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FREE VIBRATION OF LAMINATED SPHERICAL PANELS WITH RANDOM MATERIAL PROPERTIES

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Randomness in the material properties is inherent in all engineering materials. Composites exhibit a greater scatter compared to conventional materials because of the larger number of parameters associated with its fabrication and manufacturing. For accurate prediction of its behavior, the composite material properties have been modelled as random variables in the present study. Higher order shear deformation theory including rotatory inertia effects has been employed in developing the system equations and first order perturbation technique has been adopted for the solution. An approach has been presented for obtaining the analytical solution for generalized eigenvalue problem associated with free vibrations. Mean and variance of the natural frequency have been obtained for cross-ply spherical laminates projected in rectangular plan form with different boundary conditions.

1. INTRODUCTION

The use of laminated curved panels is common in many engineering fields. Composites have the specific advantage that their structural characteristics can be tailored to suit the design requirements. These are finding increased use in primary and secondary structures in aerospace projects. The composites, like most structural materials, are fabricated with appropriate quality control. The control works under finite limits due to practical and economic considerations. This results in variation in material properties, making them random. The extent and nature of the variations depend on the strictness of the quality control and the characteristics of the parameters involved with the manufacturing and fabrication process. The variations in the material properties of composite lamina are greater in comparison with the conventional materials as there are larger number of parameters involved—spatial distribution and orientation of the fibers, fibers volume fraction, void fraction, interfacial bond characteristics, thickness of laminae, curing process, etc. Variations in these parameters are reflected as variations in the basic lamina material properties. Assuming the material properties as deterministic and thus using their average values ignores such variation and introduces approximation in the analysis and design. In view of this, sometimes the design may be non-conservative.

For accurate prediction of the behavior, suitable modelling of the material properties is essential. This may be appropriately handled by modelling the properties as random variables (RVs).

1.1. LITERATURE REVIEW

Extensive literature is available on response analysis of the deterministic structures to random excitations. Nigam and Narayanan [1] have considered various types of loading in

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this class of problems. However, structural analysis with random material properties is not adequately reported.

Some literature is available for the analysis of conventional material structures with random material properties. Ibrahim [2, 3] has presented a review of structural dynamics with parameters uncertainties. Singh and Lee [4] have employed direct product technique to obtain statistical properties of natural frequency for single degree mass–damper system with randomness in damping and compared the results with Monte-Carlo simulation (MCS) and perturbation technique (PT). Chen *et al.* [5] have developed a probabilistic method to evaluate the effect of uncertainty in geometrical and material properties for truss and beam problems. The results for mean and standard deviation of displacement and rotation have been obtained. Bliven and Soong [6] have analyzed simply supported Euler–Bernoulli beam with randomly varying stiffness using PT. Mean and standard deviation of frequency have been evaluated with respect to correlation distance. Collins and Thomson [7] have studied longitudinal vibration of a four-degree-of-freedom fixed–fixed rod with uncertain area, mass and stiffness using PT. Prasthofer and Beadle [8] have evaluated the dynamic response of single-degree-of-freedom structures with uncertainty in stiffness to deterministic impulsive excitation using PT. Shinozuka and Astil [9] have employed a numerical technique to obtain statistical properties of eigen values of spring-supported columns with deterministic axial loading and random material and geometrical properties. The performance of PT has been compared with the approach used. Carvani and Thomson [10] have studied the influence of damping uncertainty on frequency response of a linear multi-degree-of-freedom system and compared the results with MCS. Chen and Soroka [11] have also studied the response of a multi-degree-of-freedom system with random properties to deterministic excitations employing PT. Liu *et al.* [12] have used probabilistic finite element to study second order statistics of the dynamic response of random truss structure. Vaicatis [13] has obtained the initial free vibration response of beams with mass and flexural rigidity as random. Chen and Zang [14] have analyzed stochastic structures subjected to deterministic excitations. The sensitivity of the response has also been obtained with random design parameters such as beam cross-sectional area, plate thickness, etc. Exact roots of frequency equation of the beam system with randomness in end conditions have been obtained by Low [15]. A method has been presented by Zang and Chen [16] for obtaining standard deviation (SD) of eigenvalues and eigenvectors for a multi-degree-of-freedom random system. Grigoriu [17] has developed a method for calculating the probabilistic characteristics of the eigenvalues of stochastic symmetric matrices. Dynamic and elasticity problems have been considered to demonstrate the approach. Gorman [18] has investigated free vibration of thin rectangular plates with variable lateral edge support by the method of superposition adopting the classical laminate theory (CLT).

Limited literature is available on analysis of composite structures with random material properties. Leissa and Martin [19] have analyzed composite material panels with variable fiber spacing using CLT. Free vibration and buckling of flat plates have been analyzed, taking them to be macroscopically orthotropic but non-homogeneous because of variable fiber spacing. Results have been obtained for glass, boron and graphite fibers with epoxy matrices for simply supported square plates. Salim *et al.* [20–22] have employed first order PT (FOPT) for analysis of composite plates using CLT with random material properties and have obtained the response statistics for static deflections, natural frequencies and buckling loads of rectangular plates. The static response statistics of graphite–epoxy composite laminates with randomness in material properties to deterministic loading have been obtained by Navneethraj *et al.* [23] using combination of finite element method and MCS. Yadav and Verma [24] have investigated free vibration and initial buckling for circular cylindrical shells with random material properties using CLT and FOPT. Numerical results for mean and SD of the first three natural frequencies have been obtained for axisymmetric vibrations of specially orthotropic and antisymmetric laminates for simply supported ends. Results have also been obtained for specially orthotropic laminates in asymmetric vibrations. Mean and SD of initial buckling load for specially orthotropic shells subjected to axial compression have been obtained by Yadav and Verma [25].

Analysis of composite spherical panels with random material properties is not reported in literature. The present study aims at developing an analysis approach to evaluate the second order statistics of eigensolution for such panels including rotatory inertia effects using FOPT [16, 20]. Higher order shear deformation theory (HSDT) has been employed to account for the transverse shear effects. This approach is valid for small dispersion of material properties. As this condition is met in most applications, it does not put any real limitations on the approach. The numerical results for mean and SD for the natural frequencies have been obtained with known second order statistics of the random material properties for cross-ply symmetric panels having square plan form with various boundary conditions.

2. PROBLEM FORMULATION

2.1. GOVERNING EQUATIONS

The system equations, derived by equilibrium and compatibility considerations, do not change in the random environment and appear similar to the deterministic case. Figure 1 shows the spherical panel element. Let (ξ_1, ξ_2, ζ) denote the orthogonal curvilinear co-ordinates (or shell co-ordinates) such that the ξ_1 and ξ_2 curves are lines of curvature on the mid-surface $\zeta = 0$, and ζ curves are straight lines perpendicular to the mid-surface. R_1 and R_2 denote the values of the principal radii of curvatures of the mid-surface. The lines of principal curvature coincide with the co-ordinate lines. The figure also shows the stress resultants M_1, M_2, M_6, N_1, N_2 and N_6.

