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MATHEMATICAL AND NUMERICAL STUDY OF THE DUFFING-HARMONIC OSCILLATOR

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Conservative non-linear oscillatory systems can often be modelled by potentials having a rational form for the potential energy [1, 2]. In addition to providing physical models of interesting non-linear dynamics, they also lead to differential equations for which the usual, expansion in a small parameter, perturbation procedures do not apply [3]. An example is the Duffing-harmonic oscillator

$$\ddot{y} + \frac{Ay^3}{B_1 + B_2 y^2} = 0, \tag{1}$$

for which the parameters (A, B_1, B_2) are non-negative. The change of variables

$$y \to x = \sqrt{\frac{B_2}{B_1}}\, x, \qquad t \to \bar{t} = \left(\frac{AB_1}{B_2}\right) t, \tag{2}$$

and the dropping of the "bar" on $\bar{t}$, gives the following non-dimensional equation which is free of non-essential parameters:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{x^3}{1 + x^2} = 0. \tag{3}$$

Note that for x, respectively, small and large, equation (3) becomes

$$x \text{ small} \quad \ddot{x} + x^3 \simeq 0, \qquad x \text{ large} \quad \ddot{x} + x \simeq 0. \tag{4a, b}$$

Thus, for x small, the equation of motion is that of a Duffing-type non-linear oscillator, while for large x, the equation of motion approximates that of a linear harmonic oscillator; hence, the name for equation (3), the Duffing-harmonic oscillator.

The main purpose of this Letter is to construct an analytical approximation to the solutions of equation (3) using the method of harmonic balance, and to both construct and analyze non-standard finite difference schemes [4] to numerically examine the solutions of equation (3).

Using the concept of phase-space [5, 6], i.e., the two-dimensional space with variables ($x, y \equiv \mathrm{d}x/\mathrm{d}t$), and applying various symmetry arguments, it is easy to demonstrate that all the curves in the phase-space corresponding to equation (3) are closed. Consequently, all motions for arbitrary initial conditions give periodic solutions. The energy function or first-integral for equation (3) is given by

$$\frac{y^2}{2} + V(x) = E, \tag{5}$$

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where the first term is the kinetic energy and the second term is the potential energy function which is given by the expression

$$V(x) = \frac{x^2}{2} - \left(\frac{1}{2}\right) \ln(1 + x^2). \tag{6}$$

An analytical approximation to the periodic solutions can be obtained by taking the following form for a first approximation based on the method of harmonic balance [3]:

$$x(t) \simeq a \cos \omega t. \tag{7}$$

With the initial conditions, $x(0) = A$ and $\dot{x}(0) = 0$, this approximate solution is given by the following values for the amplitude a and the angular frequency ω:

$$a = A, \qquad [\omega(A)]^2 = \frac{(3/4)\, A^2}{1 + (3/4)\, A^2}. \tag{8}$$

Observe that for small and large amplitudes

$$A \text{ small} \quad [\omega(A)]^2 \simeq \left(\frac{3}{4}\right) A^2, \qquad A \text{ large} \quad [\omega(A)]^2 \simeq 1, \tag{9a, b}$$

which agrees with the approximations to the equations of motion given in equations (4).

A conjecture will now be formulated regarding the exact angular frequency of the solutions to equation (3), based on the initial conditions stated above, $x(0) = A$ and $\dot{x}(0) = 0$. The exact angular frequency for the equation

$$\ddot{z} + z^3 = 0, \tag{10}$$

is [7]

$$\omega(A) = \frac{\pi A}{2F\ (1/\sqrt{2},\ \pi/2)}, \tag{11}$$

where $F\ (\lambda, \pi/2)$ is a complete elliptic integral of the first kind with modulus λ [8]. Now define the constant ϕ by

$$\phi \equiv \frac{\pi}{2F\ (1/\sqrt{2},\ \pi/2)}. \tag{12}$$

From these results and those given in equations (8) and (9), the conjecture is: the exact angular frequency for the periodic solutions to equation (3) is

