taken into account.

3.1. ALUMINUM STIFFENED CYLINDERS

Some numerical comparisons have been performed to show results obtained by using Donnell's, Love's, Sanders' and Flügge's thin shell theories in evaluating natural frequencies of stiffened cylinders with 20 rings and 20 stringers; modifications to the stiffness matrix due to the presence of stiffeners are presented in Appendix C. An aluminum cylinder ($E = 7{\cdot}1 \times 10^{10}$ N/m^2, $\rho = 2700$ kg/m^3, $\nu = 0{\cdot}31$) with wall thickness of 5 mm, radius of 1 m and length of 5 m has been considered. All natural frequencies have been evaluated for one axial half-wave, i.e., $m = 1$.

Tables 1 and 2 show normalized natural frequencies, $\omega_{1n}/\omega_{1n,Love}$, where $\omega_{1n,Love}$ is calculated by using Love's theory. The comparison of normalized values allows one to understand quickly what is the extent of the "error" associated to each theory if the result obtained by using Love's theory is assumed to be correct.

Table 1 shows normalized natural frequencies for the unstiffened cylinder and high-lights that in this case Donnell's theory provides results up to 3·5% higher with respect to the corresponding Love's natural frequencies.

Results of Table 2 are related to a structure with stringers only, rings only and both stringers and rings according to the label at the heading of the columns (in the table, $a = 5$ mm is the base of the stiffener and b is its height). Several scenarios have been considered, in which the height over base ratio ranges from 1 to 8. This table clearly shows that natural frequencies of the cylinder stiffened by stringers only and evaluated through

TABLE 1

Normalized natural frequencies $\omega_{1n}/\omega_{1n,Love}$ for an aluminum unstiffened cylinder

$n/2$	Donnell	Sanders	Flügge
1	1·0000	1·0000	1·0000
2	1·0011	0·9999	1·0000
3	1·0107	0·9997	0·9999
4	1·0322	0·9995	0·9997
5	1·0351	0·9996	0·9998
6	1·0273	0·9998	0·9999

TABLE 2

Normalized natural frequencies $\omega_{1n}/\omega_{1n,Love}$ for an aluminum stiffened cylinder

		Only stringers			Only rings			Stringers and rings		
b/a	$n/2$	Donnell	Sanders	Flügge	Donnell	Sanders	Flügge	Donnell	Sanders	Flügge
1	1	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
	2	1·0011	0·9999	1·0000	1·0011	0·9999	1·0000	1·0011	0·9999	1·0000
	3	1·0105	0·9997	0·9999	1·0111	0·9997	0·9999	1·0110	0·9997	0·9999
	4	1·0320	0·9995	0·9998	1·0334	0·9995	0·9998	1·0331	0·9995	0·9998
	5	1·0350	0·9996	0·9998	1·0356	0·9996	0·9999	1·0355	0·9996	0·9999
	6	1·0272	0·9998	0·9999	1·0274	0·9998	0·9999	1·0274	0·9998	0·9999
2	1	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
	2	1·0011	0·9999	1·0000	1·0012	0·9999	1·0000	1·0012	0·9999	1·0000
	3	1·0104	0·9997	0·9999	1·0149	0·9997	0·9999	1·0145	0·9997	1·0000
	4	1·0317	0·9995	0·9998	1·0397	0·9996	1·0000	1·0392	0·9996	1·0000
	5	1·0349	0·9996	0·9998	1·0377	0·9997	1·0001	1·0375	0·9997	1·0001
	6	1·0272	0·9998	0·9999	1·0280	0·9999	1·0001	1·0280	0·9999	1·0001
5	1	1·0000	1·0000	1·0002	0·9997	1·0000	1·0000	0·9998	1·0000	1·0002
	2	1·0010	0·9999	1·0004	1·0068	0·9999	1·0001	1·0065	0·9999	1·0005
	3	1·0099	0·9997	1·0004	1·0575	0·9998	1·0009	1·0547	0·9998	1·0012
	4	1·0305	0·9995	1·0001	1·0622	0·9999	1·0013	1·0614	0·9999	1·0014
	5	1·0344	0·9996	0·9999	1·0425	1·0000	1·0012	1·0423	1·0000	1·0012
	6	1·0271	0·9998	0·9999	1·0297	1·0000	1·0012	1·0297	1·0000	1·0012
8	1	1·0000	1·0000	1·0006	0·9995	1·0000	1·0000	0·9997	1·0000	1·0006
	2	1·0010	0·9999	1·0011	1·0265	0·9999	1·0006	1·0242	1·0000	1·0016
	3	1·0093	0·9998	1·0012	1·0991	0·9999	1·0031	1·0949	0·9999	1·0034
	4	1·0290	0·9995	1·0008	1·0686	1·0000	1·0031	1·0681	1·0000	1·0032
	5	1·0337	0·9996	1·0002	1·0445	1·0000	1·0029	1·0444	1·0000	1·0030
	6	1·0268	0·9998	1·0000	1·0313	1·0000	1·0029	1·0313	1·0000	1·0029