The spherical panel under consideration is composed of N orthotropic layers of uniform thickness. Let ζ_k and ζ_{k-1} be the top and bottom ζ co-ordinates of the kth lamina. The

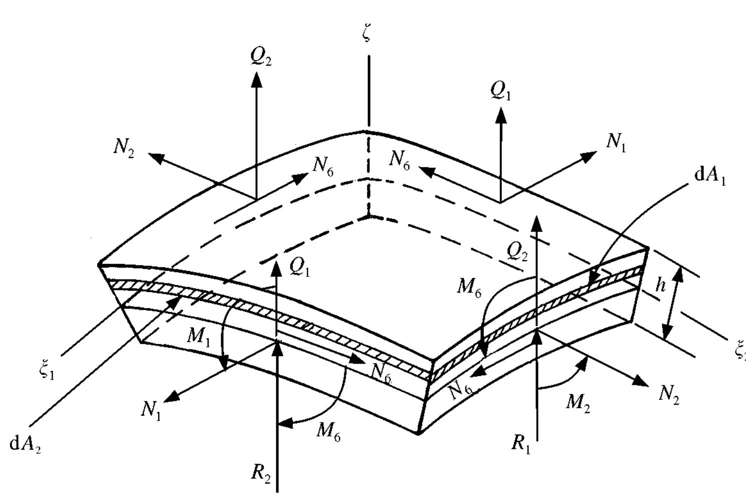

Figure 1. Spherical panel element with stress resultants.

displacement field relations according to reference [26] are

$$\bar{u}(\xi_1, \xi_2, \zeta, t) = (1 + \zeta/R_1)u + \zeta\phi_1 + \zeta^2\varphi_1 + \zeta^3\theta_1,$$
$$\bar{v}(\xi_1, \xi_2, \zeta, t) = (1 + \zeta/R_2)v + \zeta\phi_2 + \zeta^2\varphi_2 + \zeta^3\theta_2, \quad \bar{w}(\xi_1, \xi_2, \zeta, t) = w, \tag{1}$$

where t is time, $(\bar{u}, \bar{v}, \bar{w})$ are displacements along the (ξ_1, ξ_2, ζ) co-ordinates, (u, v, w) are the displacements of a point on the middle surface and ϕ_1 and ϕ_2 are the rotations at $\zeta = 0$ of normal to the mid-surface with respect to the ξ_2 and ξ_1 axes respectively. The particular choice of the displacement field in equation (1) is dictated by the desire to represent the transverse shear strains by quadratic functions of the thickness co-ordinate ζ and by the requirement that the transverse normal strains be zero.

With the above conditions the following relations are obtained:

$$\varphi_1 = \varphi_2 = 0, \qquad \theta_1 = -(4/3h^2)(\phi_1 + 1/\alpha_1(\partial w/\partial \xi_1)),$$
$$\theta_2 = -(4/3h^2)(\phi_2 + 1/\alpha_2(\partial w/\partial \xi_2). \tag{2}$$

where α_1 and α_2 are the surface metrics [26] and h is the thickness of the laminate. Substituting equation (2) into equation (1), we obtain

$$\bar{u} = (1 + \zeta/R_1)u + \zeta\phi_1 + \zeta^3(4/3h^2)[-\phi_1 - 1/\alpha_1(\partial w/\partial \xi_1)],$$
$$\bar{v} = (1 + \zeta/R_2)v + \zeta\phi_2 + \zeta^3(4/3h^2)[-\phi_2 - 1/\alpha_2(\partial w/\partial \xi_2)], \quad \bar{w} = w. \tag{3}$$

This displacement field, equation (3), is used to compute the stresses and strains. The equations of motion are derived using the dynamic analog of the principle of virtual work.

The strain-displacement relations using equation (3) with reference to curvilinear co-ordinate system are

$$\varepsilon_1 = \varepsilon_1^0 + \zeta(\kappa_1^0 + \zeta^2\kappa_1^2), \qquad \varepsilon_2 = \varepsilon_2^0 + \zeta(\kappa_2^0 + \zeta^2\kappa_2^2),$$
$$\varepsilon_4 = \varepsilon_4^0 + \zeta^2\kappa_4^1, \qquad \varepsilon_5 = \varepsilon_5^0 + \zeta^2\kappa_5^1, \qquad \varepsilon_6 = \varepsilon_6^0 + \zeta(\kappa_6^0 + \zeta^2\kappa_6^2), \tag{4}$$

where

$$\varepsilon_1^0 = \partial u/\partial x_1 + w/R_1, \qquad \kappa_1^0 = \partial\phi_1/\partial x_1, \qquad \kappa_1^2 = -(4/3h^2)(\partial\phi_1/\partial x_1 + \partial^2 w/\partial x_1^2),$$
$$\varepsilon_2^0 = \partial v/\partial x_2 + w/R_2, \qquad \kappa_2^0 = \partial\phi_2/\partial x_2, \qquad \kappa_2^2 = -(4/3h^2)(\partial\phi_2/\partial x_2 + \partial^2 w/\partial x_2^2),$$
$$\varepsilon_4^0 = \phi_2 + \partial w/\partial x_2, \qquad \kappa_4^1 = -(4/h^2)(\phi_2 + \partial w/\partial x_2), \qquad \varepsilon_5^0 = \phi_1 + \partial w/\partial x_1, \tag{5}$$
$$\kappa_5^1 = -(4/h^2)(\phi_1 + \partial w/\partial x_1), \qquad \varepsilon_6^0 = \partial u/\partial x_2 + \partial v/\partial x_1, \qquad \kappa_6^0 = \partial\phi_2/\partial x_1 + \partial\phi_1/\partial x_2,$$
$$\kappa_6^2 = -(4/3h^2)(\partial\phi_2/\partial x_1 + \partial\phi_1/\partial x_2 + 2\partial^2 w/\partial x_1 \partial x_2).$$

Here x_1, x_2 and ζ are axial, circumferential and radial Cartesian co-ordinates respectively $(\mathrm{d}x_i = \alpha_i\,\mathrm{d}\zeta_i,\ i = 1, 2)$.

The stress–strain relations for the kth lamina are given by

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \\ \sigma_4 \\ \sigma_5 \end{Bmatrix} = \begin{bmatrix} Q_{11}^{(k)} & Q_{12}^{(k)} & Q_{16}^{(k)} & 0 & 0 \\ & Q_{22}^{(k)} & Q_{26}^{(k)} & 0 & 0 \\ & & Q_{66}^{(k)} & 0 & 0 \\ & sym. & & Q_{44}^{(k)} & 0 \\ & & & & Q_{55}^{(k)} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ \varepsilon_4 \\ \varepsilon_5 \end{Bmatrix}_{(k)}, \tag{6}$$

where $Q_{ij}^{(k)}$ are the material constants of the kth lamina in the laminate co-ordinate system [26].