$$[\omega(A)]^2 = \frac{\phi^2 A^2}{1 + \phi^2 A^2}. \tag{13}$$

To preserve the conservative nature of the non-linear oscillator under investigation, any discrete (in this case, finite difference) model of the Hamiltonian or energy function must possess the property of being invariant under the interchange [9]

$$x_k \leftrightarrow x_{k-1}, \tag{14}$$

where

$$t \to t_k = (\Delta t)\, k = hk, \qquad x(t) \to x_k = x(t_k), \tag{15}$$

and $h = \Delta t$ is the time step. This implies that the discrete equations of motion do not change if [9]

$$x_{k+1} \leftrightarrow x_{k-1}. \tag{16}$$

Since the energy function or Hamiltonian for equation (3) is

$$H(x, y) = \frac{y^2}{2} + \frac{x^2}{2} - \left(\frac{1}{2}\right)\ln(1 + x^2), \tag{17}$$

the most direct way of implementing the condition of equation (14), in the construction of a discrete Hamiltonian, is to use the form [4, 9]

$$H(x_k, x_{k-1}) = \left(\frac{1}{2}\right)\left[\frac{x_k - x_{k-1}}{\psi}\right] + \frac{x_k x_{k-1}}{2} - \left(\frac{1}{2}\right)\ln(1 + x_k x_{k-1}), \tag{18}$$

where ψ is a function of step size h and has the property

$$\psi(h) = h + O(h^2). \tag{19}$$

Applying the difference operator, Δ, to $H(x_k, x_{k-1})$, where $\Delta x_k \equiv x_{k+1} - x_k$, and remembering that $H(x_k, x_{k-1}) =$ constant, gives, after simplification and rearrangement, the following expression:

$$\frac{x_{k+1} - 2x_k + x_{k-1}}{\psi^2} + x_k - x_k\left[\frac{\ln(1 + x_{k+1}x_k) - \ln(1 + x_k x_{k-1})}{x_{k+1}x_k - x_k x_{k-1}}\right]. \tag{20}$$

Note that this expression is an implicit function of x_{k+1}; consequently, a complex transcendental equation must be solved to give x_{k+1} in terms of x_k and x_{k-1}. Another possibility is to start with the equation of motion, namely, equation (3), and construct a non-standard finite difference scheme from it [4]. The simplest of such schemes which is non-local in the non-linear term [4, 9] and explicit is

$$\frac{x_{k+1} - 2x_k + x_{k-1}}{\phi^2} + \left(\frac{1}{2}\right)\left(\frac{x_k^2}{1 + x_k^2}\right)(x_{k+1} + x_{k-1}) = 0, \tag{21}$$

where now the symbol ϕ denotes $\phi = \sin(h)$. Shifting the index k upward by one unit and solving for x_{k+2} gives

$$x_{k+2} = \frac{2x_{k+1}}{[1 + (\phi^2/2)\, x_{k+1}^2/(1 + x_{k+1}^2)]} - x_k. \tag{22}$$

Given the values of x_0 and x_1, x_2 and all subsequent values of x_k can be calculated to obtain a numerical solution to equation (3). With the initial conditions, $x(0) = x_0$ and $\dot{x}(0) = 0$, a simple calculation shows that

$$x_0 = x(0), \qquad x_1 = x(h) = x_0 - \left(\frac{x_0^3}{1 + x_0^2}\right)\frac{h^2}{2} + O(h^4). \tag{23}$$

Figure 1 gives plots of x_k versus k for $h = 0{\cdot}05$, $x(0) = x_0 = 1$ and $\dot{x}(0) = 0$. Similar plots occur for other values of the initial conditions as long as the step size is selected to be sufficiently small to adequately resolve the oscillatory nature of the solution. This value can be easily estimated by using the harmonic balance prediction of the period

