Journal of Sound and Vibration (2001) **244**(1), 169–172
doi:10.1006/jsvi.2000.3467, available online at http://www.idealibrary.com on IDEAL®

KIRA IS ENERGY CONSUMING

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(*Received* 3 *October* 2000)

1. INTRODUCTION

KIRA is the numerical integration engine of STARLAB's N-body simulation program for predicting the evolution of star clusters moving under the action of gravity. It is a predictor–corrector scheme [1] that projects position, velocity, acceleration, and its rate of change, as it steps in time. KIRA is a general, explicit scheme for the numerical integration of the second order initial value problem and should be useful in tracking the state of vibrating mechanical systems. Being intended for astronomical computations over extended periods of time it must be highly accurate and efficient.

We show herein that this simple integration scheme is indeed highly accurate yet is energy consuming, invariably numerically predicting the eventual collapse of the traced system of stars or the standstill of the computed moving mechanical system.

2. PREDICT–CORRECT

Let x_0 and y_0 be the computed position and velocity at time t. Position x_p and velocity y_p are predicted at time $t+\tau$ by the Taylor expansions

$$x_p = x_0 + \tau x_0' + \tfrac{1}{2}\tau^2 x_0'' + \tfrac{1}{6}\tau^3 x_0''', \qquad y_p = y_0 + \tau y_0' + \tfrac{1}{2}\tau^2 y_0'', \tag{1}$$

where $(\ )'$ means differentiation with respect to time. Both position and velocity are corrected with

$$a_3 = 2(x_0'' - x_p'') + \tau(x_0''' + x_p'''), \qquad a_2 = -3(x_0'' - x_p'') - \tau(2x_0''' + x_p''') \tag{2}$$

to provide

$$x_1 = x_p + \tau^2(a_3/20 + a_2/12), \qquad y_1 = y_p + \tau(a_3/4 + a_2/3) \tag{3}$$

at time $t+\tau$.

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3. ACCURACY

We shall observe the accuracy and stability of this predictor–corrector scheme as it solves the pair of initial value problems $x' = y$, $y' = -x$, $x(0) = x_0$, $y(0) = y_0$. In this system $x'' = -x$, $y'' = -y$, $x''' = -y$, and it is analytically solved by

$$x(t) = x_0 \cos t + y_0 \sin t, \qquad y(t) = -x_0 \sin t + y_0 \cos t, \tag{4}$$

where $y(t) = x'(t)$.

With $x' = y$ and $y' = -x$, equations (1)–(3) become

$$x_p = x_0 + \tau y_0 - \tfrac{1}{2}\tau^2 x_0 - \tfrac{1}{6}\tau^3 y_0, \qquad y_p = y_0 - \tau x_0 - \tfrac{1}{2}\tau^2 y_0 \tag{5}$$

and

$$a_3 = 2(-x_0 + x_p) + \tau(-y_0 - y_p), \qquad a_2 = -3(-x_0 + x_p) + \tau(2y_0 + y_p) \tag{6}$$

so that

$$a_3 = (1/6)\tau^3 y_0, \qquad a_2 = (1/2)\tau^2 x_0 \tag{7}$$

with which the corrections become

$$\begin{aligned} x_1 &= (1 - \tfrac{1}{2}\tau^2 + \tfrac{1}{24}\tau^4)\, x_0 + (\tau - \tfrac{1}{6}\tau^3 + \tfrac{1}{120}\tau^5)\, y_0, \\ y_1 &= (-\tau + \tfrac{1}{6}\tau^3)\, x_0 + (1 - \tfrac{1}{2}\tau^2 + \tfrac{1}{24}\tau^4)\, y_0 \end{aligned} \tag{8}$$

at $t + \tau$.

We recall that

$$\begin{aligned} \sin\tau &= \tau - \tfrac{1}{6}\tau^3 + \tfrac{1}{120}\tau^5 - \tfrac{1}{5040}\tau^7 + \cdots, \\ \cos\tau &= 1 - \tfrac{1}{2}\tau^2 + \tfrac{1}{24}\tau^4 - \tfrac{1}{720}\tau^6 + \cdots, \end{aligned} \tag{9}$$

with which the corrected values x_1 and y_1 in equation (8) become

$$\begin{aligned} x_1 &= x_0 \cos\tau + y_0 \sin\tau + x_0 \tfrac{1}{720}\tau^6 + y_0 \tfrac{1}{5040}\tau^7, \\ y_1 &= -x_0 \sin\tau + y_0 \cos\tau + x_0 \tfrac{1}{120}\tau^5 + y_0 \tfrac{1}{720}\tau^6 \end{aligned} \tag{10}$$

as compared to the exact periodic solution of equation (4).

