



A RECURSIVE APPROACH TO COMPUTE NORMAL FORMS

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Normal forms are instrumental in the analysis of dynamical systems described by ordinary differential equations, particularly when singularities close to a bifurcation are to be characterized. However, the computation of a normal form up to an arbitrary order is numerically hard. This paper focuses on the computer programming of some recursive formulas developed earlier to compute higher order normal forms. A computer program to reduce the system to its normal form on a center manifold is developed using the Maple symbolic language. However, it should be stressed that the program relies essentially on recursive numerical computations, while symbolic calculations are used only for minor tasks. Some strategies are proposed to save computation time. Examples are presented to illustrate the application of the program to obtain high order normalization or to handle systems with large dimension.

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1. INTRODUCTION

The idea of the normal form method is to employ co-ordinate transformations to systematically construct the simplest possible form of the original autonomous differential equations [1–5]. Such transformations are given in terms of formal power series. In general, only some finite truncation of the series can be found. The simplest form, or an approximation of it obtained by some finite truncation, is expected to preserve the dynamical properties of the original system in the neighborhood of the bifurcation point [6]. Being simpler than the original equations, the normal form is more amenable to analysis. One application of normal forms is the analysis of oscillations. For example, weakly non-linear vibrations were studied by normal form theory in reference [7].

However, finding a normal form for a given non-linear system is not a simple task. Explicit formulas for higher order normal forms in terms of the coefficients of the original system are extremely involved. Therefore, several researchers have tried to develop computer programs for some specific bifurcations [8–10]. Similar complexity problems arise in asymptotic methods of non-linear analysis [11] which are known to bear a close relationship with the normal form method.

The normal form theory is usually applied in conjunction with the center manifold theory [12–17]. By successive non-linear transformations, it reduces the original system to a center manifold associated with the critical modes at a bifurcation point. The reduced system on the center manifold usually has much smaller dimension than the original system and therefore is simpler to analyze. The dynamics on a center manifold represent the asymptotic behavior of the system.

This paper focuses on the implementation of the recursive formulas developed in reference [13] as a computer program that is able to successively calculate normal forms of arbitrarily increasing order (i.e., forms that are normal up to a given polynomial order). Early attempts to implement the formulas were restricted to systems with small dimensions or to normal forms of low order, using Fortran [18]. Here, a more general implementation is achieved using the Maple symbolic computer package. However, it should be stressed that the resulting program is essentially numeric and only uses symbolic calculations in minor tasks. We believe that a relevant contribution of this paper is in showing by use of the recursive formulas how unnecessary computations can be eliminated beforehand, thus saving computation time. This is particularly meaningful for higher order normalization and/or systems with large dimensions.

2. NORMAL FORM THEORY

The problem of finding the simplest form, or normal form, to which equation (1) can be reduced by means of co-ordinate transformations, has been considered by several authors since Poincaré’s works, some hundred years ago. A recent version of the normal form was given in 1964 by Briuno [1].

Let $\psi_1(z), \dots, \psi_n(z)$ be power series in z_1, \dots, z_n without constant terms, converging in some neighborhood of $z = [z_1, \dots, z_n]^T = 0$. Then $z = 0$ is a singular point of the system

$$\dot{z}_i = \psi_i(z), \quad i = 1, \dots, n. \tag{1}$$

Theorem 1 (Briuno [1]). *For every system (1) there exists an invertible transformation*

$$z_i = x_i \sum_{q \in N_i} h_{iq} x^q, \quad i = 1, \dots, n, \tag{2}$$

reducing system (1) to the normal form

$$\dot{x}_i = x_i \sum_{q \in \lambda=0} g_{iq} x^q, \tag{3}$$

where $q = [q_1, \dots, q_n] \in N_i$; $N_i = \{q \text{ integer vector, } q_1 + \dots + q_n \geq 0 \text{ and } q_i \geq -1, q_k \geq 0 \text{ if } k \neq i\}$; $i = 1, \dots, n$; $x^q = x_1^{q_1} x_2^{q_2} \dots x_n^{q_n}$; h_{iq} and g_{iq} are constant complex numbers; $\lambda = [\lambda_1 \dots \lambda_n]^T$ is the vector of eigenvalues of the linear part of system (1). The right-hand side of equation (3) contains only resonant terms, i.e. terms satisfying $q\lambda = q_1\lambda_1 + \dots + q_n\lambda_n = 0$. Further, the linear part of the normal form (3) is in Jordan’s canonical form

When the linear part of system (1) is diagonal, the normal form (3) can be rewritten as

$$\dot{x}_i = \lambda_i x_i + \sum_{\lambda_{vi}=0} G_{iv} x^v, \quad i = 1, \dots, n, \tag{4}$$

where $v = [v_1 \dots v_n]$ is an integer vector with $v_i \geq 0$ and $\lambda_{vi} = v\lambda - \lambda_i = 0$ represents the resonance condition, which means that the summation in expression (4) includes only resonant terms.

The convergence of the normal transformation (2) was studied by Briuno. Generally, the normal transformation has to be considered in a formal sense because convergence is not always true (see also reference [19]). However, independent of convergence, the

approximations obtained by truncating higher order terms are useful as asymptotic approximations, as in the case of the celebrated Krilov–Bogoliubov’s averaging method [11].

2.1. REDUCED SYSTEM AND ASYMPTOTIC ANALYSIS

System (1) can be transformed to

$$\begin{aligned} \dot{u} &= Au + f(u, v), \\ \dot{v} &= Bv + g(u, v), \quad g(u, 0) = 0, \end{aligned} \tag{5}$$

by means of a transformation (2) where $x = [u \ v]^T$; $u \in \mathbb{R}^l$, $v \in \mathbb{R}^m$; f and g represent the non-linear terms; the eigenvalues of A and B are critical and non-critical respectively. The variables u_1, \dots, u_l are critical and v_1, \dots, v_m are non-critical. The invariant manifold $v = 0$ corresponds to a center manifold and the reduced system

$$\dot{u} = Au + f(u, 0)$$

corresponds to the restriction of the complete system (5) to this invariant manifold. It is well known that the reduced system contains the essential asymptotic properties of the complete system and the center manifold theorem allows us to reduce the dimension of a problem at a bifurcation point [12, 14].