Donnell's theory are up to 3·5% higher than corresponding values calculated with the other theories, as well as in the previous case of the unstiffened cylinder. In contrast, by considering the cylinder stiffened by rings only, the error due to Donnell's theory can be higher than in the previous case, by up to 9·9% of the corresponding Love's natural frequencies.

Finally, results listed in the last three columns of Table 2 and related to a cylinder stiffened with both stringers and frames (the previous two stiffeners are superimposed in the structure under analysis) show that the error associated with Donnell's theory can be up to 9·5% higher than the corresponding Love's natural frequency.

In all these cases, it is interesting to underline that Love's, Sanders' and Flügge's theories give very close results. Moreover, this analysis shows that errors in results obtained through Donnell's theory are related to the presence of rings.

3.2. COMPOSITE STIFFENED CYLINDERS

In this section, a family of thin composite cylinders stiffened with 20 stringers and 20 rings is considered. Moreover, geometric and material properties are: $R = 1$ m, $h = 0{\cdot}01$ m; the cylinders are composed of two graphite-epoxy layers with properties $E_1/E_2 = 40$, $G_{12}/E_2 = 0{\cdot}5$, $v_{12} = 0{\cdot}25$ and orientations of 0 and 90°. It has been assumed that both rings and stringers are made with the same graphite-epoxy material oriented along the axis of the stiffener.

In reference [17], Soldatos analyzed natural frequencies of the unstiffened composite cylinders with geometric and material properties previously listed, and demonstrated that eigenfrequencies provided by Donnell's theory are usually higher than results provided by the other theories. The magnitude of this difference depends on the length-to-radius ratio of the structure: for very short shells ($L/R < 2$) it is very low, while for longer shells (e.g., $L/R = 10$) the difference is up to 8·5%.

Accordingly, Table 3 shows normalized natural frequencies (similar to the previous example) for the unstiffened composite cylinder, with m set to 1, as a function of the circumferential number of half-waves and for length-to-radius ratios of 1 and 10, confirming that for this structure Donnell's theory gives a value up to 8·8% higher than the corresponding value obtained through Love's theory for $L/R = 10$, while for the shorter cylinder the difference is up to 1·2%.

TABLE 3

Normalized natural frequencies $\omega_{1n}/\omega_{1n,\,Love}$ for a composite unstiffened cylinder

L/R	$n/2$	Donnell	Sanders	Flügge
1	1	1·0012	1·0000	0·9999
	2	1·0021	1·0000	1·0000
	3	1·0031	0·9999	1·0001
	4	1·0052	0·9999	1·0003
	5	1·0086	0·9998	1·0009
	6	1·0122	0·9998	1·0018
10	1	1·0016	1·0000	0·9998
	2	1·0295	0·9999	0·9998
	3	1·0884	0·9999	1·0030
	4	1·0661	0·9999	1·0046
	5	1·0437	1·0000	1·0048
	6	1·0309	1·0000	1·0049

TABLE 4

Normalized natural frequencies $\omega_{1n}/\omega_{1n,Love}$ for a composite stiffened cylinder ($L/R = 1$)

