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ON THEORETICAL LIMITS OF DYNAMIC MODEL UPDATING USING A SENSITIVITY-BASED APPROACH

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The present work deals with the determination of the newly discovered conditions necessary for model updating with the eigensensitivity approach. The treatment concerns the maximum number of identifiable parameters regarding the structure of the eigenvectors derivatives. A mathematical demonstration is based on the evaluation of the rank of the least-squares matrix and produces the algebraic limiting conditions. Numerical application to a lumped parameter structure is employed to validate the mathematical limits taking into account different subsets of mode shapes. The demonstration is extended to the calculation of the eigenvector derivatives with both the Fox and Kapoor, and Nelson methods. III conditioning of the least-squares sensitivity matrix is revealed through the covariance jump.

1. INTRODUCTION

The increasing demand of realistic numerical discrete models requires accurate test-correlation procedures. In the field of vibrations several methods have been developed which allow finite element model updating by means of data from experimental modal analysis.

The present work treats the case of convergence on modal data based on the sensitivity approach. The concept of iteratively updating a structural model by using eigenvalue and eigenvector sensitivity was first proposed by Collins *et al.* [1]; this approach was followed and evolved in many other papers, dealing either with numerically simulated or with experimental data.

In the case of relatively complex structures, the error in the model can be difficult to locate. Some researchers simulate the dynamic behaviour of the total system by combining structural sub-elements identified previously through independent experiments. The inaccuracies of the model are thus attributed to the interface parameters [2–4].

An examination of the literature regarding the modal sensitivity approach shows a deficiency in the preliminary evaluation of the maximum number of parameters that can be identified. This was recently pointed out and discussed in some detail by the present authors [5, 6] also taking into account the appropriate error estimates including the effect of noisy data [7].

In some works [2, 8] a simultaneous convergence on eigenvalues and eigenvectors is proposed and the ill-conditioning effect of the sensitivity matrix is attributed to the difference in magnitude between eigenvalues and eigenvector gradients. This problem is overcome in reference [9] by balancing the sensitivity matrix; however, the limit to the maximum number of identifiable parameters is simply taken to be equal to the number of independent experimental measurements.

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Apart from these approaches, the literature shows that the evaluation of the maximum number of parameters that can be identified is left to trial and error. The goal of the present work is a more systematic treatment.

The method is based on the minimization of a quadratic measure of deviation between the calculated and experimental modal data. A general study through a least-squares solution leads to the result that the maximum number of identifiable parameters is given a theoretical limit according to the rank of sensitivity matrix which is formed by eigenvectors derivatives.

Fox and Kapoor [10] achieved the first order eigenvector derivative for a real symmetric matrix as a linear combination of the mode shapes: as this "modal method" can be computationally expensive for large problems, in practice, truncation is required and the method becomes an approximate one. A viable alternative is the method proposed by Nelson [11], which gives the eigenvector derivative for any mode shape directly by solving a linear system and dealing with the complete mass and stiffness matrices. If the Fox and Kapoor method is extended to all the mode shapes the first order derivative is exact and equivalent to the result of Nelson's method. Variations and modifications are to be found in references [12–14]. All of these have been compared in reference [15], where it is concluded that the Nelson method offers the best computational performance.

However, when all modes are included, the two methods give the same results. In the present work a theoretical demonstration to evaluate the sensitivity matrix rank is performed by using only the Fox and Kapoor method due to the simplicity of the skew symmetric formulation.

2. THEORETICAL DEVELOPMENT

In order to update a finite element model by means of experimental modal analysis the objective function to be minimized could be a quadratic measure of deviation between the calculated and the experimental modal data. Natural frequencies are more often used although these may be combined with mode shapes. A cumulative objective function can be written as follows:

$$E_a + E_w = \frac{1}{M_e} \sum_{m=1}^{M_e} \{e - a\}_m^{\mathrm{T}} \{e - a\}_m + \frac{1}{M_e} \sum_{m=1}^{M_e} (\Omega - \omega)_m^2. \tag{1}$$

Due to the fact that the objective function E_w introduces a number of M_e independent deviations, the theoretical number of identifiable parameters furnished by the matching of eigenvalues is simply equal to the number of M_e measured resonant frequencies. A more complex task is the evaluation of the limit on the parameters to be updated by the contribution of the eigenvectors. The present paper deals with the number of independent equations that are produced by the comparison of modal shapes focusing attention on the objective function E_a.