The equation of motions for forced vibrations of panels including effects of transverse shear and rotatory inertia may be written using the principle of virtual work [26] as

$$\begin{aligned} &\partial N_1/\partial x_1 + \partial N_6/\partial x_2 = \bar{I}_1\ddot{u} + \bar{I}_2\ddot{\phi}_1 - \bar{I}_3\partial\ddot{w}/\partial x_1, \\ &\partial N_6/\partial x_1 + \partial N_2/\partial x_2 = \bar{I}'_1\ddot{v} + \bar{I}'_2\ddot{\phi}_2 - \bar{I}'_3\partial\ddot{w}/\partial x_2, \\ &\partial Q_1/\partial x_1 + \partial Q_2/\partial x_2 - (4/h^2)(\partial K_1/\partial x_1 + \partial K_2/\partial x_2) \\ &\quad + (4/3h^2)(\partial^2 P_1/\partial x_1^2 + \partial^2 P_2/\partial x_2^2 + 2\partial^2 P_6/\partial x_1\partial x_2) - N_1/R_1 - N_2/R_2 \\ &\quad = \bar{I}_3\partial\ddot{u}/\partial x_1 + \bar{I}_5\partial\ddot{\phi}_1/\partial x_1 + \bar{I}'_3\partial\ddot{v}/\partial x_2 + \bar{I}'_5\partial\ddot{\phi}_2/\partial x_2 + I_1\ddot{w} - (16I_7/9h^2)(\partial^2\ddot{w}/\partial x_1^2 + \partial^2\ddot{w}/\partial x_2^2) - q, \\ &\partial M_1/\partial x_1 + \partial M_6/\partial x_2 - Q_1 + (4/h^2)K_1 - (4/3h^2)(\partial P_1/\partial x_1 + \partial P_6/\partial x_2) = \bar{I}_2\ddot{u} + \bar{I}_4\ddot{\phi}_1 - \bar{I}_5\partial\ddot{w}/\partial x_1, \\ &\partial M_6/\partial x_1 + \partial M_2/\partial x_2 - Q_2 + (4/h^2)K_2 - (4/3h^2)(\partial P_6/\partial x_1 + \partial P_2/\partial x_2) = \bar{I}'_2\ddot{v} + \bar{I}'_4\ddot{\phi}_2 - \bar{I}'_5\partial\ddot{w}/\partial x_2, \end{aligned} \tag{7}$$

where $q(x_1, x_2, t)$ is the distributed transverse load and, N_i, M_i, etc. are the stress resultants given by

$$\begin{aligned} (N_i, M_i, P_i) &= \sum_{k=1}^{N} \int_{\zeta_{k-1}}^{\zeta_k} \sigma_i^{(k)}\,(1, \zeta, \zeta^3)\,\mathrm{d}\varsigma, \quad i = 1, 2, 6, \\ (Q_1, K_1) &= \sum_{k=1}^{N} \int_{\zeta_{k-1}}^{\zeta_k} \sigma_5^{(k)}\,(1, \zeta^2)\,\mathrm{d}\varsigma, \\ (Q_2, K_2) &= \sum_{k=1}^{N} \int_{\zeta_{k-1}}^{\zeta_k} \sigma_4^{(k)}\,(1, \zeta^2)\,\mathrm{d}\varsigma. \end{aligned} \tag{8}$$

The inertias $\bar{I}'_i$ and $\bar{I}_i$, $i = 1, 2, 3, 4, 5$ are defined by the equations

$$\begin{aligned} &\bar{I}_1 = I_1 + 2I_2/R_1, \quad \bar{I}'_1 = I_2 + 2I_2/R_2, \qquad \bar{I}_2 = I_2 + I_3/R_4 - 4I_4/3h^2 - 4I_5/3h^2R_1, \\ &\bar{I}'_2 = I_2 + I_3/R_2 - 4I_4/3h^2 - 4I_5/3h^2R_2, \qquad \bar{I}_3 = 4I_4/3h^2 + 4I_5/3h^2R_1, \\ &\bar{I}'_3 = 4I_4/3h^2 + 4I_5/3h^2R_2, \\ &\bar{I}_4 = I_3 - 8I_5/3h^2 + 16I_7/9h^4, \qquad \bar{I}'_4 = I_3 - 8I_5/3h^2 + 16I_7/9h^4, \\ &\bar{I}_5 = 4I_5/3h^2 - 16I_7/9h^4, \qquad \bar{I}'_5 = 4I_5/3h^2 - 16I_7/9h^4, \\ &(I_1, I_2, I_3, I_4, I_5, I_7) = \sum_{k=1}^{N} \int_{\zeta_{k-1}}^{\zeta_k} \rho^{(k)}\,(1, \zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^6)\,\mathrm{d}\zeta, \end{aligned} \tag{9}$$

where ρ is the mass per unit volume.

The stress resultants are obtained by summing contributions of the individual layer. This yields

$$N_i = A_{ij}\varepsilon_j^0 + B_{ij}\kappa_j^0 + E_{ij}\kappa_j^2, \qquad M_i = B_{ij}\varepsilon_j^0 + D_{ij}\kappa_j^0 + F_{ij}\kappa_j^2,$$
$$P_i = E_{ij}\varepsilon_j^0 + F_{ij}\kappa_j^0 + H_{ij}\kappa_j^2, \qquad i, j = 1, 2, 6, \tag{10}$$

$$Q_2 = A_{4j}\varepsilon_j^0 + D_{4j}\kappa_j^1, \qquad Q_1 = A_{5j}\varepsilon_j^0 + D_{5j}\kappa_j^1,$$
$$K_2 = D_{4j}\varepsilon_j^0 + F_{4j}\kappa_j^1, \qquad K_1 = D_{5j}\varepsilon_j^0 + F_{5j}\kappa_j^1, \quad j = 4, 5 \tag{11}$$

where A_{ij}, B_{ij}, etc. are the laminate stiffnesses expressed as

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) = \sum_{k=1}^{N} \int_{\varsigma_{k-1}}^{\zeta_k} Q_{ij}^{(k)} (1, \varsigma, \varsigma^2, \varsigma^3, \varsigma^4, \varsigma^6)\, \mathrm{d}\varsigma, \quad i, j = 1, 2, \dots, 6. \tag{12}$$

For cross-ply spherical panels, the following laminate stiffnesses are identically equal to zero

$$A_{i6}, B_{i6}, D_{i6}, E_{i6}, F_{i6}, H_{i6} = 0, \quad i = 1, 2, \qquad A_{45}, D_{45}, F_{45} = 0. \tag{13}$$

2.2. FREE VIBRATION

The free vibration condition is obtained by setting the forcing term to zero in the governing equation (7). That is

$$q(x_1, x_2, t) = 0. \tag{14}$$

The equations of motion may now be arranged as

$$\mathbf{L}\Lambda = \mathbf{0}, \tag{15}$$

where $\mathbf{\Lambda} = [u\ v\ w\ \phi_1\ \phi_2]^{\mathrm{T}}$. The symmetric operators L_{ij} are listed in Appendix A. These are random in nature as these involve the random material properties. Consequently, equation (15) is an equation in random variables and parameters.

2.3. EIGENPROBLEM SOLUTION

The solution technique changes with the edge support conditions of the panel. When two opposite edges are simply supported with other two side edges having combination of free, fixed and simple support, a Levy-type closed-form solution is possible in conjunction with state-space approach. A detailed sequence of steps is outlined below for such a problem to obtain the second order statistics of the natural frequencies and mode shapes.

For those combinations of edge supports that are not amenable to exact solution approach, equation (15) can be transformed to a generalized eigenvalue problem by using approaches like series solutions, approximate energy and variational methods, finite element method and other numerical techniques. Beyond this point the steps required for the statistics of the natural frequencies and mode shapes would be the same as those presented below for the exact solutions.

