Journal of Sound and Vibration (2001) **244**(2), 195–209
doi:10.1006/jsvi.2000.3470, available online at http://www.idealibrary.com on IDEAL®

ON THE CONVECTION OF SOUND IN INVERSE CATENOIDAL NOZZLES

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(*Received* 11 *January* 1999, *and in final form* 19 *September* 2000)

A variational method is used to deduce the acoustic wave equation, satisfied by the potential, for quasi-one-dimensional propagation, in a duct of varying cross-section, containing a low Mach number mean flow; both wave equations, for the acoustic potential and velocity, are reduced to a 'Schrödinger' form, by using the ray approximation as a factor, in the exact solution. The latter is obtained, for the acoustic potential and velocity perturbations, both in horns (no flow) and low Mach number nozzles, of the inverse catenoidal family of ducts. The latter consists of (see Figure 1 top) the "bulged" divergent–convergent duct of profile sech, and (see Figure 1 bottom) the twin "baffles" of profile csch. The acoustic velocity perturbation, apart from one amplitude and one phase term, is common to the two cases, and is calculated in a second way, by solving a modified form of Mathieu's equation, with imaginary and hyperbolic, hence non-periodic, coefficients. These solutions are used to plot (see Figures 2–5), the amplitude (top) and phase (bottom) of the wavefields, versus distance, for several low Mach numbers, and a wide range of wavenumbers, with the compactness and the ray approximations as extremes, and intermediate values as well.

1. INTRODUCTION

One of the simplest shapes of "bulged" nozzle, is the "solitary" wave hump [1], specified by a hyperbolic secant, which matches smoothly (see Figure 1 top), divergent and convergent ducts, of exponential shape at large distance; the other inverse catenoidal duct, of cross-section specified by a hyperbolic cosecant, corresponds (see Figure 1 bottom) to "baffled" nozzles. If the wavelength is larger than the transversal dimensions of the duct, only the fundamental longitudinal acoustic mode can propagate; it is described, in a quasi-one-dimensional approximation [2, 3] by the nozzle wave equation [4, 5]. At low Mach number, it is a convected horn wave equation [6]: i.e., it combines Websters [7] horn equation [8], with the convected wave equation [9, 10]. The simpler horn wave equation, has elementary exact solutions for five shapes [11], namely, the exponential [12], catenoidal [13], sinusoidal [14], and by reciprocity [15], their inverses. The power-law [16] and Gaussian [17] shapes lead to non-elementary solutions in terms of Bessel and Hermite functions respectively. These results have been extended to low Mach number nozzles, of exponential [18], power-law [19], catenoidal, sinusoidal and Gaussian shapes [20]. The case of inverse sinusoidal nozzles was also considered [21], and the present paper addresses the case of inverse catenoidal nozzles.

2. GENERAL WAVE EQUATIONS

A variational method is used (section 2.1), to deduce wave equations for the acoustic velocity and potential (section 2.2), and reduce both of them to Schrödinger forms (section 2.3).

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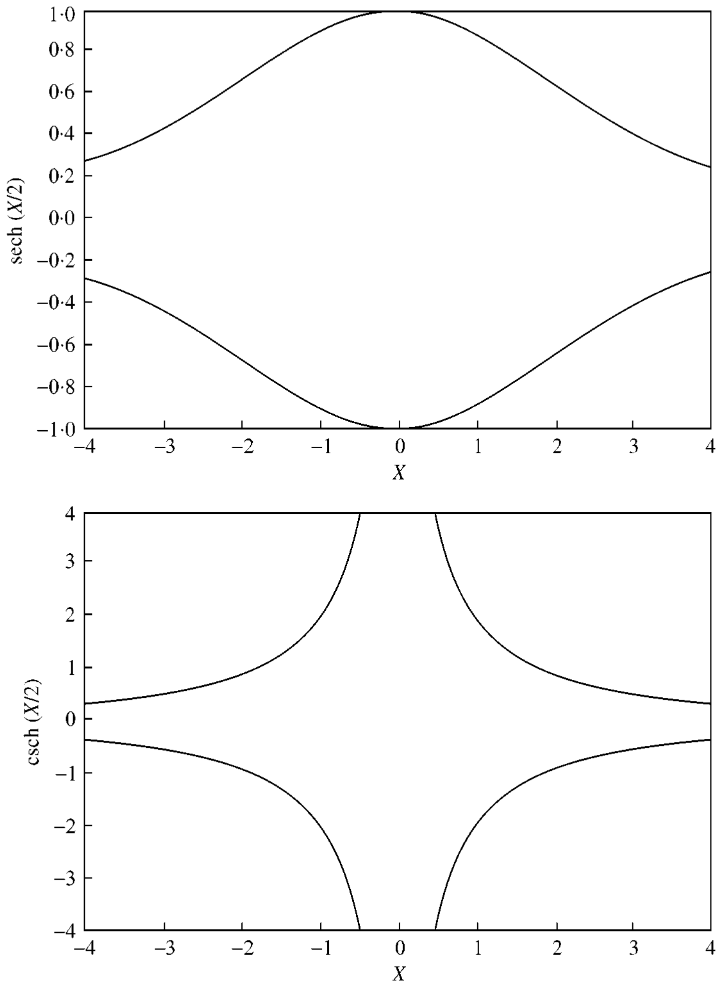

Figure 1. The inverse catenoidal ducts represent a "bulged" (a) or a "baffled" (b) nozzle.