		Only stringers			Only rings			Stringers and rings		
b/a	$n/2$	Donnell	Sanders	Flügge	Donnell	Sanders	Flügge	Donnell	Sanders	Flügge
1	1	1·0012	1·0000	0·9999	1·0013	1·0000	0·9999	1·0013	1·0000	0·9999
	2	1·0021	1·0000	1·0000	1·0022	1·0000	1·0000	1·0022	1·0000	1·0000
	3	1·0031	0·9999	1·0001	1·0035	0·9999	1·0001	1·0035	0·9999	1·0001
	4	1·0050	0·9999	1·0004	1·0062	0·9999	1·0004	1·0060	0·9999	1·0004
	5	1·0082	0·9998	1·0009	1·0103	0·9998	1·0010	1·0098	0·9998	1·0011
	6	1·0116	0·9998	1·0018	1·0142	0·9998	1·0021	1·0135	0·9998	1·0021
2	1	1·0012	1·0000	1·0000	1·0016	1·0000	0·9999	1·0016	1·0000	0·9999
	2	1·0020	1·0000	1·0001	1·0033	1·0000	1·0000	1·0032	1·0000	1·0001
	3	1·0028	0·9999	1·0003	1·0070	0·9999	1·0001	1·0064	0·9999	1·0003
	4	1·0044	0·9999	1·0006	1·0144	0·9999	1·0008	1·0126	0·9999	1·0010
	5	1·0069	0·9998	1·0011	1·0216	0·9999	1·0020	1·0188	0·9999	1·0021
	6	1·0097	0·9998	1·0019	1·0233	0·9999	1·0033	1·0209	0·9999	1·0031
5	1	1·0010	1·0000	0·9997	1·0043	1·0000	0·9998	1·0035	1·0000	0·9996
	2	1·0013	1·0000	0·9993	1·0164	1·0000	0·9999	1·0112	1·0000	0·9993
	3	1·0015	1·0000	0·9989	1·0453	0·9999	1·0022	1·0272	1·0000	1·0005
	4	1·0019	1·0000	0·9988	1·0551	1·0000	1·0063	1·0380	1·0000	1·0037
	5	1·0026	0·9999	0·9988	1·0452	1·0000	1·0084	1·0369	1·0000	1·0064
	6	1·0034	0·9999	0·9991	1·0352	1·0000	1·0091	1·0316	1·0000	1·0079
8	1	1·0006	1·0000	0·9966	1·0090	1·0000	0·9997	1·0047	1·0000	0·9967
	2	1·0006	1·0000	0·9933	1·0470	1·0000	1·0007	1·0193	1·0000	0·9943
	3	1·0006	1·0000	0·9916	1·0893	1·0000	1·0100	1·0413	1·0000	0·9990
	4	1·0007	1·0000	0·9910	1·0739	1·0000	1·0161	1·0493	1·0000	1·0073
	5	1·0009	1·0000	0·9909	1·0551	1·0000	1·0179	1·0453	1·0000	1·0130
	6	1·0012	1·0000	0·9911	1·0432	1·0000	1·0185	1·0391	1·0000	1·0158

Tables 4 and 5 show results related to the composite cylinder for both the length-to-radius ratios and stiffened with stringers only, rings only and both stringers and rings according to the label at the heading of the columns (in the table, $a = 10$ mm is the base of the stiffener and b is its height). Similar to the previous section, several scenarios have been considered, in which the height over base ratio ranges from 1 to 8.

These tables clearly show that natural frequencies of the composite cylinder stiffened by stringers only and evaluated through Donnell's theory are affected by an error with the same magnitude of the unstiffened cylinder for both L/R ratios.

In contrast, when rings are used to increase the stiffness of the cylinder under analysis natural frequencies evaluated by Donnell's theory are up to 8·9 and 33% higher than the corresponding natural frequencies evaluated with Love's theory for $L/R = 1$ and 10, respectively. It is interesting to observe that in both cases the presence of both stringers and rings leads to lower differences with respect to corresponding cases of rings only.