2.3.1. *Exact solutions approach*

The state-space concept [27, 28] is used to analyze the free vibration problem of cross-ply spherical panels. The edges $x_2 = 0$ and b are assumed to be simply supported, while the remaining ones $x_1 = 0$ and $x_2 = a$ may have an arbitrary combination of free, clamped, and simply supported edge conditions. We express the generalized displacements as products of undetermined functions and known trigonometric functions so as to satisfy identically the simply supported boundary conditions at $x_2 = 0$ and b:

$$u = w = \phi_1 = N_2 = M_2 = P_2 = 0. \tag{16}$$

Assuming the systems to vibrate in a principal mode, the displacement quantities may be written as

$$\begin{aligned} u &= U_m(x_1) f_1(x_2) \exp(\mathrm{i}\omega t), \qquad v = V_m(x_1) f_2(x_2) \exp(\mathrm{i}\omega t), \\ w &= W_m(x_1) f_3(x_2) \exp(\mathrm{i}\omega t), \qquad \phi_1 = X_m(x_1) f_1(x_2) \exp(\mathrm{i}\omega t), \\ \phi_2 &= Y_m(x_1) f_2(x_2) \exp(\mathrm{i}\omega t), \end{aligned} \tag{17}$$

where $\mathrm{i} = \sqrt{-1}$ and

$$f_1(x_2) = \sin \beta x_2, \qquad f_2(x_2) = \cos \beta x_2, \qquad f_3(x_2) = \sin \beta x_2, \qquad \beta = n\pi/b, \tag{18}$$

Substitution of equation (17) into equation (15) yields equations dependent on space co-ordinates only. These are put in the state-space form to allow use of standard solution expressions. For this, the following variables are introduced:

$$\begin{aligned} &Z_1 = U_m, \quad Z_2 = U'_m, \quad Z_3 = V_m, \quad Z_4 = V'_m, \quad Z_5 = W_m, \quad Z_6 = W'_m, \\ &Z_7 = W''_m, \quad Z_8 = W'''_m, \quad Z_9 = X_m, \quad Z_{10} = X'_m, \quad Z_{11} = Y_m, \quad Z_{12} = Y'_m, \end{aligned} \tag{19}$$

where primes over the variables indicate differentiation with respect to x_1. The system equation takes the form

$$\mathbf{Z}' = \mathbf{A}\mathbf{Z}, \tag{20}$$

where matrix $\mathbf{A}$ depends on the system stiffness, rotatory inertia and wavelength parameters and hence is random.

A formal solution to equation (20) is given by [27, 28]

$$\{\mathbf{Z}(x_1)\} = \mathrm{e}^{\mathbf{A}x_1}\{\mathbf{\Delta}\}, \tag{21}$$

where Δ is a constant column vector associated with the boundary conditions and e^{Ax_1} is given by

$$\mathrm{e}^{Ax_1} = \mathbf{S} \begin{bmatrix} \mathrm{e}^{\lambda_1 x_1} & & 0 \\ & \ddots & \\ 0 & & \mathrm{e}^{\lambda_n x_1} \end{bmatrix} \mathbf{S}^{-1}. \tag{22}$$

The value of n is 12. Here λ_i denotes the eigenvalues of $\mathbf{A}$ and $\mathbf{S}$ is the modal column matrix of eigenvectors of $\mathbf{A}$.

Substitution of equation (21) into boundary conditions associated with the remaining two opposite edges $x_1 = 0$ and a, results in a homogeneous system of equations which can be rearranged into the generalized eigenvalue problem form

$$\mathbf{K}\Delta = \lambda\mathbf{M}\Delta, \tag{23}$$

where $\mathbf{K}$ and $\mathbf{M}$ are real symmetric stiffness and inertia matrices, and Δ is the mode shape and $\lambda = \omega^2$ with ω being the frequency of natural vibration.

2.3.1.1. *Boundary conditions.* The boundary conditions for simply supported (S), clamped (C), and free (F) at the edges $x_1 = 0$ and a are

S $\quad v = w = \phi_2 = N_1 = M_1 = P_1 = 0,$

C $\quad u = v = w = \dfrac{\partial w}{\partial x_1} = \phi_1 = \phi_2 = 0,$

F $\quad N_1 = N_6 = M_1 = P_1 = \left(M_6 - \dfrac{4}{3h^2}P_6\right) = \left\{Q_1 - \dfrac{4}{h^2}R_1 + \dfrac{4}{3h^2}\left(\dfrac{\partial P_1}{\partial x_1} + \dfrac{\partial P_6}{\partial x_2}\right)\right\} = 0.$

The stiffness matrix $\mathbf{K}$ is random in nature, being dependent on the system material properties. Therefore, the eigenvalues λ and natural frequency ω are random. Hence eigenvectors and the associated displacement shape functions are also rendered random. The solution approach presented attempts to obtain the statistics of these characteristics.

2.3.1.2. *Second order statistics of the eigensolution—perturbation approach.* Without any loss of generality the random variables may be split up as the sum of a mean and a zero-mean random part.

$$\lambda_i = \bar{\lambda}_i + \lambda_i^r, \qquad \mathbf{\Delta}_i = \bar{\mathbf{\Delta}}_i + \mathbf{\Delta}_i^r, \qquad \mathbf{K} = \bar{\mathbf{K}} + \mathbf{K}^r \tag{24}$$

where

$$\bar{\lambda}_i = \bar{\omega}_i^2, \qquad \lambda_i^r = 2\bar{\omega}_i\omega_i^r + \omega_i^{r^2}, \quad i = 1, 2, \ldots, n. \tag{25}$$

The over bar denotes the mean and superscript 'r' denotes the zero-mean random part of the variables. Consider now a class of problems where the random variation is small as compared with the mean part of the material properties. This is observed in most engineering applications including composites. Further, it is quite logical to assume that the dispersion in derived quantities like λ, ω, $\mathbf{\Delta}$ and $\mathbf{K}$ are also small as compared to their mean values.

Substituting equations (24) into equation (23), expanding, collecting the same order of magnitude terms, we obtain for zero and first orders:

$$\overline{\mathbf{K}\mathbf{\Delta}_i} = \bar{\lambda}_i\mathbf{M}\bar{\mathbf{\Delta}}_i, \tag{26}$$

$$(\bar{\mathbf{K}} - \bar{\lambda}_i\mathbf{M})\mathbf{\Delta}_i^r = -(\mathbf{K}^r - \lambda_i^r\mathbf{M})\bar{\mathbf{\Delta}}_i. \tag{27}$$

Equation (26) is a deterministic equation relating the mean quantities and is the same as that obtained in the deterministic analysis. The mean eigenvalues and corresponding mean eigenvectors can be determined by conventional eigensolution procedures [27].

For the all-distinct eigenvalues, the normalized eigenvectors meet the orthogonality conditions

$$\begin{gathered}\bar{\boldsymbol{\Delta}}_i^{\mathrm{T}}\mathbf{M}\bar{\boldsymbol{\Delta}}_j = \delta_{ij},\\ \bar{\boldsymbol{\Delta}}_i^{\mathrm{T}}\overline{\mathbf{K}\boldsymbol{\Delta}}_j = \delta_{ij}\bar{\lambda}_i, \quad i,j = 1,2,\ldots,n,\end{gathered} \tag{28}$$

where δ_{ij} is the Kronecker delta.

The mean eigenvectors form a complete orthonormal set and any vector in the space can be expressed as its linear combination. Hence, the ith random part of the eigenvectors can be written as [27]

$$\boldsymbol{\Delta}_i^r = \sum_{j=1}^{n} C_{ij}^r \bar{\boldsymbol{\Delta}}_j, \quad j \neq i, \qquad C_{ii}^r = 0, \tag{29}$$

where C_{ij}^r's are random coefficients to be determined.