2.1. VARIATIONAL APPROACH TO ACOUSTICS OF LOW MACH NUMBER NOZZLES

The wave equation describing quasi-one-dimensional, longitudinal acoustic propagation, in a duct of varying cross-section, containing a low-Mach number steady mean flow, can be deduced, without eliminations, by using a variational method [3, 22, 23]; only the aspects related to the wave equation are briefly mentioned here. A one-dimensional flow always has a potential (1a),

$$v = \phi' \equiv \partial\phi/\partial x, \qquad E_v \equiv \tfrac{1}{2}\rho v^2 = \tfrac{1}{2}\rho\phi'^2, \tag{1a, b}$$

which specifies the kinetic energy (1b), where ρ is the mean flow density, and v the acoustic velocity perturbation (1a). It is related to the acoustic pressure perturbation p, by the inviscid linearized momentum equation

$$\dot{v} \equiv \partial v/\partial t, \qquad \dot{v} + (Uv)' + p'/\rho = 0, \tag{2a, b}$$

where $U(x)$ denotes the mean flow velocity; note that the effect of wall boundary layers is omitted by the assumption of inviscid quasi-one-dimensional flow. Substituting equation (1a) in equation (2b), leads to the linearized, unsteady Bernoulli equation (3a),

$$p = -\rho(\dot{\phi} + U\phi'), \qquad E_p \equiv p^2/2\rho c^2 = (\rho/2c^2)(\dot{\phi} + U\phi')^2, \tag{3a, b}$$

which relates the acoustic pressure p to the potential ϕ, and thus specifies the compression energy (3b), where c denotes the sound speed; the compression energy was calculated upon assuming that sound propagation is an adiabatic process: i.e., neglecting dissipation, not only due to viscosity but also to thermal conduction. The acoustic Lagrangian, per unit volume, is the difference of the kinetic and compression energies:

$$\mathscr{L}(x, \dot{\phi}, \phi') \equiv E_v - E_\rho = \tfrac{1}{2}\rho\{\phi'^2 - (\dot{\phi} + U\phi')^2/c^2\}. \tag{4}$$

The Lagrangian per unit length of duct $\mathscr{L}^* = \mathscr{L}S$ is, in equation (5b), the product of the Lagrangian per unit volume (4) by the cross-section of the duct $S(x)$, and the integral over space–time specifies, in equation (5a), the action,

$$A[\phi(x, t)] \equiv \int \mathrm{d}x \int \mathrm{d}t \quad \mathscr{L}^*(x, \dot{\phi}, \phi'), \qquad \mathscr{L}^* \equiv S\mathscr{L}, \tag{5a, b}$$

which is a functional of the potential. The condition of stationary action. In equation (6a)

$$\delta A = 0, \quad \partial(\partial\mathscr{L}^*/\partial\dot{\phi})/\partial t + \partial(\partial\mathscr{L}^*/\partial\phi')/\partial x = 0, \tag{6a,b}$$

leads to the Euler–Lagrange equation (6b), which specifies the wave equation. Noether's theorem would have led to the energy equation [23].

2.2. LOW MACH NUMBER NOZZLE OR CONVECTED HORN WAVE EQUATION

In the low Mach number approximation, the acoustic Lagrangian (4) simplifies to

$$M^2 \equiv U^2/c^2 \ll 1, \qquad \mathscr{L}^* = \tfrac{1}{2}\rho S[\phi'^2 - (\dot{\phi}/c)^2 - 2U\dot{\phi}\phi'/c^2], \tag{7}$$

and substitution into equation (6b) leads to the wave equation for the potential,

$$\ddot{\phi} + 2U\dot{\phi}' - (c^2/S)(S\phi')' = 0, \tag{8}$$

where it was taken into account that, at low Mach number (i) the mean flow mass density ρ and sound speed c are constant, and (ii) the volume flux is conserved as $SU = \text{const}$. An alternative, and perhaps more direct way to obtain the wave equation (8), is to substitute equations (1a) and (3a) into the linearized continuity equation for isentropic flow:

$$c^{-2}(\partial p/\partial t + U\partial p/\partial x) + (\rho/S)\partial(Sv)/\partial x = 0. \tag{9}$$

The variational approach has the advantage that it supplies not only the wave equation but also the energy equation [23].

The wave equation for the potential (8) coincides, at low Mach number, to $0(M)$ with the convected horn equation

$$\{(\partial/\partial t + U\partial/\partial x)^2 - (c^2/S)\partial/\partial x S\,\partial/\partial x\}\,\phi(x, t) = 0, \tag{10}$$

which combines the material derivative, as in the convected wave equation [9, 10], with the duct operator, as in the horn wave equation [8, 7]. Differentiating equation (8) with respect to x leads to the wave equation for the acoustic velocity (1a), viz.,

$$\ddot{v} + 2U\dot{v}' + 2U'\dot{v} - c^2\{S^{-1}(Sv)'\}' = 0, \tag{11}$$

which differs from that for the potential (8) in that (i) it has an extra term $2U'v$, non-zero for non-uniform mean flow $U' \neq 0$, which is always the case for $U' = -US'/S$ in a nozzle of varying cross-section, and (ii) the duct operators in equations (11) and (8) are different,

$$\partial/\partial x S^{-1}\partial/\partial x S - S^{-1}\partial/\partial x S\,\partial/\partial x = (S'/S)', \tag{12}$$

except for the expontential duct $S'/S = \text{const}$. The wave equation for the acoustic velocity (11) corresponds to

$$\{(\partial/\partial t + \partial/\partial x U)^2 - c^2\partial/\partial x S^{-1}\partial/\partial x S\}v(x,t) = 0, \tag{13}$$

where, besides changing the duct operator, the material derivative is also modified.