2.2. NORMAL FORM AND BIFURCATION ANALYSIS

Consider the system

$$\dot{x} = F(x, \alpha) = A(\alpha)x + f(x, \alpha), \tag{6}$$

where f is analytical in its arguments, strictly non-linear in x and $f(0, \alpha) = 0$. The system described by equation (6) depends on the control parameter vector α . Suppose that for $\alpha = \alpha^c$, the matrix $A(\alpha^c)$ has l critical eigenvalues and m non-critical eigenvalues. To study system (6) around the bifurcation value α^c , one has to retain the almost resonant terms in order to avoid small divisors [13]. To avoid the practical difficulty of calculating the normal form for varying control parameter, a very simple and efficient solution [13, 16, 17] is to think of a perturbation scheme, since α is supposed to be near the critical value α^c . Define a small parameter vector ε from

$$\alpha = \alpha^c + \varepsilon;$$

then, instead of equation (6), we consider the augmented system

$$\dot{x} = \bar{F}(x, \varepsilon), \quad \dot{\varepsilon} = 0. \tag{7}$$

A similar strategy could be used for periodic or almost periodic systems in time [13], e.g., the augmented system

$$\dot{x} = \bar{F}(x, u), \quad \ddot{u} = -\Omega^2 u, \tag{8}$$

where Ω is a positive constant, represents a periodic system with period $T = 2\pi/\Omega$ but can be seen as an autonomous (time-invariant) system.

3. THE RECURSIVE FORMULAS

This section briefly describes the recursive normalization formulas. These formulas give the normal form and the corresponding transformation which leads to a normalized reduced system of form (5), restricted to a center manifold, up to any desired order of normalization \bar{N} .

The following notation is used.

- $\delta_I = [\delta_{1I} \delta_{2I} \cdots \delta_{nI}]$; where δ_{ij} is the Kronecker's δ -function;
- $x = [x_1 \cdots x_n]^T$ or $x = \{x_i\}$, $i = 1, \dots, n$; denote n -dimensional vectors;
- $J = 1, \dots, n$; $j = 1, \dots, l$; $k = l + 1, \dots, n$; are indices with their respective ranges;
- $v = [v_1 \cdots v_n]$, $\mu = [\mu_1 \cdots \mu_l \ 0 \cdots 0]$, $\mu' = [\mu_1 \dots \mu_l]$ denote vectors of non-negative integers;
- The whole set of vectors v and μ are denoted as \mathcal{N} and \mathcal{M} , respectively, i.e., $v \in \mathcal{N}$ and $\mu \in \mathcal{M}$;
- $\phi_{Jv}^{(N)}$ and $B_{J\mu}^{(N)}$ denote complex coefficients;
- δ -notation: $\phi_{Jv}^{(N)}$ and $\phi_{JJ_1 \dots J_M}^{(N)}$ are equal if $v = \delta_{I_1} + \cdots + \delta_{I_M}$;
- $|v| = v_1 + \cdots + v_n$;
- $\sum_{|v|=N}$ is the sum over all v 's with $|v| = N$;
- $\sum_{|v|=M}^N$ is the sum over all v 's with $|v| = M, M + 1, \dots, N$;
- Other symbols: $x^v = x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n}$; $A = [\lambda_1 \cdots \lambda_l]$; $\lambda_{\mu'j} = \mu' \cdot A - \lambda_j = \mu_1 \lambda_1 + \cdots + \mu_l \lambda_l - \lambda_j$.

Assume that system (1) has l simple critical eigenvalues so that matrix A in system (5) has diagonal canonical form. The general case when A has a Jordan canonical form was considered in reference [18]. The normalization procedure begins with a linear transformation $z = Rx$, which should bring the critical part to a diagonal form:

$$\dot{x}_j = \lambda_j x_j + \sum_{|v|=1+p}^{\infty} \phi_{jv}^{(1)} x^v, \tag{9}$$

$$\{\dot{x}_k\} = A_{22} \{x_k\} + \left\{ \sum_{|v|=1+p}^{\infty} \phi_{kv}^{(1)} x^v \right\}, \tag{10}$$

where p is the *order increment* defined as follows. If the power series of system (1) have only odd powers, then $p = 2$; otherwise, $p = 1$. The critical eigenvalues are given by λ_j and the matrix A_{22} ($m \times m$) has all eigenvalues with negative real parts. The superscripts of coefficients ϕ indicate the normalization order. The order of normalization is considered to be 1 when the linear part is in block diagonal form as in equations (9) and (10). The superscripts of variable x_i in equations (9) and (10) were dropped for simplicity.

The linear transformation $z = Rx$ is constructed from the right and left critical eigenvectors u^j and v^j , $j = 1, \dots, l$, respectively, according to Appendix 2 in reference [16]. We will call as N -normal the system which is in normal form up to the N th order terms. The normalization proceeds by successive application of normal transformations (2) starting from the "diagonal" system (9) and (10), yielding increasing order of normalization. The

normal transformation to go from the $(N - p)$ -normal form, with state $x^{(N-p)}$, to the N -normal form is an almost identity transformation of the form

$$x_J^{(N-p)} = x_J^{(N)} + \sum_{|\mu|=N} B_{J\mu}^{(N)} x^{(N)\mu}, \tag{11}$$

where $N \geq 1 + p$; $p = 1$ or 2 ; $\mu_k = 0$, $k = l + 1, \dots, n$ (i.e., $\mu \in \mathcal{M}$). Note that the sum in equation (11) contains N th order monomials of critical variables only. The coefficients B of the transformation are chosen so as to normalize terms of N th order. Non-essential couplings (non-resonant terms) are then eliminated. The new system in co-ordinates $x^{(N)}$ has the same linear part as the diagonal system (9) and (10). In fact, all terms with order lower than N remain unmodified. Following this procedure, the following recursive formulas were developed in reference [13].

For $|\mu| = N$:

$$B_{j\mu}^{(N)} = \frac{\phi_{j\mu}^{(N-p)}}{\lambda_{\mu'j}}, \quad \lambda_{\mu'j} \neq 0 \quad \text{else} \quad B_{j\mu}^{(N)} = 0, \tag{12}$$

$$\{B_{k\mu}^{(N)}\} = [(\mu' A)I - A_{22}]^{-1} \{\phi_{k\mu}^{(N-p)}\}.$$

For $|\mu| \leq N$:

$$\phi_{j\mu}^{(N)} = \phi_{j\mu}^{(N-p)}, \quad \lambda_{\mu'j} = 0 \quad \text{else} \quad \phi_{j\mu}^{(N)} = 0, \tag{13}$$