4. STABILITY

System (8) is solved [2] by $x_1 = zx_0$, $y_1 = zy_0$. For z that assures satisfaction of the linear system,

$$\begin{aligned} (z - 1 + \tfrac{1}{2}\tau^2 - \tfrac{1}{24}\tau^4)x_0 + (-\tau + \tfrac{1}{6}\tau^3 - \tfrac{1}{120}\tau^5)y_0 &= 0, \\ (\tau - \tfrac{1}{6}\tau^3)x_0 + (z - 1 + \tfrac{1}{2}\tau^2 - \tfrac{1}{24}\tau^4)y_0 &= 0 \end{aligned} \tag{11}$$

for any given initial conditions x_0, y_0. To shorten the writing we put equation (11) in the concise form

$$(z+a)x_0 + by_0 = 0, \qquad cx_0 + (z+a)y_0 = 0. \tag{12}$$

The system possesses non-trivial solutions only if its determinant vanishes leading to the characteristic equation

$$z^2 + 2az + a^2 - bc = 0, \qquad z = -a \pm \sqrt{bc}. \tag{13}$$

The condition that the numerical integration scheme produce an oscillatory solution given in terms of circular functions is that the magnification factor z be complex. This happens, in view of equation (13), if $bc < 0$. Since here

$$bc = -\tau^2 + \tfrac{1}{3}\tau^4 - \tfrac{13}{360}\tau^6 + \tfrac{1}{720}\tau^8, \tag{14}$$

time step τ need to be restricted to $\tau < 1{\cdot}53$ to ensure periodicity.

Our initial value problem is conserving. Indeed, from equation (4) we have that $x^2(t) + y^2(t) = x^2(0) + y^2(0)$. The computed solution is $x_n = x_0 z^n$, $y_n = y_0 z^n$, with $n\tau = t$. Hence for the approximate position and velocity it happens that $x_n^2 + y_n^2 = (x_0^2 + y_0^2)z^{2n}$ and the integration scheme is energy producing if $|z| > 1$, and is energy consuming if $|z| < 1$. From characteristic equation (13) we have that $|z|^2 = a^2 - bc$, and

$$|z|^2 = 1 - \tfrac{1}{180}\tau^6 - \tfrac{1}{2880}\tau^8 \tag{15}$$

or

$$|z| = 1 - \tfrac{1}{360}\tau^6 \tag{16}$$

if $\tau \ll 1$. Clearly, $|z| < 1$ for any $\tau > 0$ and the scheme is energy consuming.

5. NUMERICAL EXAMPLES

Let $x_0 = 1$, $y_0 = 0$, then according to equation (4) $x = \cos t$, $y = -\sin t$ and the motion is harmonic with period $T = 2\pi$. Also $x^2(t) + y^2(t) = 1$ which is the equation of a unit circle in the xy plane. After n steps the computed radius of this circle is $r_n = |z|^n$.

Let n be now the number of time steps per period and m the number of periods. With $\tau = 2\pi/n$ we have from equation (16) that

$$|z|^{mn} = \left(1 - \frac{171}{n^6}\right)^{mn} \tag{17}$$

or

$$|z|^{mn} = [(1-\varepsilon)^{1/\varepsilon}]^{171m/n^5}, \tag{18}$$

where $\varepsilon = 171/n^6$. For a sufficiently small ε we may replace $(1-\varepsilon)^{1/\varepsilon}$ by e^{-1} to have

$$|z|^{mn} = e^{-171m/n^5}. \tag{19}$$

Say $n = 36$. Then after some 37 600 periods the computed radius drops to 90% of its original value.

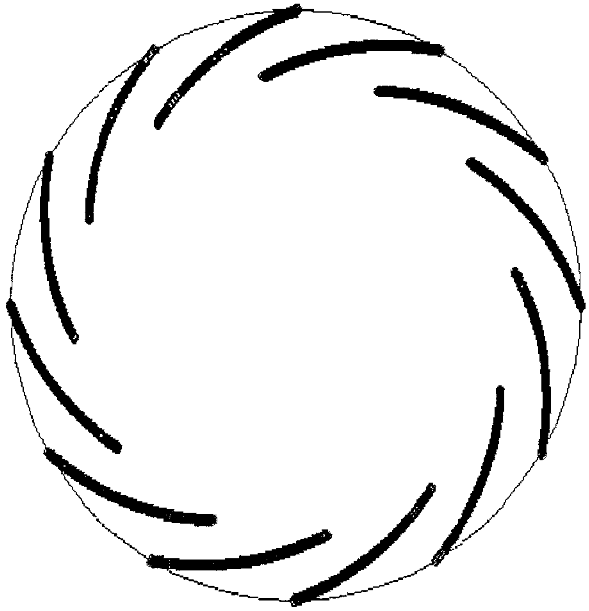

Figure 1. Positions of a planet as computed with the scheme of equations (1)–(3).

Figure 1 shows the positions of a planet as computed with the scheme of equations (1)–(3). Under the initial conditions of $x_0 = 1$, $y_0 = 0$ the planet is theoretically moving clockwise in a perfectly circular orbit in a periodic motion completing a revolution in 2π units of time. Computations were done with $n = 12$ steps per revolution and were extended over $m = 360$ revolutions. The computational errors in both amplitude and period are clearly manifested in the apparent spiral trajectory of the planet as it heads to the center.

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