2.3. REDUCED WAVE EQUATION FOR ACOUSTIC POTENTIAL AND VELOCITY

Since the mean flow is steady, the frequency is conserved, and it is convenient to use a Fourier representation,

$$\phi, v(x,t) = \int_{-\infty}^{+\infty} e^{-i\omega t}\Phi, V(x;\omega)\,d\omega, \tag{14a, b}$$

where Φ is the potential, and V the velocity perturbation spectrum, for a wave of frequency ω, at position x. They satisfy respectively equations (8) and (11), namely,

$$\Phi'' + (1/L + 2ikM)\Phi' + k^2\Phi = 0, \tag{15a}$$

$$V'' + (1/L + 2ikM)V' + [k^2 + (1/L)' + 2ikM']V = 0, \tag{15b}$$

where k is the wavenumber and M the Mach number,

$$k \equiv \omega/c, \qquad M(x) \equiv U(x)/c, \tag{16a, b}$$

and L plus (minus) the length scale for changes in cross-section (mean flow velocity or Mach number):

$$L \equiv S/S' = -U/U' = -M/M'. \tag{17}$$

The "original" wave equations (15a, b), may be reduced to "Schrödinger's" form, without the term involving the first derivative, via the standard [24, 25] change of dependent variable,

$$\Phi, V(x;\omega) = \exp\left\{-\frac{1}{2}\int^x [1/L(\xi) + 2ikM(\xi)]\,d\xi\right\}\Psi, W$$

$$= [S(x)]^{-1/2}\exp\left\{-ik\int^x M(\xi)\,d\xi\right\}\Psi, W(x;\omega), \tag{18a, b}$$

which leads to

$$\Psi'' + [k^2 - (1/2L)' - 1/4L^2]\,\Psi = 0, \tag{19a}$$

$$W'' + [k^2 + (1/2L)' - 1/4L^2 - 2\mathrm{i}kM/L]\,W = 0, \tag{19b}$$

The reduced wave equation for the potential (19a), is simpler than that for the velocity (19b), in that it omits dependence on the mean flow, and thus takes the same form for nozzles as for horns. In the ray approximation, when the wavelength is much shorter than the length scale $k^2L^2 \gg 1$, upon bearing in mind equation (17), that $1/L' \sim 1/L^2 \ll k^2$, the first square bracket in equation (19a) reduces to k^2 in equation (21a),

$$k^2L^2 \gg 1, \quad \Psi'' + k^2\Psi = 0, \tag{21a}$$

$$k^2L^2 \sim 1/M^2 \gg 1, \quad W'' + k^2W = 0 \tag{21b}$$

and since the low Mach number approximation assumes $1/M^2 \gg 1$, it is reasonable to scale $kL \sim 1/M$, so that the square brackets in equation (19b) also reduce to k^2 in equation (21b). The solutions $\mathrm{e}^{\pm \mathrm{i}kx}$ of equations (21a, b) show that in the case (14a, b; 18a, b) of the ray approximation,

$$\phi, v(x, t) \sim \mathrm{e}^{-\mathrm{i}\omega(t \pm x/c)}[S(x)]^{-1/2} \exp\left\{-\mathrm{i}k\int^x M(\xi)\,\mathrm{d}\xi\right\}, \tag{22a, b}$$

there are plane waves propagating in the positive or negative x direction, with amplitude proportional to the inverse square root of the cross-section (to converse the energy flux $\rho c|v|^2S$), and a phase shift equal to the integrated Doppler effect.

3. THE INVERSE CATENOIDAL NOZZLES

The solution of the wave equation, for arbitrary low frequencies (section 3.1), in the inverse catenoidal ducts (section 3.2), shows some interesting relations between (section 3.3) horns and nozzles.

3.1. ACOUSTIC FIELDS IN THE RAY APPROXIMATION AND FOR LOW FREQUENCY

The ray approximation (22a, b) applies to high frequencies $\omega^2 \gg c^2/L^2$, as long as the wavelength $\lambda > d$ remains larger than the transverse dimension d of the duct, namely $\omega = kc = 2\pi c/\lambda < 2\pi c/d$. Bearing in mind that the transverse dimensions of the duct scale on the cross-sectional areas as $d \sim \sqrt{S(x)}$, and recalling the length scale (17) for variations in cross-section, indicates that the ray approximation (22a, b) is restricted to the range of frequencies

$$cS'(x)/S(x) < \omega < 2\pi c/\sqrt{S(x)}, \tag{23}$$

which depends on nozzle shape. The exact solution of the quasi-one-dimensional wave equations (19a, b) would not be restricted by the lower bound in equation (23): i.e., would apply to arbitrary low frequencies. In the case of low-frequency sound $\omega \leqslant c/L$ or $kL \leqslant 1$, the exact solution of the reduced wave equations (19a, b) depends on the particular shape of the duct. Examples are the inverse catenoidal nozzles

$$S_\pm(x) = S_0 \operatorname{sech}^2, \operatorname{csch}^2(x/2l), \tag{24a, b}$$

where the parameter l is a length scale for the shape of nozzle, which may be interpreted as follows: (i) both nozzles match a divergent duct for $x<0$ to a convergent duct for $x>0$, both with exponential form or constant length scale, at large distance compared to the length scale l:

$$|x| \gg l, \quad S_\pm(x) = S_0 \exp(-|x|/l); \tag{25}$$

(ii) in the former case (24a), the matching (see Figure 1 top) is smooth, through a "bulge" where the cross-section is maximum and equal to S_0,

$$x^2 \ll l^2, \quad S_+(x) = S_0(1 - x^2/4l^2); \tag{26a}$$

(iii) in the latter case (24b), the cross-section diverges at the origin,

$$x^2 \ll l^2, \quad S_-(x) = 4S_0 l^2/x^2 \tag{26b}$$

and there are two "baffled" nozzles (see Figure 1 bottom). The mean flow remains of low Mach number,

$$M_\pm(x) = M_0 \cosh^2, \sinh^2(x/2l) \tag{27a, b}$$

provided that $M_0^2 e^{|x|/l} \ll 1$, which excludes far, narrow sections of the nozzle. The length scale for variations in cross-section (17),

$$L_\pm(x) = -l \coth, \tanh(x/2l), \tag{28a, b}$$

coincides with the length scale for nozzle shape $|L| \sim l$ in modulus at large distance $|x| \gg l$, and increases to $L_+(0) = \infty$ (decreases to $L_-(0) = 0$), at the origin, for "bulged" ('baffled') nozzles. The ray solution (22a, b) for the inverse catenoidal nozzles (24a, b) is

$$\phi, v(x,t) = e^{-i\omega(t \pm x/c)} \cosh, \sinh(x/2l) \exp\{-i(klM_0/2)[\sinh(x/l) \pm x/l]\}, \tag{29a, b}$$

and is restricted to the frequency range specified by equations (23, 24a, b):

$$(c/l)\tanh, \coth(x/2l) < \omega < (2\pi c/\sqrt{s_0})\cosh, \sinh(x/2l). \tag{30a, b}$$

Only the upper bound in equation (30b) restricts one to the exact solution of the wave equations, which is considered next.