$$\phi_{k\mu}^{(N)} = 0 \quad \forall \mu,$$

and the remaining coefficients ϕ are given by

$$\begin{aligned} \phi_{j\bar{v}}^{(N)} = & \phi_{j\bar{v}}^{(N-p)} - \sum_{i=1}^l \sum_{|\nu|=|\bar{v}|-N+1} (\bar{v}_i - \nu_i + 1) \phi_{i\nu}^{(N)} B_{J\bar{v}-\nu+\delta_i}^{(N)} \\ & + \sum_{M=1+p}^{|\bar{v}|-p} \sum_{I_1 \dots I_M} \phi_{JI_1 \dots I_M}^{(N-p)} C_{IML} \left[\sum_{|\mu(i')|=N; i'=1, \dots, L} B_{i_1\mu(1)}^{(N)} \dots B_{i_L\mu(L)}^{(N)} \right], \end{aligned} \tag{14}$$

where \mathcal{C}_{IML} denotes the sum of all L - combinations i_1, \dots, i_L of I_1, \dots, I_M , with L being an integer satisfying $0 < L \leq M$ and $L = (|\bar{v}| - M)/(N - 1)$ (if $(|\bar{v}| - M)/(N - 1)$ is not an integer then the combination is void), and moreover,

$$\bar{v} = \mu(1) + \dots + \mu(L) + \delta_{i_{L+1}} + \dots + \delta_{i_M}, \tag{15}$$

where $\mu(1), \dots, \mu(L)$ are integer non-negative vectors of the type $\mu(j) = [\mu_1(j) \dots \mu_l(j) 0 \dots 0]$ and $|\mu(1)| = \dots = |\mu(L)| = N$.

4. COMPUTATIONAL ISSUES

Some simple measures to save computation time are: (1) the parameter equation $\dot{\varepsilon} = 0$ needs no update in the normalizing procedure; (2) only one equation of a complex conjugate pair of equations needs update; (3) if the \bar{N} -normal form is required, the only terms of order $\bar{N} - 1$ that need update are those independent of non-critical variables (i.e., terms of the

form x^μ). However, some more difficult problems in the implementation of a numerically efficient program were found [18]:

1. define an adequate ordering of integer vectors v corresponding to the powers of a multivariable monomial of the form $x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n}$ (with non-negative powers);
2. determine the vectors $v = \mathcal{N}$ and the vectors $\mu(1), \dots, \mu(L) \in \mathcal{M}$ that enter the sums, in the second and third terms of formula (14), respectively;
3. for each normalizing transformation, skip the updating of coefficients that will not affect the final \bar{N} th normal form.

In what follows, we discuss each of these problems and propose some possible solutions.

4.1. ORDERING INTEGER VECTORS

Problem 1 is important in the reduction of the memory space required to handle the series coefficients B and ϕ . For example, with $l = n = 5$ and $\bar{N} = 9$ the memory space required by the conventional solution is $5 \times 9^5 = 295\,245$ units. If the ordering approach described below is employed the total number of monomials of order up to 9 is found to be 1206; thus, the memory space required is $5 \times 1206 = 6030$, approximately 1/49 of the memory space required by the conventional solution. The integer vectors $v = [v_1 \ v_2 \ \cdots \ v_n]$, can be generated and ordered according to reference [18] by introducing an auxiliary matrix H :

$$H_{i,1} = H_{n,k} = 1, \quad i = 1, \dots, n, \quad k = 1, \dots, \bar{N},$$

$$H_{i,k} = H_{i+1,k} + H_{i,k-1}, \quad i = 1, \dots, n-1, \quad k = 2, \dots, \bar{N}.$$

Note that, given n and \bar{N} , the matrix H , which is formed in a manner similar to Pascal's triangle of combinatorial analysis, is calculated only once in the program. For example, with $n = 3$ and $\bar{N} = 5$ we have the auxiliary matrix

$$H = \begin{bmatrix} 1 & 3 & 6 & 10 & 15 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

An important property of matrix H is that the sum of the elements of its N th column gives the number of monomials of order N (in n variables), i.e.,

$$\sum_{i=1, \dots, n} H_{i,N} = \text{number of monomials of order } N \text{ in } n \text{ variables.}$$

For example, the number of monomials of order 5 in 3 variables is $21 = 1 + 5 + 15$. The order number n_v of all vectors v of dimension n and order $|v| = N$ is given by first defining $n_v = 1$ if $v = [N \ 0 \ \cdots \ 0]$. Now, let $v_\delta = [I_1 \ I_2 \ \cdots \ I_N]$ be the δ -representation of integer vector v . Then, given v , N and n the corresponding n_v is calculated from expression

$$n_v = 1 + \sum_{i=1}^N \left[\sum_{r=1}^{I_i - I_{i-1}} H_{r, N-i+1} \right], \quad I_0 = 1. \tag{16}$$

The algorithm of Figure 1 implements the computation of equation (16).

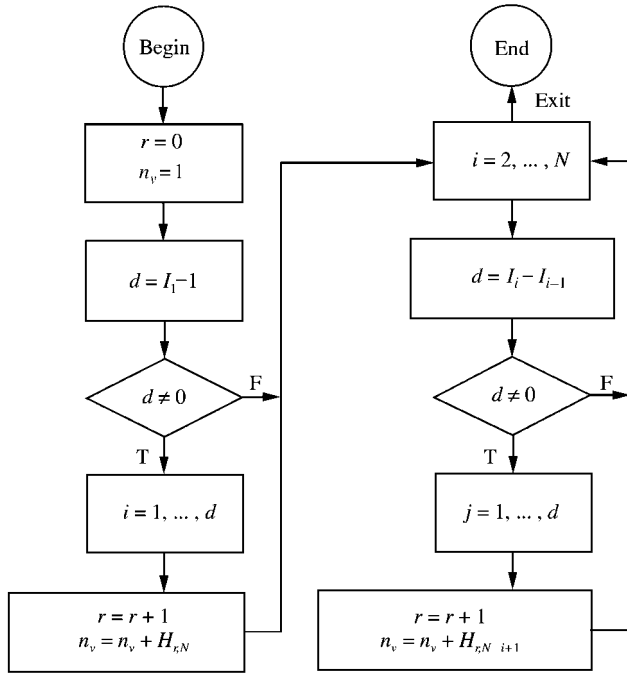


Figure 1. Program to calculate the ordering number n_v of all vectors of non-negative integers v , with $|v| = N$.

TABLE 1

Ordering all vectors of non-negative integers v with $n = 3$ and $|v| = 2, 3$

v	v_δ	$I_1 - 1$	$I_2 - I_1$	$I_3 - I_2$	N_v
[2 0 0]	[1 1]	—	—	—	1
[1 1 0]	[1 2]	0	1	—	2
[1 0 1]	[1 3]	0	2	—	3
[0 2 0]	[2 2]	1	0	—	4
[0 1 1]	[2 3]	1	1	—	5
[0 0 2]	[3 3]	2	0	—	6
[3 0 0]	[1 1 1]	—	—	—	7
[2 1 0]	[1 1 2]	0	0	1	8
⋮	⋮	⋮	⋮	⋮	⋮
[0 0 3]	[3 3 3]	2	0	0	16

It is clear from Table 1 that the differences $I_i - I_{i-1}$; $i = 1, \dots, N$ determine those elements of matrix H that enter expression (16). If $I_i - I_{i-1} = 0$, then no element of matrix H is added.