Substituting equation (29) into the first order equation (27), pre-multiplying by $\bar{\boldsymbol{\Delta}}_i^{\mathrm{T}}$ and $\bar{\boldsymbol{\Delta}}_k^{\mathrm{T}}$ ($i \neq k$), respectively and invoking orthogonality, we have

$$\lambda_i^r = \bar{\boldsymbol{\Delta}}_i^{\mathrm{T}}\mathbf{K}^r\bar{\boldsymbol{\Delta}}_i, \tag{30}$$

$$\boldsymbol{\Delta}_i^r = \sum_{j=1}^{n} \bar{\boldsymbol{\Delta}}_j \frac{\bar{\boldsymbol{\Delta}}_j^{\mathrm{T}}\mathbf{K}^r\bar{\boldsymbol{\Delta}}_i}{\bar{\lambda}_i - \bar{\lambda}_j}, \quad j \neq i. \tag{31}$$

For the l multiple eigenvalues $\bar{\lambda}_i$, λ_i^r is determined by [16]

$$D\eta - \lambda_i^r C\eta = 0, \tag{32}$$

where D and C are the matrices with elements

$$d_{kj} = \bar{g}_k^{\mathrm{T}}\mathbf{K}^r\bar{g}_j, \qquad c_{kk}\ \bar{\mathbf{g}}_k^{\mathrm{T}}\mathbf{M}\bar{\mathbf{g}}_k. \tag{33}$$

$$\bar{\boldsymbol{\Delta}}_i = \sum_{j=1}^{m} \eta_i^j \bar{g}_j, \quad i = 1, 2, \ldots, l \tag{34}$$

We get the l eigenvectors $\bar{\boldsymbol{\Delta}}_i$, $i = 1,2,\ldots,l$, and $\boldsymbol{\Delta}_k^r$ can be expressed by

$$\boldsymbol{\Delta}_k^r \begin{cases} = \displaystyle\sum_{j=1}^{n} \bar{\boldsymbol{\Delta}}_j \frac{\bar{\boldsymbol{\Delta}}_j^{\mathrm{T}}\mathbf{K}^r\bar{\boldsymbol{\Delta}}_k}{\bar{\lambda}_k - \bar{\lambda}_j}, \quad k = i,\ i+1,\ldots,i+l-1,\ j = 1,\ldots,i+1,\ldots,n \\ = 0, \quad k,\ j = i,\ldots,i+l-1,\ j \neq k \end{cases} \tag{35}$$

For the present case λ, ω, $\boldsymbol{\Delta}$ and $\mathbf{K}$ are random because the material properties, as detailed earlier, are random. Let $b_1, b_2, \ldots, b_m$ denote the random material properties. The b_j can also be expressed as

$$b_j = \bar{b}_j + b_j^r. \tag{36}$$

According to first order Taylor's rule, when b_j^r are small compared with their mean values, we can expand the dependent quantities λ, $\mathbf{\Delta}$ and $\mathbf{K}$ about their mean values, giving their random parts as

$$\lambda_i^r = \sum_{j=1}^{m} \bar{\lambda}_{i,j} b_j^r, \qquad \mathbf{\Delta}_i^r = \sum_{j=1}^{m} \bar{\mathbf{\Delta}}_{i,j} b_j^r, \qquad \mathbf{K}^r = \sum_{j=1}^{m} \bar{\mathbf{K}}_{,j} b_j^r, \tag{37}$$

where $_{,j}$ denotes partial differentiation with respect to b_j and the derivatives are evaluated at $\bar{b}_j$.

For the distinct eigenvalue systems, we have [27]

$$\bar{\lambda}_{i,j} = \bar{\mathbf{\Delta}}_i^{\mathrm{T}} \bar{\mathbf{K}}_{,j} \bar{\mathbf{\Delta}}_i, \tag{38}$$

$$\bar{\mathbf{\Delta}}_{i,j} = \sum_{\substack{s=1 \\ s \neq i}}^{n} \bar{\mathbf{\Delta}}_s \frac{\bar{\mathbf{\Delta}}_s^{\mathrm{T}} \mathbf{K}_{,j} \bar{\mathbf{\Delta}}_i}{\bar{\lambda}_i - \bar{\lambda}_s}. \tag{39}$$

For the multiple eigenvalue system, $\bar{\lambda}_{i,j}$ satisfies the following equation:

$$A\eta - \bar{\lambda}_{i,j} C\eta = 0, \tag{40}$$

where C is the same matrix as in equation (32). The elements of $\mathbf{A}$ are

$$A_{rs} = \bar{g}_r^{\mathrm{T}} \bar{\mathbf{K}}_{,j} \bar{g}_s, \quad r, s = 1, 2, \ldots, l. \tag{41}$$

$\bar{\mathbf{\Delta}}_{i,j}$ can be expressed as

$$\bar{\mathbf{\Delta}}_{i,j} = \sum_{s=1}^{n} \bar{\mathbf{\Delta}}_s \frac{\bar{\mathbf{\Delta}}_s^{\mathrm{T}} \mathbf{K}_{,j} \bar{\mathbf{\Delta}}_i}{\bar{\lambda}_i - \bar{\lambda}_s}, \quad s \neq i, i+1, \ldots, i+l-1. \tag{42}$$

Using equations (37) the eigenvalue and mode shape covariance [29, 30] are obtained as

$$Var(\lambda_i) = \sum_{j=1}^{m} \sum_{k=1}^{m} \bar{\lambda}_{i,j} \bar{\lambda}_{i,k} Cov(b_j, b_k), \qquad Var(\mathbf{\Delta}_i\ \mathbf{\Delta}_i^{*\mathrm{T}}) = \sum_{j=1}^{m} \sum_{k=1}^{m} \bar{\mathbf{\Delta}}_{i,j}^{l} \bar{\mathbf{\Delta}}_{i,k}^{l} Cov(b_j, b_k), \tag{43}$$

where $Cov(b_j, b_k)$ is the covariance between b_j and b_k. The variances of λ_i and $\mathbf{\Delta}_i$ can be evaluated from equation (43) with the help of equations (38), (39), (41) and (42).

3. RESULTS AND DISCUSSION

The outlined approach is validated by comparison with MCS results and then used to evaluate the second order statistics for the natural frequencies of two layer antisymmetric cross-ply [0°/90°] graphite–epoxy spherical panels with SSSS, SCSC, SSSC, SFSS, and SFSC boundary conditions. The lamina material properties E_{11}, E_{22}, G_{12}, G_{13}, G_{23} and

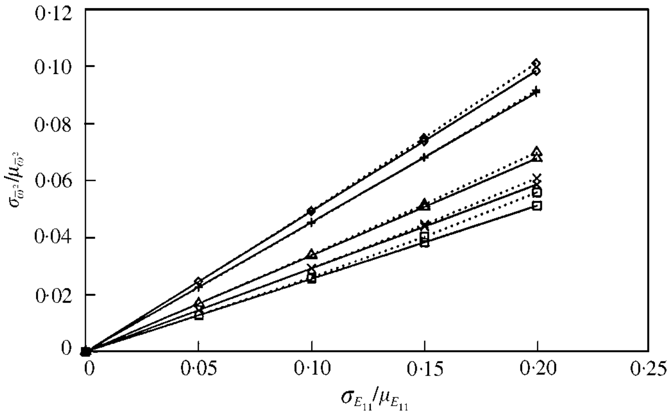

Figure 2. Comparison of results from Monte-Carlo simulation with the present approach, [0/90/90/0°] laminate, with $R/a = 5$, $a/b = 1$ and $a/h = 10$ for SSSS. Key: ◇, first mode; +, second mode; □, third mode; ×, fourth mode; △, fifth mode. ——, PT; ······, MCS.