$$T = \frac{2\pi}{\omega} = \frac{4\pi}{\sqrt{3}A}, \qquad A = x_0 \tag{24}$$

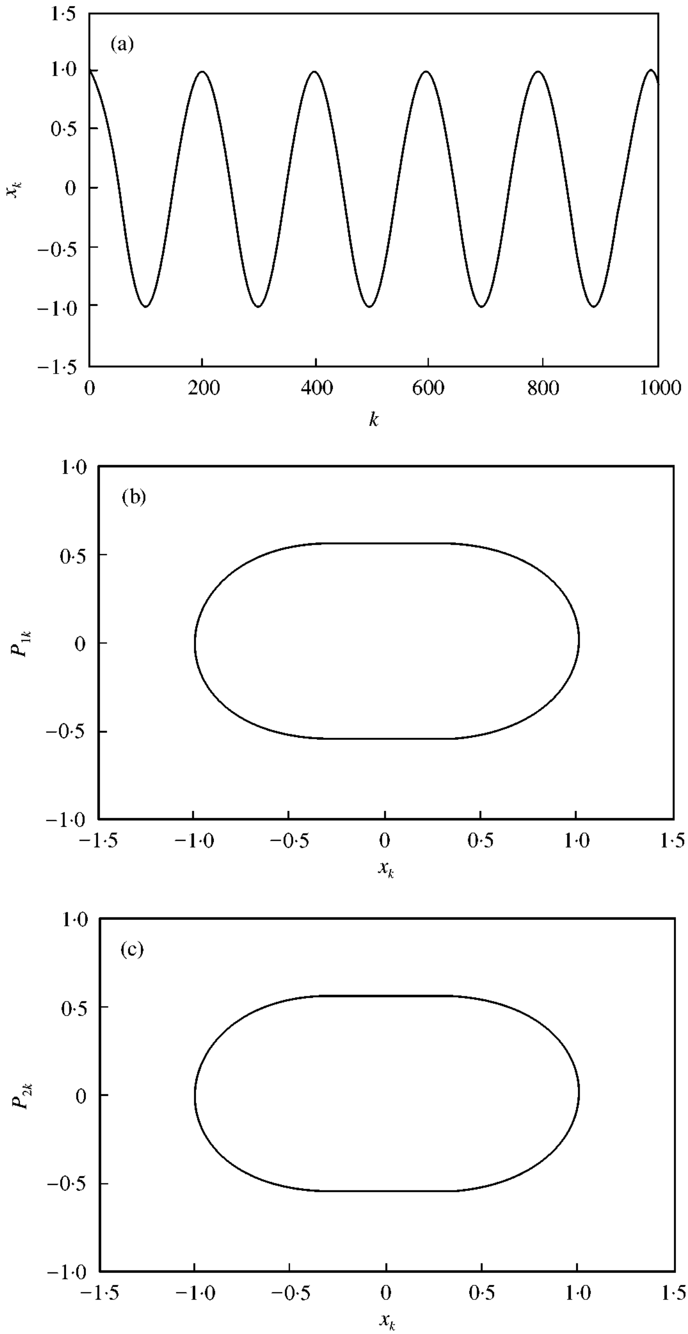

Figure 1. (a) Numerical solution of the Duffing-harmonic non-linear oscillator for $x(0) = 1$ and $\dot{x}(0) = 0$. The middle and bottom graphs are phase-space plots, where the discrete momenta are taken to be (b) $p_{1k} = (x_k - x_{k-1})/\phi$ and (c) $p_{2k} = (x_{k+1} - x_{k-1})/2\phi$.

and choosing $\Delta t = h$ to be

$$h = \frac{T}{20} = \frac{\pi}{5\sqrt{3}x_0}, \tag{25}$$

i.e., h is (approximately) one-twentieth the period. Using the conjectured exact angular frequency, equation (13), in the harmonic balance approximation, equation (7), good agreement is found between the analytical approximate solution and the numerical results. In particular, all the curves in phase-space were found to be closed as expected from our analysis.

In summary, the Duffing-harmonic oscillator has all periodic solutions and harmonic balance gives a good estimate for the angular frequency, i.e., equation (8) and its generalization, equation (13). Two non-standard finite difference schemes were constructed and the explicit scheme was used to numerically integrate the equation of motion. Further work on the Duffing-harmonic equation will center on trying to prove the relation conjectured in equation (13).

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