3.3. COMPARISON WITH FE RESULTS

The last numerical comparison has been performed by using NASTRAN to develop a finite element model of a simple fuselage structure without the floor. The model has been built by using plate elements for both the cylinder and stiffeners, the latter being connected to the cylinder through the use of rigid-body element (RBE) allowing to establish the

TABLE 5

Normalized natural frequencies $\omega_{1n}/\omega_{1n,Love}$ for a composite stiffened cylinder ($L/R = 10$)

		Only stringers			Only rings			Stringers and rings		
b/a	$n/2$	Donnell	Sanders	Flügge	Donnell	Sanders	Flügge	Donnell	Sanders	Flügge
1	1	1·0016	1·0000	0·9998	1·0019	1·0000	0·9998	1·0019	1·0000	0·9998
	2	1·0287	0·9999	0·9999	1·0382	0·9999	0·9999	1·0372	0·9999	1·0000
	3	1·0871	0·9999	1·0030	1·0960	0·9999	1·0034	1·0948	0·9999	1·0034
	4	1·0659	0·9999	1·0046	1·0670	1·0000	1·0047	1·0668	1·0000	1·0047
	5	1·0437	1·0000	1·0048	1·0440	1·0000	1·0049	1·0439	1·0000	1·0049
	6	1·0309	1·0000	1·0049	1·0311	1·0000	1·0049	1·0311	1·0000	1·0049
2	1	1·0016	1·0000	0·9999	1·0047	1·0000	0·9998	1·0046	1·0000	0·9999
	2	1·0279	0·9999	1·0002	1·1064	0·9999	1·0009	1·1016	0·9999	1·0011
	3	1·0855	0·9999	1·0031	1·1177	1·0000	1·0041	1·1166	1·0000	1·0042
	4	1·0656	0·9999	1·0046	1·0693	1·0000	1·0046	1·0692	1·0000	1·0046
	5	1·0437	1·0000	1·0048	1·0448	1·0000	1·0046	1·0448	1·0000	1·0046
	6	1·0309	1·0000	1·0049	1·0318	1·0000	1·0046	1·0318	1·0000	1·0046
5	1	1·0016	1·0000	1·0005	1·0504	1·0000	0·9998	1·0478	1·0000	1·0005
	2	1·0252	0·9999	1·0018	1·2952	1·0000	1·0079	1·2872	1·0000	1·0080
	3	1·0793	0·9999	1·0040	1·1321	1·0000	1·0094	1·1317	1·0000	1·0094
	4	1·0644	0·9999	1·0048	1·0745	1·0000	1·0095	1·0744	1·0000	1·0095
	5	1·0434	1·0000	1·0049	1·0495	1·0000	1·0095	1·0495	1·0000	1·0095
	6	1·0308	1·0000	1·0049	1·0364	1·0000	1·0095	1·0364	1·0000	1·0095
8	1	1·0016	1·0000	1·0013	1·1788	1·0000	0·9998	1·1639	1·0000	1·0013
	2	1·0226	0·9999	1·0042	1·3354	1·0000	1·0177	1·3305	1·0000	1·0176
	3	1·0717	0·9999	1·0058	1·1400	1·0000	1·0186	1·1399	1·0000	1·0186
	4	1·0623	0·9999	1·0053	1·0818	1·0000	1·0187	1·0818	1·0000	1·0187
	5	1·0429	1·0000	1·0050	1·0568	1·0000	1·0188	1·0568	1·0000	1·0188
	6	1·0307	1·0000	1·0049	1·0432	1·0000	1·0188	1·0435	1·0000	1·0188

TABLE 6

Natural frequencies (Hz) for a simple fusolage structure evaluated through both a FE model and the considered shell theories

(m, n)	NASTRAN	Donnell	Sanders	Love	Flügge
(1, 2)	21·91	24·92(1·13)	21·97(1·00)	21·97(1·00)	21·97(1·00)
(1, 4)	36·99	51·27(1·39)	39·27(1·06)	39·27(1·06)	38·87(1·05)
(2, 2)	71·57	73·05(1·02)	71·87(1·00)	71·87(1·00)	71·87(1·00)
(2, 4)	46·37	59·24(1·28)	48·83(1·05)	48·83(1·05)	48·57(1·05)
(3, 4)	68·24	78·26(1·15)	70·59(1·03)	70·59(1·03)	70·43(1·03)

junction between different structural elements. To ensure a correct description of the dynamic behavior of the fuselage up to 200 Hz, about 30 000 elements have been used.