3.2. FILTERING FUNCTION AND CUT-OFF FREQUENCY FOR CATENOIDAL DUCTS

In the case of inverse catenoidal nozzles (24a, b; 27a, b; 28a, b), the wave equation for the potential (15a) = (31a) is simpler than for the velocity (15b) = (31b), namely,

$$\Phi''_\pm + \lfloor -2\tanh, \coth(z) + 2iKM_0 \cosh^2, \sinh^2(z) \rfloor \Phi'_\pm + K^2 \Phi_\pm = 0, \tag{31a}$$

$$V''_\pm + \lfloor -2\tanh, \coth(z) + 2iKM_0 \cosh^2, \sinh^2(z) \rfloor V'_\pm + \lfloor K^2 + 2iKM_0 \sinh(2z) + 2\operatorname{sech}^2, -\operatorname{csch}^2(z) \rfloor V_\pm = 0, \tag{31b}$$

where the prime denotes the derivative with respect to the dimensionless co-ordinate (32a).

$$z \equiv x/2l, \quad K \equiv 2kl = kx/z = 2\omega l/c \equiv \Omega, \tag{32a, b}$$

and a dimensionless frequency Ω or reference wavenumber K was introduced (32b) satisfying $\Omega z = kx$. The reduced wave equations (19a, b) are

$$\Psi''_{\pm} + \lfloor \Omega^2 + 1 - \coth^2, \tanh^2(z) \rfloor \Psi_{\pm} = 0, \tag{33a}$$

$$W''_{\pm} + \lfloor \Omega^2 - 1 + 2\mathrm{i}\Omega M_0 \sinh(2z) \rfloor W_{\pm} = 0 \tag{33b}$$

and are simpler than the unreduced forms (31a, b). The equation for the velocity (33a) takes the same form for both kinds of ducts (horns and nozzles). The simplest method of solution is to start from the wave equation for the velocity (33b), in the absence of flow, when it simplifies to

$$M_0 = 0, \ \mathrm{d}^2 f/\mathrm{d}x^2 + (\omega^2/c^2 - 1/4l^2) f = 0, \tag{34}$$

which specifies a filtering function

$$f(x;\omega) = \begin{cases} A\mathrm{e}^{\alpha x/2l} + B\mathrm{e}^{-\alpha x/2l} & \text{if } \omega < \omega, & (35\mathrm{a}) \\ Ax + B & \text{if } \omega = \omega_* \equiv c/2l, & (35\mathrm{b}) \\ A\mathrm{e}^{\mathrm{i}\beta x} + B\mathrm{e}^{-\mathrm{i}\beta x} & \text{if } \omega > \omega, & (35\mathrm{c}) \end{cases}$$

which is linear at the cut-off frequency (35b), monotonic (35a) below, and propagating (35c) above:

$$\alpha \equiv \sqrt{1 - (\omega/\omega_*)}^2, \qquad \beta = (\omega/c)\sqrt{1 - (\omega_*/\omega)}^2. \tag{36a, b}$$

The constants of integration A, B are determined from boundary or radiation conditions, e.g., specifying the acoustic velocity, pressure or potential at two positions. In the case of an infinite duct, in the positive x direction, a bounded acoustic field, below the cut-off frequency (35a), requires $A = 0$; wave propagation, above the cut-off frequency (35c), in the positive x direction, requires $B = 0$, and in the negative x direction requires $A = 0$.

3.3. ACOUSTIC FIELDS IN THE INVERSE CATENOIDAL HORNS AND NOZZLES

When considering, for example, waves propagating (35c) in the positive x direction, the acoustic velocity perturbation is given, in reduced form, by

$$W^0_{\pm}(x;\omega) = f(x; \omega > \omega_*) = w_0 \mathrm{e}^{\mathrm{i}\beta x}, \tag{37}$$

or, in non-reduced form (18b), for inverse catenoidal horns (24a, b) by

$$V^0_{\pm}(x;\omega) = w_0\, s_0^{-1/2} \cosh, \sinh(x/2l)\mathrm{e}^{\mathrm{i}\beta x}, \tag{38}$$

implying, for the potential,

$$\Phi^0_{\pm}(x;\omega) = \int^x V^0_{\pm}(\xi;\omega)\,\mathrm{d}\xi$$

$$= (w_0/2\sqrt{s_0})\{[\exp(x/2l\beta x)]/(1/2l + \mathrm{i}\beta x) \pm [\exp(-x/2l + \mathrm{i}\beta x)]/(-1/2l + \mathrm{i}\beta)\}. \tag{39}$$

Thus, the reduced potential for a horn is

$$\Psi^0_\pm(x;\omega) = \sqrt{s_0}\,\mathrm{sech},\mathrm{csch}(x/2l)\,\Phi^0_\pm(x;\omega) \equiv \Psi_\pm(x;\omega) \tag{40}$$

and it coincides with the reduced potential for a nozzle, since the latter does not depend on the mean flow, as follows from equation (33a). Substituting equation (40) in equation (18a, b) with equations (24a, b), yields the unreduced potential in a nozzle as

$$\Phi_\pm(x;\omega) = \Psi^0_\pm(x;\omega)\,s_0^{-1/2}\cosh,\sinh(x/2l)\exp\left\{-\,\mathrm{i}kM_0\int^x \cosh^2,\sinh^2(\xi/2l)\,\mathrm{d}\xi\right\} \tag{41}$$