A global order number N_v for all integer vectors v with $|v| = 2, \dots, \bar{N}$ can be obtained by modifying n_v . The number N_v is calculated by simply adding to n_v all entries of the matrix H , except the first column, to the left of the $|v|$ th column. This can be expressed as

$$\text{start } N_v \leftarrow n_v ; \text{ then, for } k = 1 + p, 1 + 2p, \dots, |v| - p, \quad i = 1, \dots, n, \quad N_v \leftarrow N_v + H_{i,k} \tag{17}$$

The global ordering obtained for all v with $|v| = 2, 3$ is given in Table 1. A noteworthy property of the proposed ordering is that, if the vectors in Table 1, in direct (v) or delta (v_δ) representations, are regarded top-down as cardinal numbers, they are strictly decreasing or increasing respectively.

By construction, the correspondence between integer vector v and order number N_v is one to one. The normal form program starts by constructing a table containing all integer vectors v with $|v| = 1 + p, 1 + 2p, \dots, \bar{N}$.

4.2. PARTITIONING INTEGER VECTORS

This section relates to problem 2. Consider the second term of equation (14). Given \bar{v} and δ_i , the problem is to find vectors $v \in \mathcal{N}$, with $|v| = |\bar{v}| - N + 1$ and such that $\bar{v} - v + \delta_i = \mu \in \mathcal{M}$. This problem is equivalent to determining the complete list of vectors $\mu \in \mathcal{M}$, $|\mu| = N$ such that $\bar{v} + \delta_i - \mu = v \in \mathcal{N}$. In turn, this is equivalent to partitioning $\bar{v} + \delta_i \in \mathcal{N}$ into the sum of two terms $\mu + v$, where $\mu \in \mathcal{M}$. To solve the problem we say that (with some abuse of language and notation) μ is contained in v , denoted as $v \supseteq \mu$, if $v - \mu \in \mathcal{N}$. Thus, the list consists of all μ contained in the given $\bar{v} + \delta_i \in \mathcal{N}$ and can be obtained by extracting from the given vector all possible μ following the order defined in problem 1. A partitioning algorithm following this procedure is described in Appendix A.

The partitioning technique is illustrated by some examples given in Table 2. As a further example, note that for $v = [1 \ 1 \ 0 \ 4 \ 0]$ and $l = 3$ and $|\mu| = 3$, the result is an empty list.

Consider now the calculation of the last term of equation (14). The inner sum covers all $\mu(i')$, $i' = 1, \dots, L$ such that equation (15) holds. This is equivalent to partitioning the vector $\bar{v} - (\delta_{i_{l+1}} + \dots + \delta_{i_m})$ as a sum of L vectors in \mathcal{M} . This can be done by first testing if $\bar{v} - (\delta_{i_{l+1}} + \dots + \delta_{i_m}) \in \mathcal{N}$. If this holds, we write $v = \bar{v} - (\delta_{i_{l+1}} + \dots + \delta_{i_m})$ and find a first list $\{\mu(1)\}$ of all $\mu(1) \subseteq v$, and also the corresponding list $\{c(1)\}$, where $c(1) = v - \mu(1)$. We repeat this procedure for each element of $\{c(1)\}$ and obtain the new lists $\{\mu(2)\}$ and $\{c(2)\}$, and so forth, until the lists $\{\mu(L)\}$ and $\{c(L)\}$ are achieved. This tree-like generation of μ gives all partitions of the given v . For more details see reference [18].

This partitioning procedure was implemented in the early FORTRAN program (NORFOR) developed in reference [18]. However, in the present Maple version (NORFORM) a “reverse” strategy is adopted, that is to say, instead of starting from a given \bar{v} and then finding the partitions defined by equation (15), we rather calculate each product in the inner sum of equation (14) and then update the corresponding coefficient $\phi_{j\bar{v}}^{(N)}$ with \bar{v} satisfying equation (15). This simpler method is possibly less efficient than the partitioning procedure. Therefore, the latter should be implemented in the future.

TABLE 2

The list of vectors μ ($l = 3$), contained in two given vectors v

$v = [2 \ 1 \ 2 \ 1 \ 0], \mu = 3$	$v = [5 \ 1 \ 2 \ 0 \ 2], \mu = 5$
[2 1 0 0 0]	[5 0 0 0 0]
[2 0 1 0 0]	[4 1 0 0 0]
[1 1 1 0 0]	[4 0 1 0 0]
[1 0 2 0 0]	[3 1 1 0 0]
[0 1 2 0 0]	[3 0 2 0 0]
	[2 1 2 0 0]

4.3. REDUCING COEFFICIENT UPDATES

It is easily verified that, in order to achieve an \bar{N} -normal reduced system with finite \bar{N} , it is not necessary to update all coefficients from the current to the desired normalization order \bar{N} . Before stating Theorem 2, we define a *bounding* function of $|v|$ defined in the interval $[N + 1, H]$, for integers H, N with $H \geq N + 1$ as

$$f^{H,N}(|v|) = \frac{H - |v|}{N} \quad \text{if } \frac{H}{N+1} < |v|,$$

$$f^{H,N}(|v|) = |v| \quad \text{if } \frac{H}{N+1} \geq |v|.$$
(18)

The following theorem [20] is useful to discard unnecessary updates and reduce the computation time significantly.

Theorem 3. *Assume that system (1) is in $N - 1$ normal form ($N \geq 1$, and for $N = 1$ the system is the original one with linear normal transformation yet to be performed). Let $v'' = [v_{l+1}, \dots, v_n]$. Then the coefficients $\phi_{iv}^{(N)}$ ($|v| \geq N + 1$) to be updated in order to get the \bar{N} -normal reduced system ($\bar{N} \geq N - 1$) are those satisfying*

$$|v''| \leq f^{\bar{N},N}(v), \quad i = 1, \dots, l,$$

$$|v| \leq f^{\bar{N}-1,N}(v), \quad i = l + 1, \dots, n.$$
(19)

As an example, assume that we wish to obtain the 3-normal form. Then, for the linear transformation we need to update only the coefficients

- (1) $\phi_{jv}(|v| = 2, |v''| \leq 1, j = 1, \dots, n)$,
- (2) $\phi_{k\mu}(|\mu| = 2, \mu \in \mathcal{M}, k = l + 1, \dots, n)$,
- (3) $\phi_{j\mu}(|\mu| = 3, \mu \in \mathcal{M}, j = 1, \dots, l; \lambda_{\mu^j} = 0)$.