Due to the incompleteness of the measured mode shapes the number of G_a d.o.f.s of the numerical model is usually much larger than that of the experimental grid G_e. In order to avoid the fictitious conditioning effect related to the expansion methods [16, 17] the numerical mode vector is reduced to the subset of measured G_e d.o.f. in the following way:

$$\{\hat{a}\} = [T]\{a\}, \tag{2}$$

where $[T]$ is a $G_e * G_a$ reduction matrix whose terms $t_{i,j} = 1$ establish the correspondence between the jth numerical d.o.f. with the ith experimental d.o.f.. D.o.f.s missing in $\{\hat{a}\}$ are associated with terms $t_{i,j} = 0$.

Therefore, the modal shape error function is conveniently written in the following vector form [6]:

$$E_a = \frac{1}{M_e} \sum_{m=1}^{M_e} \{e - \hat{a}\}^{\mathrm{T}} \{e - \hat{a}\}, \tag{3}$$

where the summation is limited to the number of M_e measured mode shapes.

The minimum of the euclidean norm expressed in equation (3) may be found iteratively through the convergence algorithm such as optimization procedures. At convergence, a run of the least-squares algorithm will provide the covariance matrix of the estimated parameters, and the ill-conditioning of the least-squares sensitivity matrix can be thus revealed numerically.

3. LEAST-SQUARES SOLUTION

The condition that function E_a be a minimum for any jth structural parameter

$$\frac{\partial E_a}{\partial p_j} = -\frac{2}{M_e} \sum_{m=1}^{M_e} \left(\frac{\partial \{\hat{a}\}_m}{\partial p_j} \right)^{\mathrm{T}} \{e - \hat{a}\}_m = 0, \tag{4}$$

which after linearization of mode shape vectors around the convergence value for each structural parameter p_n

$$\{\hat{a}\}_m = \{\hat{a}\}_m^0 + \sum_{n=1}^{N} \frac{\partial \{\hat{a}\}_m^0}{\partial p_j} (p_n - p_n^0) \tag{5}$$

becomes

$$\{p - p^0\} = ([S]^{\mathrm{T}}[S])^{-1}[S]^{\mathrm{T}}\{b\} \tag{6}$$

which corresponds to a classical least-squares problem where $([S]^{\mathrm{T}}[S])^{-1}[S]^{\mathrm{T}} = [a]^{+}$ is the generalized inverse matrix [18] where:

$$\{b\}^{\mathrm{T}} = \left\{ \{e - \hat{a}^0\}_1^{\mathrm{T}}, \ldots, \{e - \hat{a}^0\}_m^{\mathrm{T}}, \ldots, \{e - \hat{a}^0\}_{M_e}^{\mathrm{T}} \right\} \tag{7}$$

and

$$[S] = \begin{bmatrix} \dfrac{\partial \{\hat{a}\}_1^0}{\partial p_1} & \cdots & \dfrac{\partial \{\hat{a}\}_1^0}{\partial p_N} \\ \vdots & & \vdots \\ \dfrac{\partial \{\hat{a}\}_{M_e}^0}{\partial p_1} & \cdots & \dfrac{\partial \{\hat{a}\}_{M_e}^0}{\partial p_N} \end{bmatrix}. \tag{8}$$

Taking into account the incompleteness of the experimental data the sensitivity matrix expressed in equation (8) has $M_e * G_e$ rows; as noted also in reference [9], the maximum number of parameters must satisfy the following relation:

$$N < M_e G_e \tag{9}$$

which is not correlated to the expression of the sensitivity coefficient, being just a count of available equations correlated to the experimental data.

The following will show that the number of parameters N is further limited according to the structure of mode derivatives. At this point it is opportune to stress that the limitations found in this paper are necessary conditions, and which means that the limit of the number of parameters will be identified. As has been investigated at length elsewhere, [5, 6, 19], the numerical values of the derivatives in the matrix $[S]$ establish the degree of dependence of mode variations on parameter variations, and therefore the covariance of the estimated parameters.