The fuselage, made in aluminum ($E = 7{\cdot}1 \times 10^{10}$ N/m^2, $\rho = 2700$ kg/m^3), has length of 16 m, radius of 1·3 m and wall thickness of 1·2 mm. It is stiffened by 40 rings and 40 stringers that are equally spaced; the former have I-shaped cross-section with the following properties: position of the center or mass, 0·045 m (referred to the fuselage sidewall middle plane); moment of inertia, $2{\cdot}966 \times 10^{-7}$ m^4; area, $2{\cdot}52 \times 10^{-4}$ m^2. Stringers have T-shaped cross-section with the following properties: position of the center of mass, 0·00734 m (referred to the fuselage sidewall middle plane); moment of inertia; $1{\cdot}577 \times 10^{-9}$ m^4; area, $4{\cdot}88 \times 10^{-5}$ m^2.

In Table 6, natural frequencies of corresponding eigenmodes evaluated by using NASTRAN and the four theories under analysis are compared. In order to permit a faster comparison, analytical results have been normalized (in parentheses) with respect to the corresponding result evaluated through NASTRAN.

It is evident that results obtained by using Donnell's theory are inaccurate, giving rise to a maximum difference of about 39% in mode (1, 4). Conversely, the other three theories provide results in good agreement with those obtained by using NASTRAN.

4. DISCUSSION

In the previous section, it has been demonstrated through numerical evidence that Donnell's theory may give poor results with respect to the other three theories in some cases.

To understand the reason for such errors, it is interesting to look at Appendix B, where the elements of the stiffness matrix are shown. Clearly, to justify the poor results provided by Donnell's theory it is necessary to look at terms that are neglected by Donnell and considered by the other three theories. Moreover, it is necessary to pay attention to terms affected by the presence of rings and stringers: A'_{11}, B'_{11}, D'_{11}, A'_{22}, B'_{22} and D'_{22} according to Appendix C. It is easy to see that such terms may be D'_{22}/R^4 and B'_{22}/R^3, both related to the presence of rings.

Furthermore, it should be recalled that Donnell's theory provides results within an engineering accuracy of 5% in the case of homogeneous isotropic shells: for such structures the previous two terms should be negligible.

As a consequence, to understand when Donnell's theory can be used, it is possible to evaluate the ratio between D'_{22}/R^4 and A'_{22}/R^2, the first being neglected by Donnell with respect to the second according to terms $K'_{2,2}$ and $K'_{2,3}$ of the stiffness matrix in Appendix B. Indeed, in the particular case of unstiffened homogeneous isotropic shells and upon assuming $\delta_F = 0$,

$$\frac{D'_{22}/R^4}{A'_{22}/R^2} = \frac{Eh^3/12(1-\nu^2)R^4}{Eh/(1-\nu^2)R^2} = \frac{(h/R)^2}{12} \ll 1. \tag{16}$$

As a result, the following ratio can be introduced:

$$\gamma_D = \frac{D'_{22}/R^4}{A'_{22}/R^2}\,\frac{12}{(h/R)^2} = \frac{12\,D'_{22}}{A'_{22}h^2}. \tag{17}$$

This is equal to 1 for unstiffened homogeneous isotropic shells (when D'_{22} and $A'_{22}h^2$ have the same order of magnitude) and can be used to determine the accuracy of Donnell's theory if applied to orthotropic shells.

When composite or stiffened cylinders are considered, γ_D is calculated by using both material and geometric properties and its value will no longer be equal to unity. In particular, if $\gamma_D \gg 1$ the term D'_{22} will have a greater magnitude than $A'_{22}h^2$ such that Donnell's theory will skip large terms in the stiffness matrix giving rise to errors in the evaluation of the dynamic properties of the structure. In contrast, if γ_D is low Donnell's theory will not consider only negligible terms.