Thus, a large number of coefficients can be discarded since, in general, l is much smaller than n .

5. NUMERICAL EXAMPLES

Some examples are presented to illustrate the applicability of the normalization method for higher order normalization using the Maple program that we will call NORFORM. All the results were obtained by running the NORFORM on a 400 MHz PC Pentium II. The computation time refers to the calculation of the normal transformations, the normal forms and the periodic solution (first two examples). To avoid confusion with the index i , we use the notation $I = \sqrt{-1}$.

5.1. VAN DER POL OSCILLATOR

We use the following version of Van der Pol's equation:

$$\ddot{x} - (\varepsilon^2 - x^2)\dot{x} + x = 0.$$
(20)

This version is convenient because it has all non-linear terms with degree 3. Thus, the order increment is $p = 2$ and we can directly obtain odd-normal forms. We rewrite equation (20) in state-space form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 x_3^2 - x_1^2 x_2 \\ 0 \end{bmatrix},$$

where the variable x_3 corresponds to parameter ε . This is a fully critical system $l = n = 3$ with all three eigenvalues on the imaginary axis. The recursive normalization can be applied up to any required order; however, the computation time increases very fast as the order of normalization increases. We have obtained normalization up to the 17th order with NORFORM. The program NORFORM took about 30s of CPU time to calculate the 11-normal form. Note that in reference [21], only the 3-normal form was obtained. The limit cycles obtained by increasing normalization order, from 3 to 17 for $\varepsilon = 1$, are shown in Figures 2–6. The limit cycles approximated by different orders of normal forms are plotted with 60 points. The result of the oscillator simulation is also presented with the initial condition $x_1(0) = 1$.

Figure 3 shows that high order normalization effectively gives better approximation, in comparison with the simulation result obtained by numerical integration of the differential equations. This encourages us to look for increasing order of normal forms.

Figures 2–6 show that the convergence of the approximations to the true periodic solution of Van der Pol’s oscillator ($\varepsilon = 1$) is not uniform. The 13-normal form leads to a much more precise periodic oscillation than the 3-normal form. However, the 15- and 17-normal forms are slightly less precise than the 13-normal form. This may be related to the non-convergence of the power series involved in the normal form.

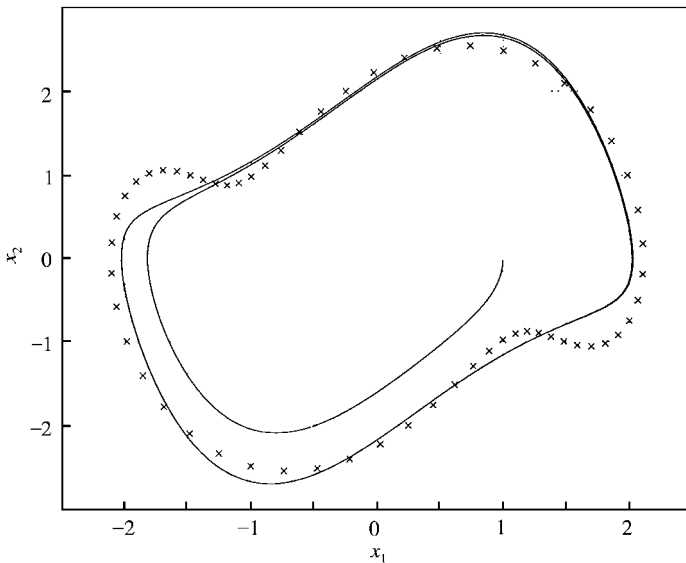


Figure 2. Limit cycles of the Van Der Pol oscillator with $\varepsilon = 1$, computed from numerical simulation (solid line) and from the 3-normal form obtained via NORFORM (plotted with “x”).

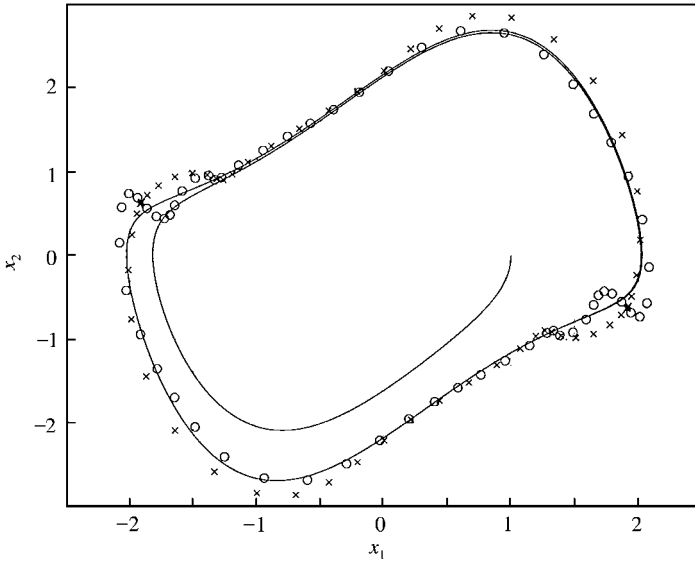


Figure 3. Limit cycles of the Van Der Pol oscillator ($\epsilon = 1$) obtained via NORFORM (plotted with “x” and “o”, for 5- and 7-normal forms respectively).

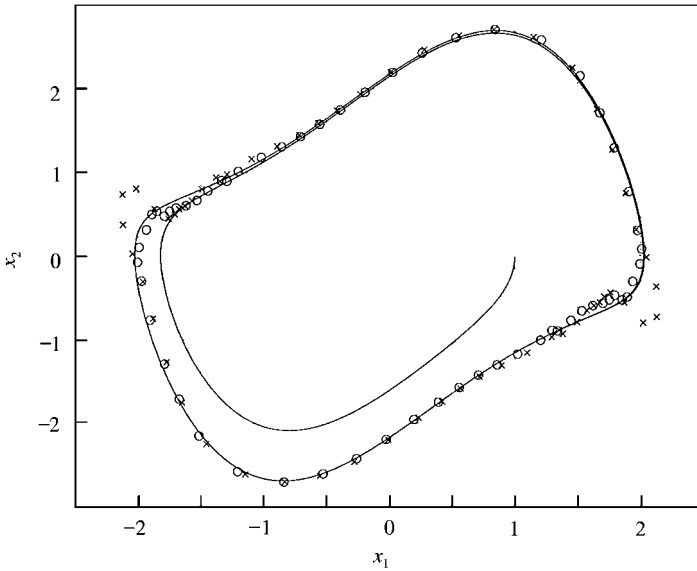


Figure 4. Limit cycles of the Van Der Pol oscillator ($\epsilon = 1$) obtained via NORFORM (plotted with “x” and “o”, for 9- and 11-normal forms respectively).