ν_{12} are modelled as basic RVs. Here E_{11} and E_{22} are longitudinal elastic and transverse elastic moduli respectively. G_{12} is in-plane shear modulus, G_{13} and G_{23} are out-of-plane moduli and ν_{12} is the Poisson ratio. The relationship between their mean values is assumed to be as follows [26]:

$$\bar{E}_{11} = 25\bar{E}_{22}, \qquad \bar{G}_{12} = \bar{G}_{13} = 0{\cdot}5\bar{E}_{22}, \qquad \bar{G}_{23} = 0{\cdot}2\bar{E}_{22}, \qquad \bar{\nu}_{12} = 0{\cdot}25, \qquad \rho = 1.$$

3.1. VALIDATION STUDY

The results obtained by the outlined approach have been compared with MCS. Figure 2 shows the comparison for a [0/90/90/0°] laminate with $R/a = 5$, $a/b = 1$, $a/h = 10$ and only E_{11} is assumed as random, other material properties being deterministic for all edges simply supported (SSSS). For the MCS technique, the material property samples are obtained by generating a set of random numbers to fit the desired mean, and standard deviation (SD). The number of samples simulated for simulation based on convergence is 15 000. These values are used in equation (23), which is solved repeatedly to generate a sample of the natural frequencies. This sample is processed to obtain the second order statistics of the first five natural frequencies. For the range of SD considered in the variable E_{11} the results from the present approach come very close to MCS results. One can conclude that the FOPT adopted for the present analysis is sufficient to give accurate results for the level of variations considered in the basic random variables.

3.2. TWO LAYERED ANTISYMMETRIC CROSS-PLY [0/90°] LAMINATE

The effect of the randomness in the material properties on the panel natural frequency has been obtained by allowing the ratio of the SD to mean to vary from 0 to 20% for laminated cross-ply [0/90°] spherical panels with $R_1 = R_2 = R$. All the variances have been normalized with the corresponding mean values. The panel geometry used is $R/a = 5$, $a/b = 1$ and $a/h = 10$. Results have been obtained for the mean and the variances of the natural frequencies and mode shapes for different edge support conditions.

TABLE 1

Non-dimensionalized mean natural frequencies of laminated spherical panels with $R_1 = R_2 = R, a/b = 1, a/h = 10, R/a = 5$ *and* $\varpi = (\omega\, a^2\sqrt{\rho/E_{22}})/h$: *Stacking sequence*: [0/90°] *for all edges simply supported* (*SSSS*)

Natural frequency, ϖ					
Mode	1	2	3	4	5
ϖ	9·3412	22·0381	22·2027	30·4377	39·9879

TABLE 2

Non-dimensionalized mean fundamental frequency of [0/90°] *spherical panels with* $R_1 = R_2 = R$ *and* $R/a = 5$, $a/b = 1$, $a/h = 10$ *and* $\varpi = (\omega a^2\sqrt{\rho/E_{22}})/h$ *with various boundary conditions*

Fundamental frequency, ϖ				
SCSC	SSSC	SFSS	SFSC	SSSS
14·5235	14·4838	6·4025	6·4245	9·3412

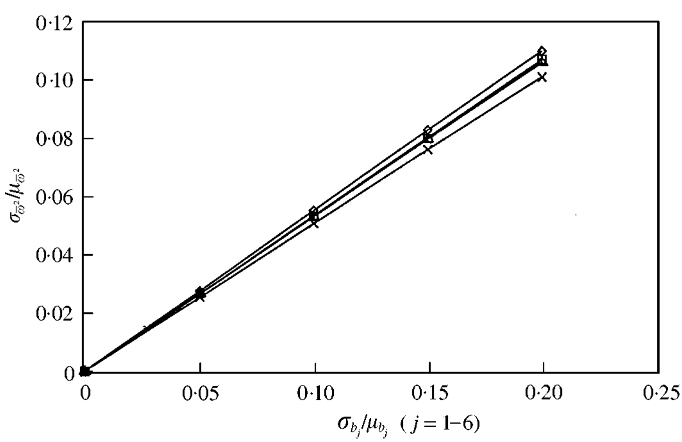

Figure 3. Variation of SD of the first five natural frequencies with SD of basic material properties, [0/90°] laminate, $R/a = 5$, $a/h = 10$ and $a/b = 1$, with all basic material properties changing simultaneously for SSSS. Key: as in Figure 2.

3.2.1. *Mean frequency*

The mean values of the natural frequencies and mode shapes have been obtained as the solution of the deterministic eigenvalue problem in equation (26). The frequency has been non-dimensionalized using material and geometric parameters as $\varpi = (\omega a^2\sqrt{\rho/\bar{E}_{22}}/h$. Table 1 shows the first five non-dimensionalized mean natural frequencies for the laminate for all edges simply supported (SSSS). The difference in mean natural frequencies between the first and second modes is greater than that for any other consecutive modes. The first five natural frequencies are associated with predominantly radial oscillations as indicated by the mode shapes (not presented here).

To examine the effects of different support conditions on natural frequency, the mean values of the non-dimensionalized fundamental frequency for the laminate with SCSC, SSSC, SFSS, and SFSC boundary conditions are presented in Table 2. The results show that the fundamental frequencies for SCSC and SSSC are greater than that for any other support conditions. It also shows that the fundamental frequencies for SFSS and SFSC are smaller than that for any other support conditions.

3.2.2. *Frequency variance*

The variances of the square of the non-dimensionalized natural frequencies have been obtained for different SD of the basic material properties. Figure 3 represents the variation