The investigation in the present work concerns the algebraic structure of this array of derivatives and not their numerical values.

4. EIGENDERIVATIVES SKEW SYMMETRIC PROPERTIES

According to the modal method [10], the eigenvector derivatives of the mth mode shape can be written as follows:

$$\frac{\partial\{a\}_m}{\partial p_n} = \sum_{j=1}^{G_a} d^n_{m,j}\{a\}_j = [a]\{d\}^n_m, \tag{10}$$

where

$$d^n_{m,j} = \begin{cases} 0 & j = m \\ \{a\}_j^{\mathrm{T}} \dfrac{\partial[K]}{\partial p_n}\{a\}_m & n = 1, N \quad j = 1, G_a \\ \dfrac{}{2\pi^2(\omega_m^2 - \omega_j^2)} & j \neq m \end{cases} \tag{11}$$

According to Fox and Kapoor [10] the derivative is exact if, as indicated in equation (10), the summation is performed over the complete set of the G_a calculated mode shapes. For the purpose of this work it is crucial to underline the skew-matrix structure of the $d^n_{m,j}$ coefficients. While this feature was useful [4] in reducing the computational effort when the modal method is implemented, its importance is stressed here in view of the evaluation rank of the sensitivity matrix.

Taking into account the incompleteness of the experimental data equation (10) becomes

$$\frac{\partial\{\hat{a}\}_m}{\partial p_n} = [\hat{a}]\{d\}^n_m, \tag{12}$$

where matrix $[\hat{a}]$ has G_a columns and G_e rows and $\{d\}^n_m$ is an expedient to order the coefficients $d^n_{m,j}$ in a column vector of length G_a.

5. RANK EVALUATION OF THE SENSITIVITY MATRIX

Taking into account equation (12) the sensitivity matrix of equation (8) becomes

$$[S] = \begin{bmatrix} [\hat{a}]\{d\}^1_1 & \cdots & [\hat{a}]\{d\}^N_1 \\ \vdots & & \vdots \\ [\hat{a}]\{d\}^1_{M_e} & \cdots & [\hat{a}]\{d\}^N_{M_e} \end{bmatrix}. \tag{13}$$

In order to evaluate the limiting effect due to the incompleteness of the experimental data and the skew symmetric properties of the derivative of the eigenvectors, let the matrices $[C]$ and $[D]$ denote the sub-matrices in which the sensitivity matrix can be decomposed as

$$[S] = \begin{bmatrix} [\hat{a}] & & [0] \\ & \ddots & \\ [0] & & [\hat{a}] \end{bmatrix} \begin{bmatrix} \{d\}_1^1 & \cdots & \{d\}_1^N \\ \vdots & & \vdots \\ \{d\}_{M_e}^1 & \cdots & \{d\}_{M_e}^N \end{bmatrix} = [C][D]. \tag{14}$$

Through a well-known matrix theorem [20], the *rank* of a product matrix is equal to the minimum rank of the sub-matrices:

$$rank\,[S] = \min(rank\,[C],\, rank\,[D]). \tag{15}$$

Matrix $[C]$ is defined as a banded matrix, by repeating the reduced mode shape sub-matrix $[\hat{a}]$ on the diagonal. From equation (14) it follows that the number of diagonal elements is M_e; hence, the total number of rows is M_eG_e and the total number of columns is M_eG_a. All the non-square matrices on the diagonal are formed by linear independent eigenvectors; their rank is:

$$rank[\hat{a}] = \min(G_e, G_a) = G_e \tag{16}$$

since it has already been assumed that $G_e \ll G_a$.

Therefore, the rank of matrix $[C]$ equals M_e times the rank of $[\hat{a}]$.

$$rank[C] = \min(M_eG_e, M_eG_a) = M_eG_e. \tag{17}$$

The experimental G_e d.o.f. must be chosen so as to be sufficient to display the measured mode shapes univocally; thus $M_e{*}G_e$ will not include a dependent variable.