The importance of the ratio γ_D is demonstrated by the previous numerical examples. Indeed, for the aluminum cylinder with rings analyzed in section 3.1 and $b/a = 8$, the ratio γ_D is higher than 33† and for the composite cylinder of section 3.2 γ_D is equal to 24·43 and

†According to equation (17) and Appendix A, γ_D is given by

$$\gamma_D = 12\,\frac{D_{22} + D^{+}_{22}}{(A_{22} + A^{+}_{22})h^2},$$

due to the presence of the rings; by recalling geometrical and material properties of the structure it follows that $A_{22} = Eh/(1-\nu^2) = 3{\cdot}927 \times 10^8$ N/m, $D_{22} = Eh^3/12(1-\nu^2) = 818{\cdot}2$ N m and, according to Appendix C, $A^{+}_{22} = N_r Eba/L = 56{\cdot}8 \times 10^6$ N/m, $D^{+}_{22} = N_r Eab^3/3L = 30293{\cdot}2$ N m so that $\gamma_D = 33{\cdot}22$.

85·56 for ratios b/a of 5 and 8, respectively; in these cases, D'_{22}/R^4 has great importance is elements $K'_{2,2}$ and $K'_{2,3}$.

Finally, the case of the fuselage structure analyzed in section 3.3. shows the importance of γ_D: for this stiffened cylinder, representing a real engineering solution, $\gamma_D = 8592$ so that the stiffness in the circumferential direction is due almost entirely to rings. It follows that results provided by Donnell's theory are highly inaccurate.

5. CONCLUSIONS

In this article, several results obtained through Love's, Donnell's Flügge's and Sanders' thin shell theories to evaluate natural frequencies of stiffened cylinders have been presented.

It is clearly demonstrated that in some cases Donnell's theory leads to very high errors in the evaluation of eigenfrequencies of such structures, while the other three theories provide results that are in good agreement. Moreover, numerical evidence and the introduction of the ratio γ_D show that this inaccuracy is mainly related to the stiffening effect due to rings.

This investigation is completed with the analysis of the dynamic behavior of a simple fuselage structure stiffened with both rings and stringers; by taking advantage of NASTRAN to develop an FE model of the analyzed structure and referring to natural frequencies provided by this software, it is shown that errors up to 40% can be obtained if Donnell's theory is used, while the other theories provided results within the engineering accuracy.

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REFERENCES

1. M. Mukhopadhyay and G. Sinha 1992 *Shock and Vibration Digest* **24**, 3–13. A review of dynamic behaviour of stiffened shells.
2. D. M. Egle and J. L. Sewall 1968 *American Institute of Aeronautics and Astronautics Journal* **6**, 518–526. An analysis of free vibration of orthogonally stiffened cylindrical shells with stiffeners treated as discrete elements.
3. M. Baruch and J. Singer 1963 *Journal of Mechanical Engineering Science* **5**, 23–27. The effect of eccentricity of stiffeners on the general instability of stiffened cylindrical shells under hydrostatic pressure.
4. J. A. McElman, M. M. Mikulas Jr and M. Stein 1966 *American Institute of Aeronautics and Astronautics Journal* **4**, 887–894. Static and dynamic effects of eccentric stiffening of plates and cylindrical shells.
5. J. Singer, M. Baruch and O. Harari 1966 *Journal of Mechanical Engineering Science* **8**, 363–373. Inversion of the eccentricity effect in stiffened cylindrical shells buckling under external pressure.
6. A. Rosen and J. Singer 1974 *Journal of Sound and Vibration* **34**, 357–378. Vibrations of axially loaded stiffened cylindrical shells.
7. The Apollo Program Office 1966 *Structural Systems and Program Decisions*, Vol. 1. NASA SP-6008.
8. L. R. Koval 1980 *Journal of Sound and Vibration* **71**, 511–521. On sound transmission into a stiffened cylindrical shell with rings and stringers treated as discrete elements.
9. L. D. Pope, E. G. Wilby, C. M. Willis and W. H. Mayes 1983 *Journal of Sound and Vibration* **89**, 371–417. Aircraft interior noise models: sidewall trim, stiffened structures and cabin acoustics with floor partition.
10. J. E. Cole III, A. W. Stokes, J. M. Garrelick and K. F. Martini 1988 *Analytical Modeling of the Structureborne Noise Path on a Small Twin-Engine Aircraft*. NASA CR 4136.