which simplifies, by equation (40), to

$$\Phi_\pm(x;\omega) = \Phi^0_\pm(x;\omega)\exp\{-\,\mathrm{i}(\Omega/4)M_0[\sinh(x/l) \pm x/l]\} \tag{42}$$

and thus the acoustic potential in the inverse catenoidal nozzle (42), is the same as for the corresponding horn (39), with an extra integrated Doppler shift, due to the non-uniform mean flow. The acoustic velocity in the nozzle is given by

$$\begin{aligned} V_\pm(x;\omega) &= \mathrm{d}\Phi_\pm/\mathrm{d}x = \exp\{-\,\mathrm{i}\{\Omega/4)M_0[\sinh(x/l) \pm x/l]\}\{V^0_\pm(x;\omega) \\ &\quad -\,\mathrm{i}(\Omega/4l)M_0\Phi^0_\pm(x;\omega)[\cosh(x/l) \pm 1]\} \end{aligned} \tag{43}$$

and thus the acoustic velocity in the inverse catenoidal nozzle differs from that in the corresponding horn, not just by a Doppler shift, as in equation (42), but also by an extra term, specified by the mean flow, the reason being that the latter appears in the wave equations for the unreduced (19b) and reduced (33b) velocity. The latter, i.e. the reduced velocity, is given, from equations (43; 18b; 24a, b), by

$$W_\pm(x;\omega) = w_0\mathrm{e}^{\mathrm{i}\beta x} - \mathrm{i}(\Omega/4l)M_0\,\mathrm{sech},\mathrm{csch}(x/2l)\quad \Phi^0_\pm(x/\omega)[\cosh(x/l) \pm 1], \tag{44}$$

where the first term applies to a horn (37), and the second is the effect of the mean flow. In this way, the solution (44) of equation (33b) has been obtained, indirectly. Although this completes the physical solution of the problem, equation (33b) is mathematically interesting enough to be considered directly, and it may arise in other wave problems.

4. THE MODIFIED MATHIEU EQUATION

Equation (33b) is a modified form of the well-known Mathieu equation, with imaginary and non-periodic coefficients. It can be transformed into an equation with polynomial coefficients (section 4.1), and its solutions obtained in terms of Frobenius–Fuchs series (section 4.2), converging in the neighbourhood of one (of two) regular singularities, and they are used to plot the wave amplitude and phase, as a function of distance, for several values of low Mach number, and a wide range of values of dimensionless wavenumber (section 4.3).

4.1. TRANSFORMATION OF THE MODIFIED, IMAGINARY, NON-PERIODIC MATHIEU EQUATION

Since the reduced wave equation for the velocity (33b), is the same for both catenoidal ducts (24a, b; 27a, b; 28a, b), it appears (18b) in the acoustic velocity for both cases:

$$V_\pm(x;\omega) = W(x;\omega)\cosh,\sinh(x/2l)\exp\{-\,\mathrm{i}(\Omega/4)M_0[\sinh(x/l) \pm x/l]\}, \tag{45}$$

The function $W(x;\omega)$ may be obtained directly, by solving equation (33b), which resembles a Mathieu equation [26, 27],

$$Y'' + [a + b\cos(2\theta)]\,Y = 0, \tag{46}$$

with two important differences: (i) one of the two constants is imaginary; (ii) the circular function is replaced by a hyperbolic function, so that for real z, the coefficients are not periodic. Thus Floquet's theory [25] does not apply to equation (33b), since the solutions are not periodic. A useful result from this comparison, is the analogy with Lindemann's transformation [26] of Mathieu's equation (46), to polynomial coefficients, via the change of independent variable $\zeta = \cos^2 z$. In order to do this, the coefficient of equation (33b) is transformed from hyperbolic to circular, via an imaginary change of variable:

$$\mathrm{i}z = \theta + \pi/4, \qquad \mathrm{i}\sinh(2z) = \cos(2\theta). \tag{47a, b}$$

Thus the function (48a),

$$F(\theta) \equiv W(z;\omega) = W(-\mathrm{i}\theta - \mathrm{i}\pi/4;\omega), \tag{48a}$$

satisfies a Mathieu equation

$$F'' + \lfloor 1 - \Omega^2 - 2\Omega M_0\cos(2\theta)\rfloor F = 0, \tag{48b}$$

with coefficients which are circular functions. The change of variable

$$\zeta = \cos^2\theta, \qquad G(\zeta) \equiv F(\theta) = W(-\,\mathrm{i}\arg\cos\sqrt{\zeta} - \mathrm{i}\pi/4;\omega), \tag{49a, b}$$

leads to a differential equation with polynomial coefficients,

$$\zeta(1-\zeta)G'' - (\zeta - 1/2)G' - \lfloor \Omega M_0\zeta - (1 - \Omega^2 - 2\Omega M_0)/4\rfloor G = 0, \tag{50}$$

and three singular points $\zeta = 0, 1, \infty$.

4.2. CONVERGENT FROBENIUS–FUCHS SERIES IN THE NEIGHBOURHOOD OF REGULAR POINTS

The points zero and unity $\zeta = 0, 1$ are regular singularities of the differential equation (50), in their neighbourhood converging series expansions exist, respectively in powers of ζ, $1-\zeta$. The series expansion in the neighbourhood of $\zeta = 0$, proceeds in powers ζ given by equations (49a, 47a), namely,

$$\zeta = \cos^2(\mathrm{i}z - \pi/4) = (\cosh z + \mathrm{i}\sinh z)^2/2 = \{1 + \mathrm{i}\sinh(2z)\}/2, \tag{51}$$

and converges for

$$1 > |\zeta| = \sqrt{1 + \sinh^2(2z)}/2, \qquad x < l\arg\sinh(\sqrt{3}) \equiv x_0. \tag{52a, b}$$