Matrix $[D]$ contains column vectors $\{d\}_m^n$ ordered in rows according to the structural parameter, and in columns according to the experimental mode shape. Matrix $[D]$ consists of M_eG_a rows and N columns and the rank is the minimum number between the total number of parameters N and the number of independent rows.

Linear independence of rows is strictly linked to the skewness as pointed out in equation (11). First, each column vector $\{d\}_m^n$ has an element equal to zero when $j = m$; moreover, each column vector $\{d\}_m^n$ contains pairs of elements equal in value but opposite in sign:

$$d_{m,j}^n = \frac{\{a\}_j^{\mathrm{T}}(\partial[K]/\partial p_n)\{a\}_m}{2\pi^2(\omega_m^2 - \omega_j^2)} = -\,d_{j,m}^n. \tag{18}$$

Both the zeros and the pairs are positioned in all columns of $[D]$ in a single row, because this position does not depend on the structural parameter N; therefore the number of rows containing zeros in the matrix $[D]$ is equal to M_e, and the number of linearly dependent rows due to the equal pairs is $(M_e/2)\,(M_e - 1)$. Over the M_eG_a total number of rows, those linearly independent are

$$rows = M_eG_a - M_e - \frac{M_e}{2}(M_e - 1). \tag{19}$$

The rank of the matrix $[D]$ is calculated as

$$rank\,[D] = \min\left\{N, \frac{M_e}{2}\,[2G_a - (M_e + 1)]\right\}. \tag{20}$$

Finally, taking into account equation (15) and substituting equations (17) and (20) in it, the rank of the sensitivity matrix $[S]$ is given by

$$rank\,[S] = \min\begin{cases} M_e G_e \\ N \\ \dfrac{M_e}{2}\,[2G_a - (M_e + 1)] \end{cases}. \tag{21}$$

6. NECESSARY CONDITIONS

The limit to the number of identifiable parameters is then obtained from equation (21) as

$$N \leqslant \min\left\{G_e M_e, \frac{M_e}{2}\,[2G_a - (M_e + 1)]\right\}. \tag{22}$$

The inequality can be applied to the case when the eigenvector derivatives are calculated by equation (10) with all G_a modes or with the equivalent Nelson method.

As in the case of "modal method" the derivatives are normally calculated with a truncated set of eigenvectors. In order to extend to this case the validity of the necessary conditions, it is possible to decompose he sensitivity matrix in a two-factor product and evaluate the rank of each factor's submatrices.

The summation of equation (10) is limited to a subset of M_a modes:

$$\frac{\partial\{\hat{a}\}_m}{\partial p_n} = \sum_{j=1}^{M_a} d^n_{m,j}\{\hat{a}\}_j = [\hat{a}]^{M_a}\{d\}^n_m. \tag{23}$$

Therefore, including the truncation limiting on the calculated eigenvectors derivatives, the rank of matrix $[\hat{a}]^{M_a}$ is

$$rank[\hat{a}]^{M_a} = \min(G_e, M_a). \tag{24}$$

The number of numerical mode shapes M_a used to calculate the eigenvector derivatives may be equal or slightly greater than M_e.

The subset G_e d.o.f. is the best choice in order to show not only the M_e experimental mode shapes but also the M_a numerical modes. The proper selection of G_e prevents M_a eigenvectors from becoming linearly dependent by coincidence, due to incompleteness (see equation (2)).

From this it follows:

$$rank[C] = \min(M_e G_e, M_e M_a). \tag{25}$$

Repeating the same limiting condition for the linear independence of the rows, of the matrix $[D]$ and as $M_e M_a$ is now the total number of rows, then it follows that the linearly

independent rows are

$$rows = M_e M_a - M_e - \frac{M_e}{2}(M_e - 1) \tag{26}$$

So the rank of matrix $[D]$ is calculated through the relation

$$rank[D] = \min\left\{N, \frac{M_e}{2}[2M_a - (M_e + 1)]\right\}. \tag{27}$$

The more restrictive formula that gives the rank of the sensitivity matrix for the approximate eigenvector derivative is

$$rank\,[S] = \min\begin{cases} M_e G_e \\ M_e M_a \\ N \\ \frac{M_e}{2}[2M_a - (M_e + 1)] \end{cases} \tag{28}$$

which, observing that the fourth relation is always lower than the second, finally gives the limit to the identifiable parameters:

$$N \leqslant \min\left\{G_e M_e, \frac{M_e}{2}[2M_a - (M_e + 1)]\right\}. \tag{29}$$

The inequality can be used either in the *a priori* planning of the number of measurements to be taken or for an *a posteriori* determination of the maximum number of parameters that can be identified.