11. C. R. FULLER, J. P. MAILLARD, M. MERCADAL and A. H. VON FLOTOW 1997 *Journal of Sound and Vibration* **203**, 754–761. Control of aircraft interior noise using globally detuned vibration absorbers.
12. R. M. JONES 1975 *Mechanics of Composite Materials*. New York: McGraw-Hill.
13. K. KRAUS 1967 *Thin Elastic Shells*. New York: Wiley.
14. V. V. NOVOZHILOV 1964 *Thin Elastic Shells*. Groningen: P. Noordhoff.
15. J. N. REDDY 1988 *Proceedings Mechanics of Composite Materials and Structures Conference, Portugal* 1–87. Theory and analysis of laminated composite plates and shells.
16. E. CARRERA 1991 *Journal of Sound and Vibration* **150**, 405–433. The effects of shear deformation and curvature on buckling and vibrations of cross-ply laminated composite shells.
17. K. P. SOLDATOS 1984 *Journal of Sound and Vibration* **97**, 305–319. A comparison of some shell theories used for the dynamic analysis of cross-ply laminated circular cylindrical panels.
18. J. E. COLE III 1997 *Computers and Structures* **65**, 385–393. Vibrations of a framed cylindrical shell submerged in and filled with acoustic fluids: spectral solution.
19. S. A. RINEHART and J. T.-S. WANG 1972 *Journal of Sound and Vibration* **24**, 151–163. Vibration of simply supported cylindrical shells with longitudinal stiffeners.
20. W. FLÜGGE 1973 *Stresses in Shells*. Berlin: Springer-Verlag.

APPENDIX A

The stiffness submatrices $\bar{A}'$, $\bar{B}'$ and $\bar{D}'$ are given in terms of the extensional, coupling and bending submatrices defined in equation (14) as

$$A'_{11} = A_{11} + A^+_{11} + \delta_F\left(\left(\frac{1}{R_\beta} - \frac{1}{R_\alpha}\right)(B_{11} + B^+_{11}) - \frac{D_{11} + D^+_{11}}{R_\alpha R_\beta}\right),$$

$$A'_{12} = A_{12},$$

$$A'_{22} = A_{22} + A^+_{22} + \delta_F\left(\left(\frac{1}{R_\alpha} - \frac{1}{R_\beta}\right)(B_{22} + B^+_{22}) - \frac{D_{22} + D^+_{22}}{R_\alpha R_\beta}\right),$$

$$A'_{66} = A_{66} + \delta_F\left(D_{66}\left(\frac{1}{R^2_\alpha} + \frac{1}{R_\alpha R_\beta} + \frac{1}{R^2_\beta}\right)\right),$$

$$B'_{11} = B_{11} + B^+_{11} + \delta_F\left(\frac{1}{R_\beta} - \frac{1}{R_\alpha}\right)(D_{11} + D^+_{11}),$$

$$B'_{12} = B_{12},$$

$$B'_{22} = B_{22} + B^+_{22} + \delta_F\left(\frac{1}{R_\alpha} - \frac{1}{R_\beta}\right)(D_{22} + D^+_{22}),$$

$$B'_{66} = B_{66} - 3/2\delta_F D_{66}\left(\frac{1}{R_\alpha} + \frac{1}{R_\beta}\right),$$

$$D'_{11} = D_{11} + D^+_{11},$$

$$D'_{12} = D_{12},$$

$$D'_{22} = D_{22} + D^+_{22},$$

$$D'_{66} = D_{66},$$

where additional terms A^+, B^+ and D^+ are related to the stiffening elements as outlined in Appendix C.