The series expansion in the neighbourhood of $\zeta = 1$, proceeds in powers of

$$\xi \equiv 1 - \zeta = \{1 - \mathrm{i}\sinh(2z)\}/2, \qquad H(\xi) \equiv G(\zeta), \tag{53a, b}$$

and converges for

$$1 > |\xi| = \sqrt{1 + \sinh^2(2z)/2}, \qquad x < l\arg\sinh\sqrt{3} \equiv x_0. \tag{54a, b}$$

Thus, both series expansions can be obtained the same way, and have the same region of convergence; choosing to perform the change of variable (53a, b) in equation (50) leads to

$$\xi(1-\xi)H'' + (\xi - 1/2)H' + \lfloor \Omega M_0 \xi + (\Omega^2 - 1 + 6\Omega M_0)/4 \rfloor H = 0. \tag{55}$$

In the neighborhood of the regular singularity at $\zeta = 1$ or $\xi = 0$, there is a convergent Frobenius–Fuchs expansion,

$$H_\sigma(\xi) = \xi^\sigma \sum_{n=0}^{\infty} a_n(\sigma)\xi^n, \tag{56}$$

with index σ, and coefficients a_n, to be determined. Substituting equation (56) in equation (55), and equating to zero the coefficients of powers of ξ, leads to the recurrence formula for the coefficients:

$$(n+\sigma+1)(n+\sigma-1/2)a_{n+1}(\sigma) = \lfloor \lambda - (n+\sigma)^2 \rfloor a_n(\sigma) - \Omega M_0 a_{n-1}(\sigma), \tag{57a}$$

$$4\lambda \equiv \Omega^2 - 1 + 6\Omega M_0. \tag{57b}$$

The case $n = -1$, with $a_{-1} = a_{-2} = 0$, leads to $a_0\sigma(\sigma - 3/2) = 0$; if $a_0 = 0$, then from equation (57a) it follows that $a_n = 0$ for all n, and by equation (56) these results a trivial solution $H_\sigma(\xi) = 0$. For a non-trivial solution to exist $a_0 \neq 0$, implying that $\sigma = 0,\ 3/2$, which specifies two particular integrals,

$$H_0(\xi) = \sum_{n=0}^{\infty} a_n(0)\xi^n, \qquad H_{3/2}(\xi) = \sum_{n=0}^{\infty} a_n(3/2)\xi^{n+3/2}, \tag{58a, b}$$

which are linearly independent; thus the general integral is a linear combination

$$H(\xi) = A\,H_0(\xi) + B\,H_{3/2}(\xi) \tag{59}$$

and a_0 may be incorporated into the arbitrary constants of integration, A, B by setting $a_0(0) = 1 = a_0(3/2)$.

4.3. EFFECT OF MACH AND WAVENUMBER ON WAVE AMPLITUDE AND PHASE

The waveforms are plotted against dimensionless distance (60a),

$$X \equiv x/l, \qquad J_\pm(X) \equiv Z_{0,3/2}(X)/Z_{0,3/2}(0), \tag{60a, b}$$

where

$$Z_\sigma(X) \equiv H_\sigma(\xi) = \sum_{n=0}^{\infty} a_n(\sigma)\{(1 - \mathrm{i}\sinh X)/2\}^{n+\sigma} \tag{61}$$

Since equation (60b) is complex, there are separate plots of the modulus $|J_\pm|$, i.e., the ratio of amplitudes at X and $X = 0$, and the argument $\arg(J_\pm)$, i.e. phase shift between $X = 0$ and X, respectively at the top and bottom of Figures 2–5. The Mach number is given three low values (62a):

$$M = 0{\cdot}1,\ \underline{0{\cdot}2},\ 0{\cdot}3, \qquad \Omega = 0{\cdot}2,\ \underline{1},\ 5 \tag{62a, b}$$

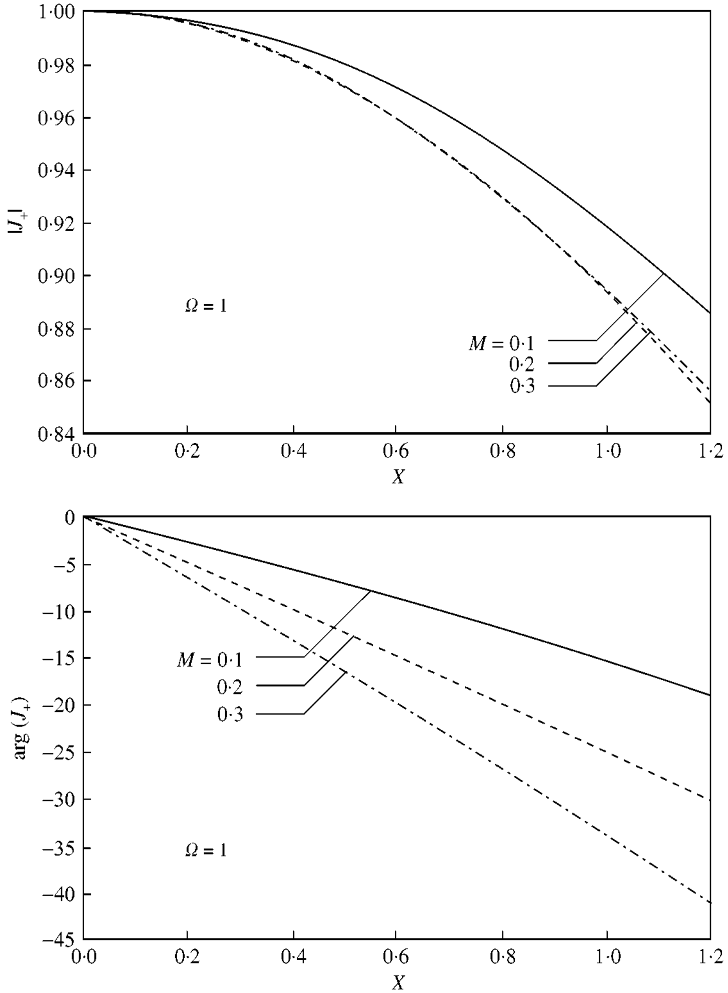

Figure 2. Ratio of amplitudes (top) and phase difference (bottom) of the reduced acoustic velocity, at dimensionless distance $X \equiv x/l$, relative to initial value at $X = 0$, for wave function J_+ finite at $X = 0$, with fixed wavenumber $K = 1$ and three values of Mach number.