5.2. DOUBLE PENDULUM

The double pendulum system shown in Figure 7 consists of two rigid weightless links of equal length which carry two concentrated masses under the action of springs, dampers and a follower force.

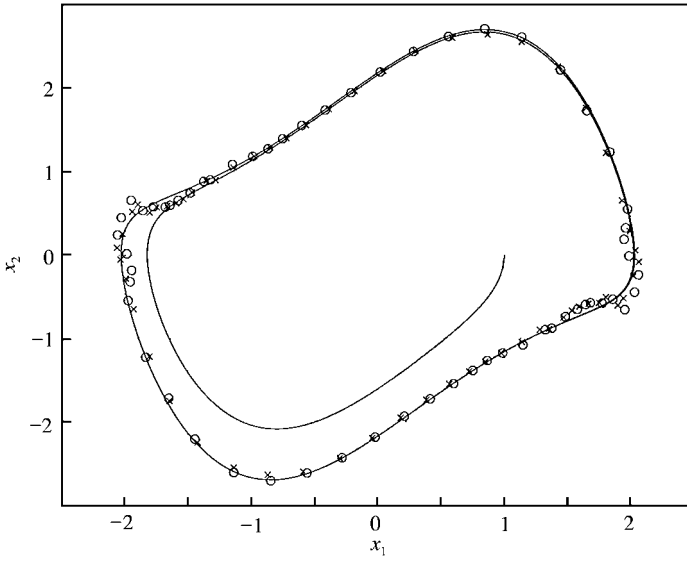


Figure 5. Limit cycles of the Van Der Pol oscillator ($\epsilon = 1$) obtained via NORFORM (plotted with “x” and “o”, for 13- and 15-normal forms respectively).

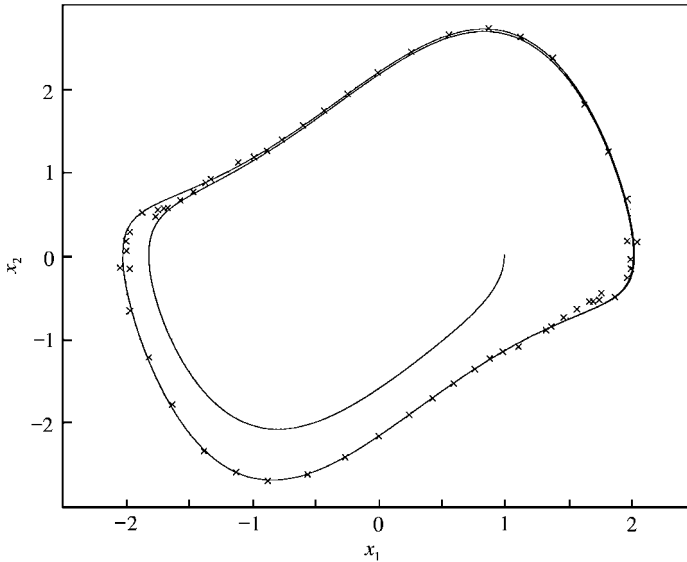


Figure 6. Limit cycle of the Van Der Pol oscillator ($\epsilon = 1$), computed from the 17-normal form (plotted with “x”) obtained via NORFORM.

As in reference [22], the double pendulum is described by a set of first order differential equations with non-linear terms up to third order. We will assume that the equations are exact in order to allow comparison with the results of reference [22]. These are also the equations used in the numerical simulations.

The critical point is defined by a set of parameters (f_1, f_2, f_3, f_4 and $\eta = \eta_c + \mu$). At the critical point the linear part of the system has two critical eigenvalues at $\pm \frac{1}{2}I$ and a zero

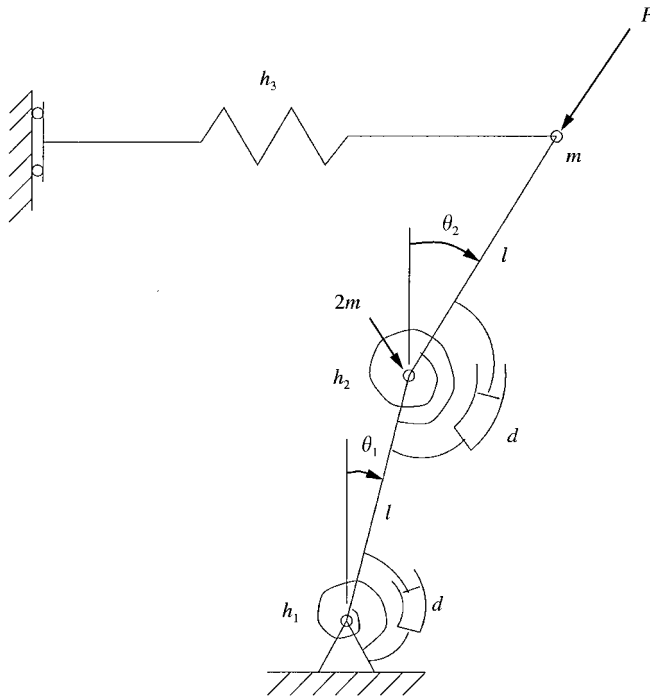


Figure 7. The double-pendulum system.

eigenvalue due to parameter μ . Thus, the dimension of the critical part is $l = 3$ and the non-critical part has dimension 2 with eigenvalues located at $-\frac{1}{2}$ and $-\frac{5}{4}$. This corresponds to a Hopf bifurcation.

The state variable z is defined as $z_1 = \varepsilon, z_2 = \theta_1, z_3 = \theta_2, z_4 = \dot{\theta}_1, z_5 = \dot{\theta}_2$, where $\varepsilon^2 = \mu$. This results in a system with odd non-linearities ($p = 2$). The variable z_3 is different from z_3 in reference [22] to avoid the singularity of a linear transformation matrix S_{11} [16].

The NORFORM program took about 5 min 30s to achieve the 9-normal form approximation. Close to the bifurcation ($\mu = 0.05$), the 3- and 5-normal forms give good approximations of the post-critical limit cycle, compared to the numerical simulation, as shown in Figures 8 and 9. However, far from the bifurcation ($\mu = 0.1$), a better approximation is obtained with higher order normalization, as seen in Figures 10 and 11.