APPENDIX B

The stiffness matrix $[K]$ of the cylinder with simply supported boundary conditions is given where displacements have expression (15); in the following $[K'] = [K]/(\pi RL/2)$ and $j = n/2$.

$$K'_{1,1} = \frac{m^2\pi^2 A'_{11}}{L^2} + \frac{j^2 A'_{66}}{R^2} + \delta_S\left(-\frac{j^2 B'_{66}}{R^3} + 1/4\,\frac{j^2 D'_{66}}{R^4}\right),$$

$$K'_{1,2} = \frac{m\pi j A'_{12}}{LR} - \frac{m\pi j A'_{66}}{LR} + \frac{m\pi j B'_{12}}{LR^2}(\delta_L + \delta_S + \delta_F) + \frac{m\pi j B'_{66}}{LR^2}(\delta_L + \delta_S + 2\delta_F)$$
$$- 3/4\,\frac{m\pi j D'_{66}\delta_S}{LR^3},$$

$$K'_{1,3} = \frac{m\pi j^2 D'_{66}\delta_S}{LR^3} - \frac{m\pi j^2 B'_{12}}{LR^2} - 2\frac{m\pi j^2 B'_{66}}{LR^2} - \frac{m^3\pi^3 B'_{11}}{L^3} - \frac{m\pi A'_{12}}{LR},$$

$$K'_{2,2} = \frac{j^2 A'_{22}}{R^2} + \frac{m^2\pi^2 A'_{66}}{L^2} + \frac{m^2\pi^2 B'_{66}}{L^2 R}(4\delta_F + 3\delta_S + 2\delta_L)$$
$$+\left(2\frac{j^2 B'_{22}}{R^3} + \frac{j^2 D'_{22}}{R^4}\right)(\delta_F + \delta_L + \delta_S) + \frac{m^2\pi^2 D'_{66}}{L^2R^2}(\delta_L + 9/4\delta_S + 4\delta_F),$$

$$K'_{2,3} = -\frac{j^3 B'_{22}}{R^3} - \frac{j A'_{22}}{R^2} - 2\frac{m^2\pi^2 j B'_{66}}{L^2 R} - \frac{m^2\pi^2 j B'_{12}}{L^2 R} - \frac{m^2\pi^2 j D'_{66}}{L^2R^2}(2\delta_L + 3\delta_S + 4\delta_F)$$
$$-\left(\frac{j^3 D'_{22}}{R^4} + \frac{m^2\pi^2 j D'_{12}}{L^2R^2} + \frac{j B'_{22}}{R^3}\right)(\delta_F + \delta_L + \delta_S),$$

$$K'_{3,3} = 2\frac{j^2 B'_{22}}{R^3} + \frac{m^4\pi^4 D'_{11}}{L^4} + 2\frac{m^2\pi^2 B'_{12}}{L^2 R} + \frac{j^4 D'_{22}}{R^4} + \frac{A'_{22}}{R^2} + 2\frac{m^2\pi^2 j^2 D'_{12}}{L^2R^2} + 4\frac{m^2\pi^2 j^2 D'_{66}}{L^2R^2}.$$

APPENDIX C

The effect of both rings and stringers can be taken into account as shown by McElman *et al.* [4]; the same procedure was applied by NASA during the design of the Saturn V launch vehicle [7].

The stiffening effect can be represented by adding the average potential energy of deformation due to rings and stringers to the energy related to the cylinder. Moreover, according to Novozhilov [14], it is assumed that rings possess bending and extensional stiffness in their own plane, such that resistance to displacements perpendicular to their plane and torsion can be neglected. Accordingly, stringers possess bending and extensional stiffness, while torsion is neglected.

Under these assumptions, it can be demonstrated that the stiffening effect due to stringers and rings can be represented by introducing additional terms into the extensional, coupling and bending submatrices. If N_r equally spaced rings are present, the additional terms are

$$[A^+_{22},\ B^+_{22},\ D^+_{22}] = \frac{N_r E_r}{L}\int_{A_r}[1,\ z,\ z^2]\,\mathrm{d}A_r.$$

Similarly, the presence of N_s equally spaced stringers gives the following additional terms:

$$[A^+_{11},\ B^+_{11},\ D^+_{11}] = \frac{N_s E_s}{2\pi R}\int_{A_s}[1,\ z,\ z^2]\,\mathrm{d}A_s.$$