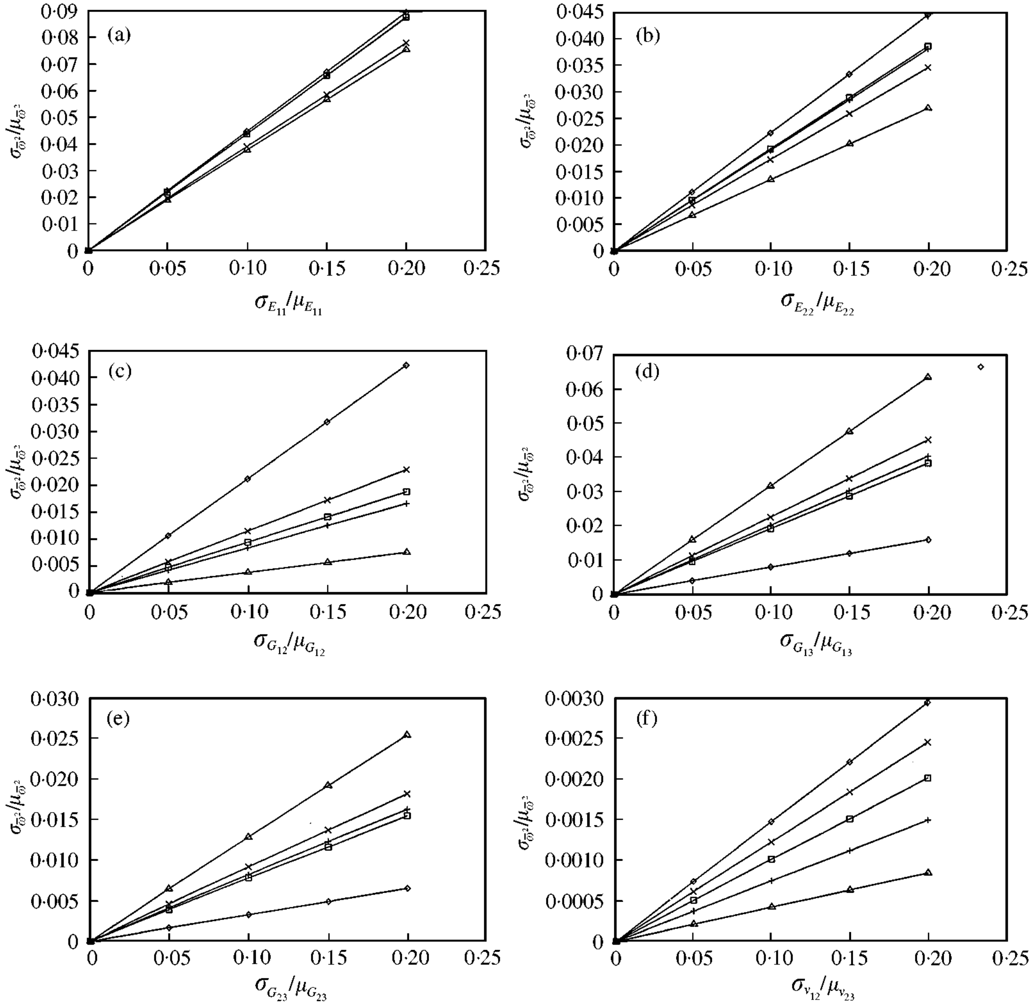

Figure 4. Variation of SD of the first five natural frequencies with SD of basic material properties. [0/90°] laminate, with $R/a = 5$, $a/b = 1$ and $a/h = 10$ for SSSS. (a) Only E_{11} varying; (b) only E_{22} varying; (c) only G_{12} varying; (d) only G_{13} varying; (e) only G_{23} varying; (f) only v_{12} varying. Key: as in Figure 2.

of the normalized SD of the first five natural frequencies of the laminate to normalized SD of the material properties for all edges simply supported (SSSS) while Figure 4(a)–(f) show the plots with only one material property SD changing at a time, others being held constant at zero level. This is equivalent to only one property being random while the others are modelled as deterministic. These graphs show that the change in the normalized SD of ϖ^2 is linear with the change in the material property SD. The dispersion in the first five natural frequencies shows close growth rates with simultaneous variations in the material properties. The effect of simultaneous changes in all basic RVs on the fundamental frequency is found to be greater than that for any other natural frequencies for the thickness ratio considered. Compared to the others, the fundamental frequency shows higher sensitivity to the individual variation of E_{11}, G_{12}, E_{22} and ν_{12}. Amongst the material properties the effects of E_{11} on natural frequencies is found to be the strongest.

For comparison of the effect of edge supports Figure 5 shows the variation of the normalized SD of the non-dimensionalized fundamental frequency of the laminate to SD of the material properties changing simultaneously for SSSS, SCSC, SSSC, SFSS, and SFSC. It is observed that the effect of simultaneous changes in the material properties on the fundamental frequency is highest for SFSS and lowest for SSSS. SFSC lies in between SSSC and SCSC. However, these differences in dispersion are very small. It is further observed that almost equal sensitivity is shown by pairing of SFSS with SFSC and SSSC with SCSC. Figure 6(a)–(f) show the variation of the fundamental frequency of the laminate with the change in only one material property at a given time for SSSS, SCSC, SSSC, SFSS, and SFSC. From examination of the results, it is observed that the effect of E_{11} on dispersion in the fundamental frequency for SFSS and SFSC is the highest while, the dispersions for SSSS is lowest. It is also observed that the influence of G_{12} for SSSS is greater compared to any other support conditions considered in this study. The supports with free boundary condition are also more sensitive towards changes in E_{22} as compared to other boundary conditions. In general, the fundamental frequency is most affected by scatter in E_{11} and shows significant sensitivity to scatter in G_{12}, E_{22} and G_{13} for all support conditions considered. The fundamental frequency is least affected by the changes in ν_{12}. The SCSC and SSSC are more sensitive towards changes in G_{23} as compared to any other boundary conditions.

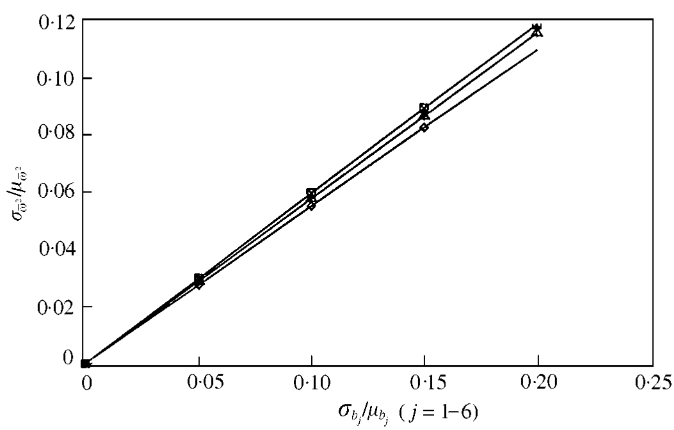

Figure 5. Variation of SD of the fundamental frequency with SD of basic material properties. [0/90°] laminate, with $R/a = 5$, $a/h = 10$ and $a/b = 1$, with all basic material properties changing simultaneously. Key: ◇, SSSS; +, SCSC; □, SFSC; ×, SFSS; △, SSSC.