A further way to check the quality of the approximations is to compare the corresponding oscillation frequencies with that of the exact periodic solution obtained by simulation. Table 3 shows the frequencies obtained by simulation and by the N -normal forms for $\mu = 0.05$ and 0.1. For comparison, we also present the frequencies calculated from the results of reference [22]. It is clear that our recursive normalization method gives superior approximations to the true frequency.

6. AN EXAMPLE WITH LARGE DIMENSION

The NORFORM (Maple) program was applied to an example with dimension $n = 10$. The purpose is to evaluate the processing time for systems with large dimensions.

We have not yet implemented the simplification possible with Theorem 2. So only an estimate of the effectiveness of the latter will be given.

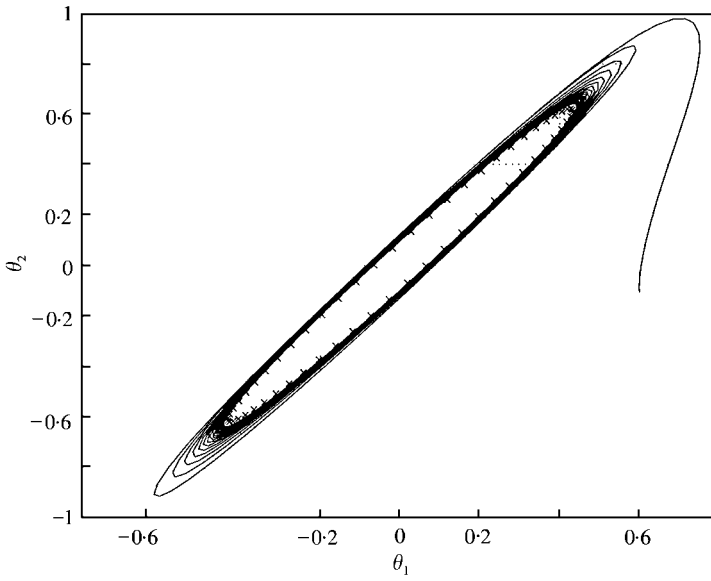


Figure 8. Limit cycle of the double pendulum with $\eta = 1.55$ obtained with the 3-normal form. Solid line corresponds to the simulation result.

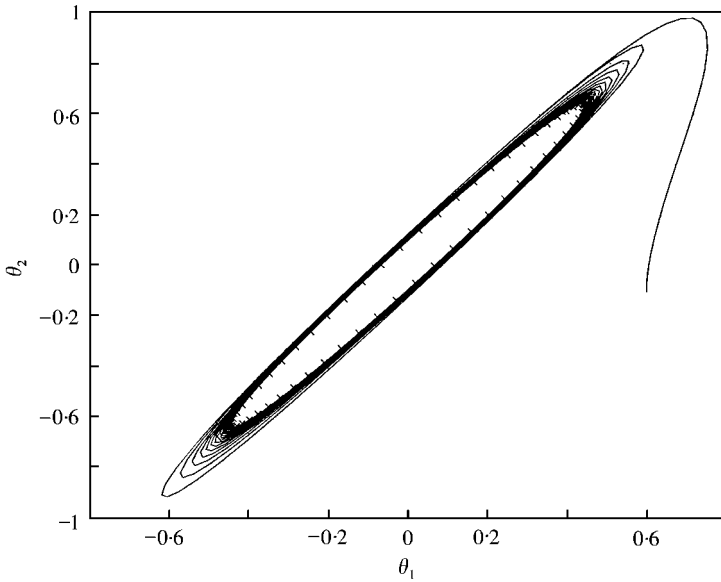


Figure 9. Limit cycle of the double pendulum with parameter $\eta = 1.55$, obtained with the 5-normal form.

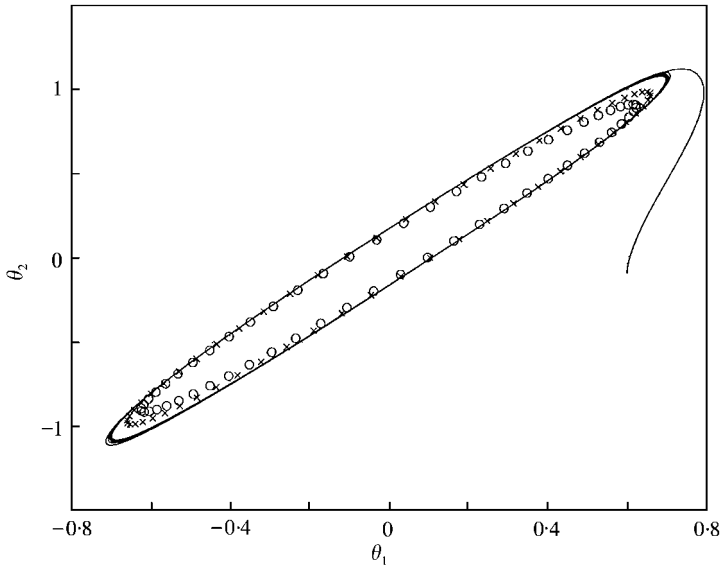


Figure 10. Limit cycles of the double pendulum with parameter $\eta = 1.60$, obtained with 3,5-normal forms, plotted with "O", "x" respectively.

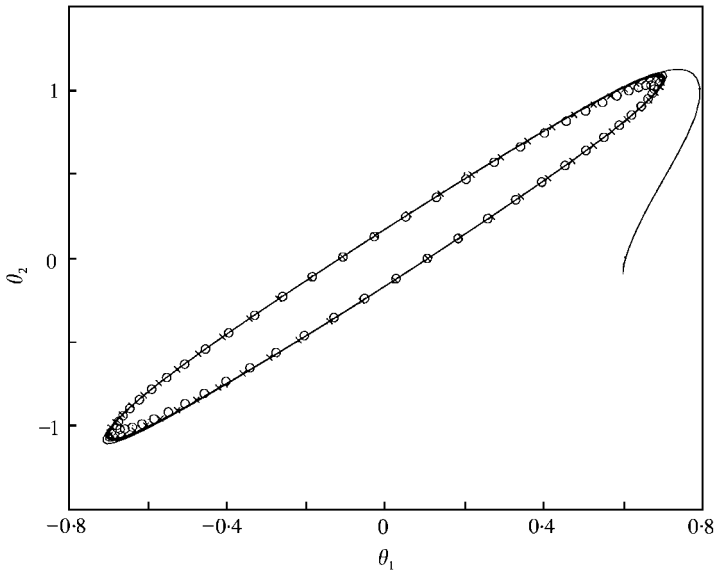


Figure 11. Limit cycle of the double pendulum with parameter $\eta = 1.60$, obtained with 7,9-normal forms, plotted with "O", "x" respectively.