and the frequency a wider range of values (62b) ranging from ray theory or high frequency $\Omega^2 \gg 1$, to compactness or low frequency $\Omega^2 \ll 1$ through the intermediate frequency $\Omega \sim 1$. Figures 2 and 3 apply to J_+ and Figures 4 and 5 to J_-, with the first of each pair having fixed $\Omega = 1$ and varying M (Figures 2 and 4), and the second fixed $M = 0{\cdot}3$ and varying Ω (Figures 3 and 5), i.e., the baseline case is underlined in equations (62a, b). From the figures, it is seen that: (Figure 2) the amplitude (top) and phase (bottom) decrease always, faster for increasing Mach number, in the case of J_+; (Figure 3) also for J_+, the amplitude (top) increases strongly with distance for low frequencies $\Omega^2 \ll 1$, and decays slightly for intermediate and high frequencies $\Omega \geqslant 1$, whereas the phase (bottom) shows the same trend reversed, i.e., increases slightly at low frequencies, and decays strongly at intermediate and high frequencies; (Figure 4) for J_-, the amplitude (top) increases, and the phase (bottom) decreases with distance, with weak dependence on Mach number, for fixed intermediate frequency $\Omega = 1$; (Figure 5) also for J_-, the dependence on frequencies is much stronger,

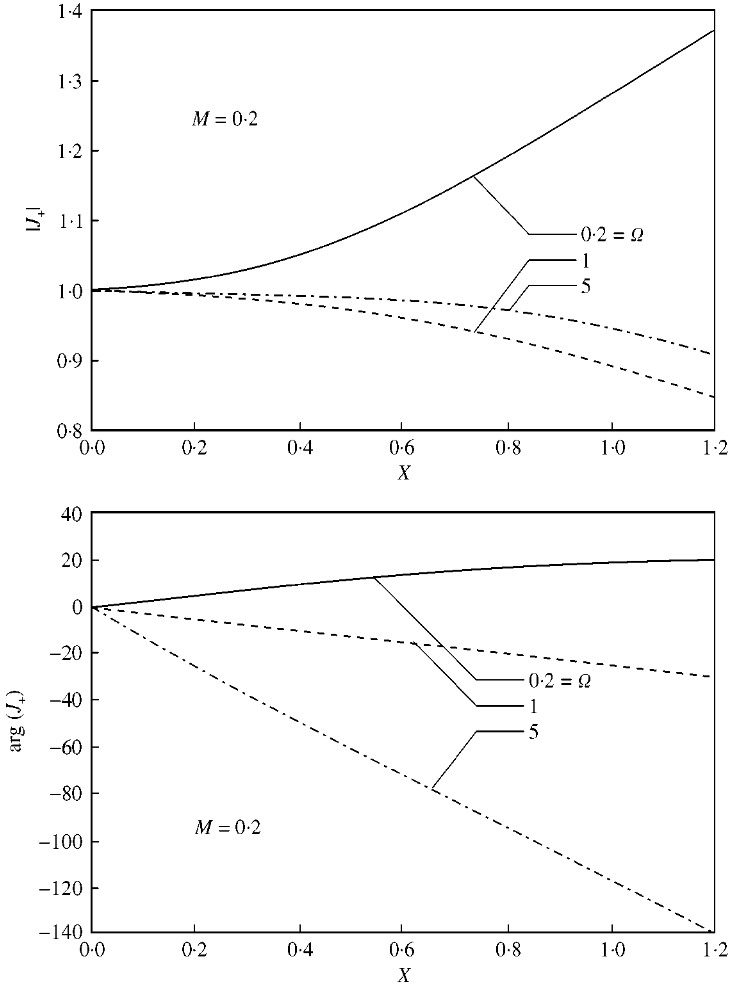

Figure 3. As Figure 2, with fixed Mach number $M = 0{\cdot}3$, and three values of wavenumber.

with amplitude (top) increasing less, and phase (bottom) decreasing more, as the frequency increases. The interpretation of these results is as follows: (i) in the ray approximation (22a, b), the sound amplitude is larger where the cross-section is smaller, and here also the mean flow velocity is larger, and hence larger phase shifts occur; (ii) the amplitude and phase, in the ray approximation, are distinct for the two inverse catenoidal ducts (45), but the correction for non-ray effects (33b, 44) is the same; (iii) this correction has been plotted in the Figures 2–5, showing where the ray theory overestimates $|J_\pm| < 1$ or underestimates $|J_\pm| > 1$ the amplitude, and where the phase shift is more $\arg(J_\pm) > 0$ or less $\arg(J_\pm) < 0$ than the prediction of ray theory.

5. DISCUSSION

The present theory assumes nearly flat wavefronts, and thus applies away from large changes of cross-section, i.e. this excludes the baffles around $X = 0$ for the csch-duct