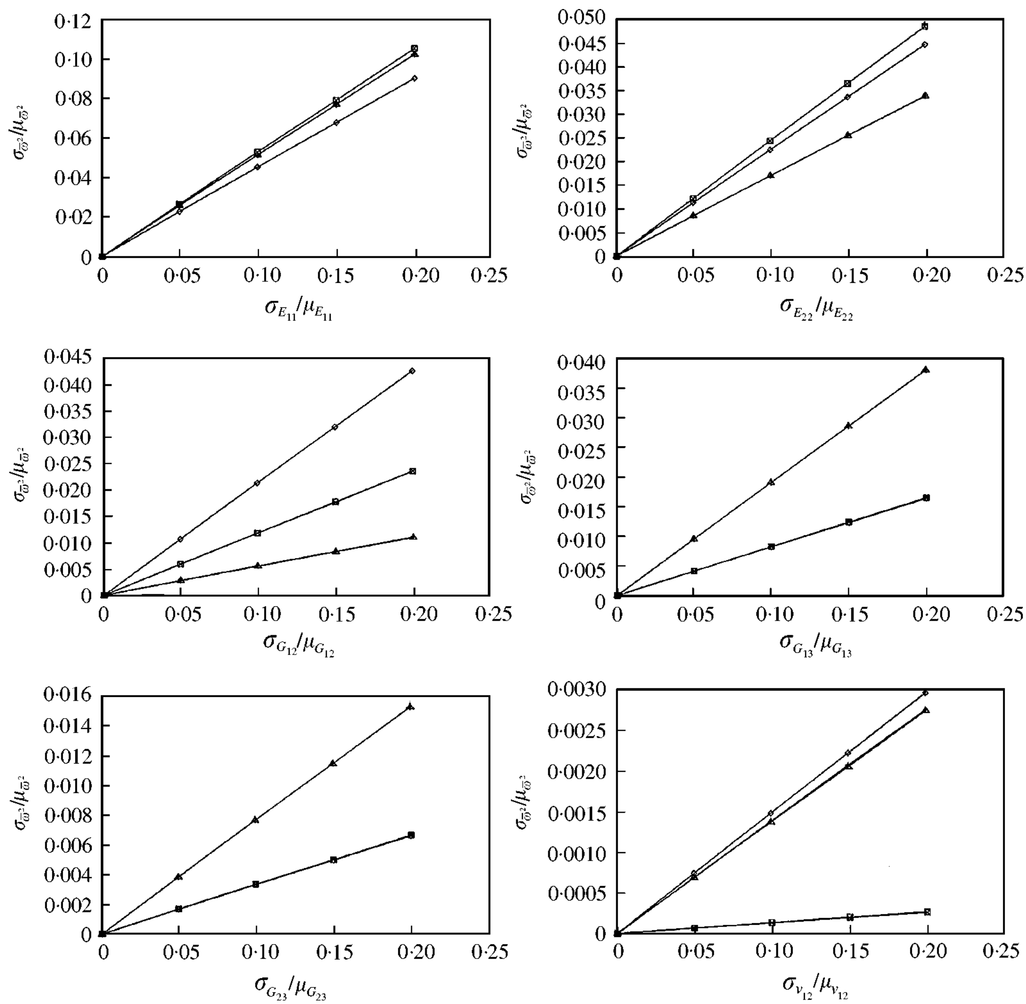

Figure 6. Variation of SD of the fundamental frequency with SD of basic material properties. [0/90°] laminate, with $R/a = 5$, $a/b = 1$ and $a/h = 10$. (a) Only E_{11} varying; (b) only E_{22} varying; (c) only G_{12} varying; (d) only G_{13} varying; (e) only G_{23} varying; (f) only v_{12} varying.
Key: as in Figure 5.

4. CONCLUSIONS

An approach has been outlined and used to evaluate the second order statistics of the natural frequencies for spherical laminates with rectangular plan form and different edge support conditions. The following conclusions can be drawn from the results obtained for graphite–epoxy laminated cross-ply spherical panels.

(1) The SD of the square of the natural frequency changes linearly with SD of the material properties.
(2) The fundamental frequency is most affected by simultaneous changes in SD of the material properties as compared to subsequent four natural frequencies for simply supported laminates .

(3) For the SSSS laminate the effect of E_{11} is most dominant on dispersion in the natural frequencies and effect of ν_{12} is least dominant. Out of all the natural frequencies, the fundamental frequency is most sensitive to changes in E_{11}.
(4) The SFSS and SFSC are most sensitive while the SSSS is least sensitive to simultaneous changes in the material properties.
(5) The effect of dispersion in longitudinal elastic modulus, E_{11} on the scatter in the fundamental frequency is most important for all support conditions considered while the effect of ν_{12} is least important.

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APPENDIX A

$$L_{11} = A_{11}d_1^2 + A_{66}d_2^2 - \bar{I}_1 d_t^2,$$

$$L_{12} = (A_{12} + A_{66})d_1 d_2,$$

$$L_{13} = - c_2[E_{11}d_1^3 + (E_{12} + 2E_{66})d_1 d_2^2] + A_{11}d_1/R_1 + A_{12}d_1/R_2 + \bar{I}_3 d_1 d_t^2,$$

$$L_{14} = (B_{11} - c_2 E_{11})d_1^2 + (B_{66} - c_2 E_{66})d_2^2 - \bar{I}_2 d_t^2,$$

$$L_{15} = [B_{12} + B_{66} - c_2(E_{12} + E_{66})]d_1 d_2,$$

$$L_{22} = A_{66}d_1^2 + A_{22}d_2^2 - \bar{I}_1' d_t^2,$$

$$L_{23} = - c_2[E_{22}d_2^3 + (E_{12} + 2E_{66})d_1^2 d_2] + A_{12}d_2/R_1 + A_{22}d_2/R_2 + \bar{I}_3' d_2 d_t^2,$$

$$L_{24} = L_{15},$$

$$L_{25} = (B_{22} - c_2 E_{22})d_2^2 + (B_{66} - c_2 E_{66})d_1^2 - \bar{I}_2' d_t^2,$$

$$\begin{aligned} L_{33} = {} & [c_1(D_{55} - c_1 F_{55}) - (A_{55} - c_1 D_{55})]d_1^2 + [c_1(D_{44} - c_1 F_{44}) - (A_{44} - c_1 D_{44})]d_2^2 \\ & + c_2^2[H_{11}d_1^4 + (2H_{12} + 4H_{66})d_1^2 d_2^2 + H_{22}d_2^4] - 2c_2(E_{12}d_1^2 + E_{22}d_2^2)/R_2 \\ & - 2c_2(E_{11}d_1^2 + E_{12}d_2^2)/R_1 - (A_{11}/R_1^2 + 2A_{12}/R_1 R_2 + A_{22}/R_2^2) \\ & + I_1 d_t^2 - c_2^2 I_7(d_1^2 d_t^2 + d_2^2 d_t^2), \end{aligned}$$

$$\begin{aligned} L_{34} = {} & [c_1(D_{55} - c_1 F_{55}) - (A_{55} - c_1 D_{55})]d_1 - c_2[(F_{11} - c_2 H_{11})d_1^3 + [2F_{66} + F_{12} \\ & - c_2(2H_{66} + H_{12})]d_1 d_2^2 + [B_{11}/R_1 + B_{12}/R_2 - c_2(E_{11}/R_1 + E_{12}/R_2)]d_1 + \bar{I}_5 d_1 d_t^2, \end{aligned}$$

$$\begin{aligned} L_{35} = {} & [c_1(D_{44} - c_1 F_{44}) - (A_{44} - c_1 D_{44})]d_2 - c_2[(F_{22} - c_2 H_{22})d_2^3 + [2F_{66} + F_{12} \\ & - c_2(2H_{66} + H_{12})]d_1^2 d_2] + [B_{12}/R_1 + B_{22}/R_2 - c_2(E_{12}/R_1 + E_{22}/R_2)]d_2 + \bar{I}_5' d_2 d_t^2, \end{aligned}$$

$$L_{44} = c_1(D_{55} - c_1 F_{55}) - (A_{55} - c_1 D_{55}) + [D_{11} - 2c_2 F_{11} + c_2^2 H_{11}] d_1^2$$
$$+ [D_{66} - 2c_2 F_{66} + c_2^2 H_{66}] d_2^2 - \bar{I}_4 d_t^2,$$
$$L_{45} = [D_{12} + D_{66} - 2c_2(F_{12} + F_{66}) + c_2^2(H_{12} + H_{66})] d_1 d_2,$$
$$L_{55} = c_1(D_{44} - c_1 F_{44}) - (A_{44} - c_1 D_{44}) + [D_{22} - 2c_2 F_{22} + c_2^2 H_{22}] d_2^2$$
$$+ [D_{66} - 2c_2 F_{66} + c_2^2 H_{66}] d_1^2 - \bar{I}_4' d_t^2,$$

where

$$c_1 = 4/h^2, \qquad c_2 = c_1/3, \qquad d_j^i = \partial^i/\partial x_j^i, \qquad d_t^i = \partial^i/\partial t^i.$$