TABLE 3

Frequency comparison, RN stands for recursive normalization

Simulation ω (rad/s)	$\mu = 0.05$ $\omega = 0.440$ rad/s		$\mu = 0.1$ $\omega = 0.333$ rad/s	
	RN	[22]	RN	[22]
3-normal	0.454	0.464	0.408	0.428
5-normal	0.447	0.470	0.376	0.445
7-normal	0.445	0.469	0.357	0.443
9-normal	0.444	0.469	0.340	0.442

Consider the following system taken from references [17, 18]:

$$\dot{x} = Ax + \begin{bmatrix} 0 \\ x_2x_3 + \alpha_1x_2^2x_3 \\ x_2x_3 + \alpha_2x_1x_1^2 \\ x_3x_5 \\ x_4x_6 \\ x_5x_7 \\ x_6x_8 \\ x_7x_9 \\ x_8x_{10} \\ x_9x_2 \end{bmatrix}, \tag{21}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

with $x \in \mathcal{R}^{10}$. The first equation corresponds to the bifurcation control parameter. Consider, for simplicity, that $\alpha_1 = \alpha_2 = 1$. Note also that this example presents quadratic and cubic non-linear terms so that the order increment is $p = 1$. The critical eigenvalues are $0, \pm I$ ($l = 3$) and the non-critical eigenvalues are $-1, -1 \pm \sqrt{2}I, -1 \pm \sqrt{-2 + \sqrt{2}}, -1 \pm \sqrt{-2 - \sqrt{2}}$ ($m = 7$).

The 3-, 5- and 7-normal forms results were obtained in 6 s, 4 min 6 s and 4 h 51 min, respectively, without the simplification of Theorem 2. The 7-normal form of the reduced system is presented as below ($I = \sqrt{-1}$):

$$\begin{aligned} \dot{x}_2 = & Ix_2 - \frac{1}{2}Ix_1^2x_2 + \frac{1}{6}Ix_2^2x_3 - \frac{1}{8}Ix_1^4x_2 \\ & - \frac{1}{6}Ix_1^2x_2^2x_3 + \frac{7}{432}Ix_2^3x_3^2 - \frac{1}{16}Ix_1^6x_2 \\ & - \frac{17}{96}Ix_1^4x_2^2x_3 + \frac{25}{864}Ix_1^2x_2^3x_3^2 - \frac{31003}{155520}Ix_2^4x_3^3. \end{aligned} \quad (22)$$

Only an estimate of the reduction in processing time by application of Theorem 2 was obtained by restricting the number of coefficients ϕ to be updated, according to the number predicted by Theorem 2. The modified NORFORM took 1 h and 12 s CPU time to perform the 7-normal form computation, thus reducing computation time by a factor of more than 4. We believe that this is a pessimistic estimate and that further reduction can be achieved through Theorem 2.

7. CONCLUSION

Based on a set of previously developed recursive formulas, a Maple program denominated as NORFORM was successfully implemented. It allows the computation of normal forms and normal transformations, up to an arbitrary order. Numerical examples were presented to compare the non-linear oscillations computed from the normal forms of increasing order with solutions obtained by numerical simulation. Larger systems tend to be quite time consuming if higher order normalization is required. However, at present, NORFORM does not fully exploit the processing time reduction that could be achieved by skipping the update of a large class of coefficients according to Theorem 2 and also by following the partitioning procedure of section 4.2. These simplifications are being presently incorporated in NORFORM. We expect that the computational load will be drastically reduced in this way.

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APPENDIX A: PARTITION ALGORITHM

The following Figure A1 displays a computational scheme for the partition algorithm described in problem 2, section 4.2.

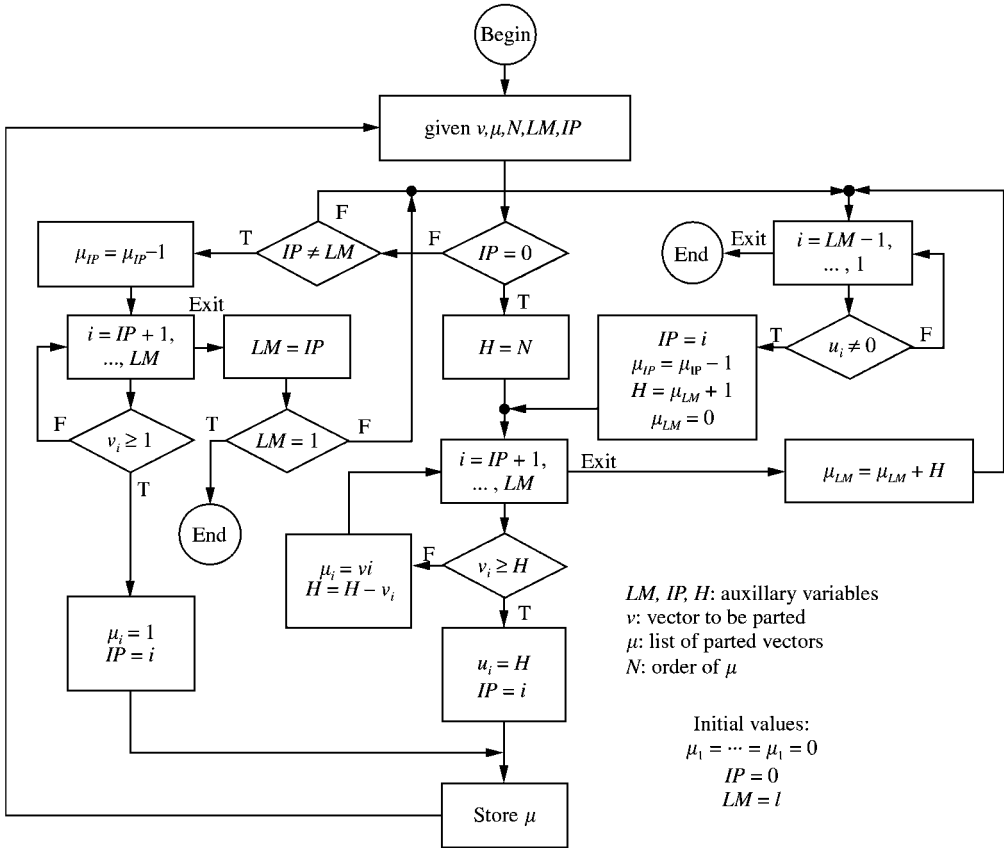


Figure A1. Implementation of the partition algorithm described in problem 2, section 4.2