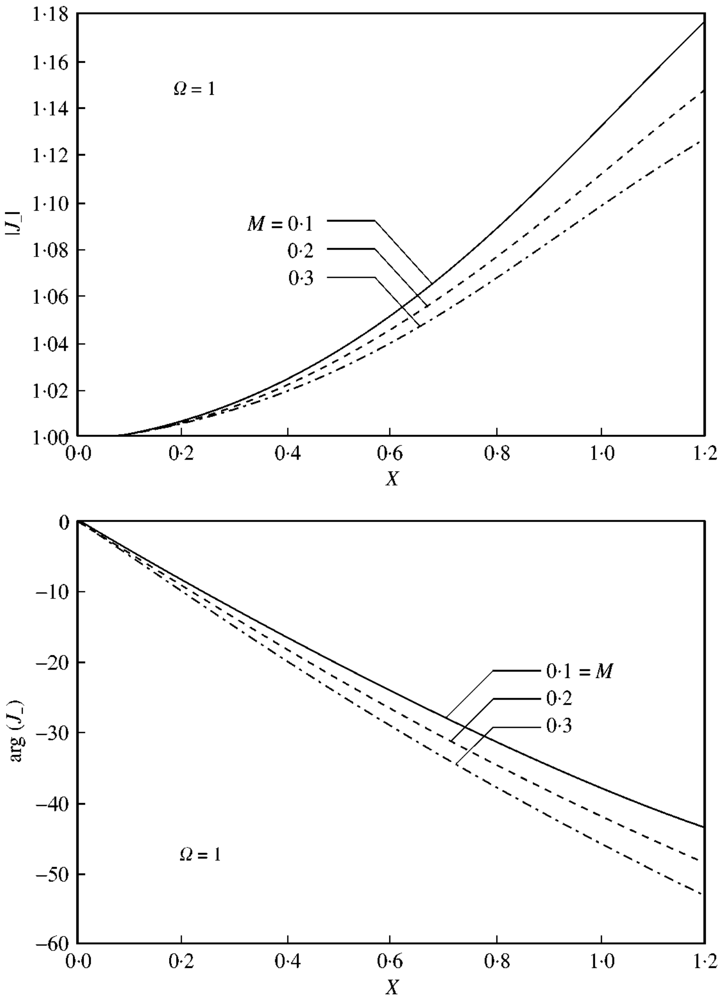

Figure 4. As Figure 1, for wave function J_- which vanishes at $X = 0$.

(Figure 1, bottom), but not the bulge around $X = 0$ for the sech-duct (Figure 1, top). In both cases the mean flow Mach number must be low, so that the theory does not apply too far down into the narrow sections. The ray solution, which applies at high frequencies, and neglects reflections from the walls, corresponds to the factor of W in equation (45). It consists (i) of an amplitude varying like the inverse square root of the cross-section, i.e., a "focussing" in converging ducts and "defocussing" in diverging ducts; (ii) of a phase shift specified by the integrated Doppler effect, i.e., a lead for propagation downstream and a lag for propagation upstream. The function W, which is plotted in Figures 2–5, as modulus (top) and argument (bottom), versus axial position, thus representing the amplitude (top) and phase (bottom) correction, relative to ray theory, when the frequency Ω is not high. For the first wavefunction $W \equiv J_+$ and frequency $\Omega = 1$ in Figure 2, the amplitude increase into the converging duct $X > 0$, is less than predicted by ray theory (top), and the latter also overestimates the phase shift (bottom), more so for increasing Mach number. For fixed Mach number $M = 0{\cdot}2$ in Figure 3, the effect on the first wave function J_+, of decreasing frequency, is an increasing amplitude (top) and decreasing phase (bottom) relative to the

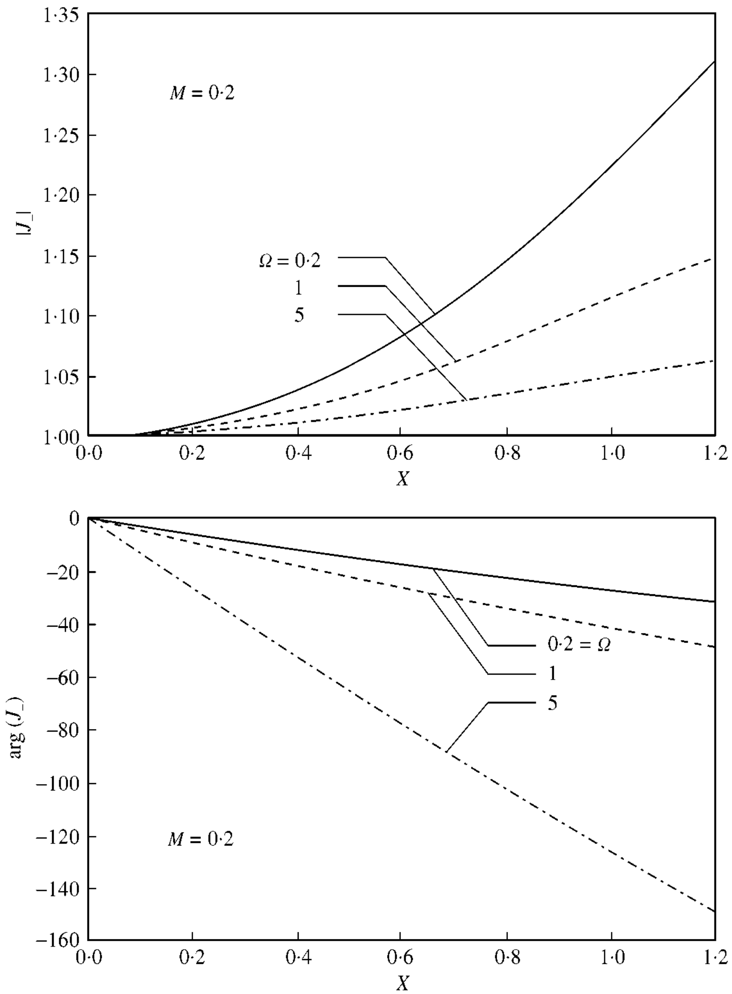

Figure 5. As Figure 2, for J_-.

prediction of ray theory; the latter overestimates amplitude at high frequency $\Omega > 1$ and underestimates at low frequency $\Omega < 1$ (top), whereas for phase (bottom) ray theory underestimates at low frequency and overestimates at high frequency. For the second wavefunction $W \equiv J_-$ (Figures 4 and 5) the situation is simpler: ray theory requires a correction factor greater than unity for amplitude (top) and a negative phase correction (bottom), i.e. it underestimates amplitude and overestimates phase; the effect is more pronounced, i.e. a larger correction is needed, as Mach number increases (Figure 4) or frequency decreases (Figure 5).

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