7. EXAMPLE CASE

In order to validate the previous theoretical demonstrations, a simple mass–spring example is proposed. Table 1 shows the mass and the stiffness values. A schematic representation of this test structure is presented in Figure 1.

TABLE 1

Mass and spring values of the example structure

Mass	(kg)	Spring	(N/m)
m_1	5×10^{-3}	k_{11}	10
m_2	10×10^{-3}	k_{22}	15
m_3	3×10^{-3}	k_{33}	20
m_4	2×10^{-3}	k_{44}	35
m_5	5×10^{-3}	k_{55}	40
	$i \neq j$; $i + j =$ odd	k_{ij}	15
	$i \neq j$; $i + j =$ even	k_{ij}	25

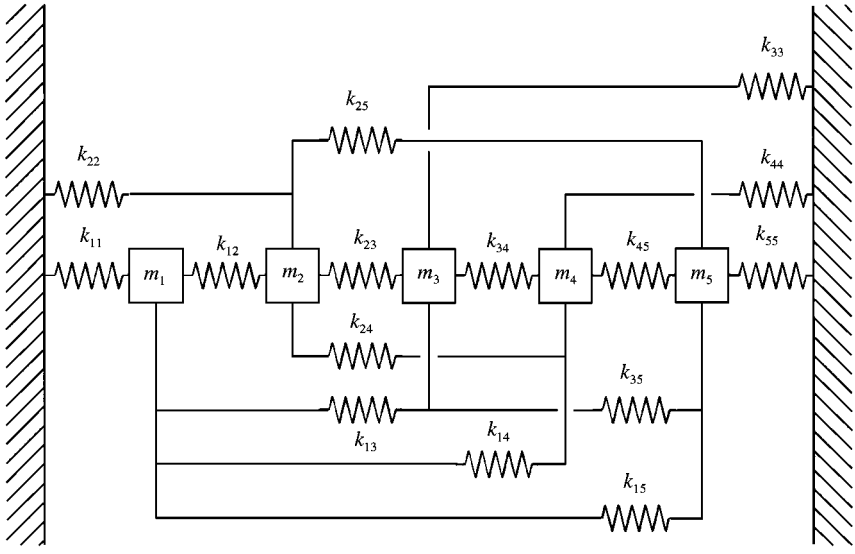

Figure 1. Schematic representation of the example structure.

If the stiffness parameters are changed slightly, then the least-squares problem of equation (6) is solved. The variable stiffness parameters are those listed above. The fictitious "experimental" eigenvectors are those relative to the correct model. Due to the simplicity of the problem, it suffices here to iterate the least-squares algorithm.

The model updating procedure is stable and converges for different starting percentage errors of the unknown parameter versus the reference stiffness. The identification procedure diverges only when the number of parameters to be identified does not satisfy the condition given by equations (22) and (29).

The transition point between the convergence and divergence is related to the ill-condition of the least-squares solution. This condition is expressed in terms of the covariance matrix [19]; the unscaled covariance matrix is

$$[C] = ([S]^{\mathrm{T}}[S])^{-1} \tag{30}$$

and the global measure of its value is defined as follows:

$$\overline{\mathrm{cov}} = tr[C]/N = \sum_{i=1}^{N} C_{ij}/N \tag{31}$$

Using this formula three different cases of convergence can be compared increasing the number of parameters to be identified and varying the number of experimental modes:

(1) eigenvector derivatives calculated through the modal method with a complete set of modal shapes, equation (10);
(2) eigenvector derivatives calculated through the algebraic Nelson method;
(3) eigenvector derivatives calculated through the modal method with a truncated set of modal shapes, equation (12).

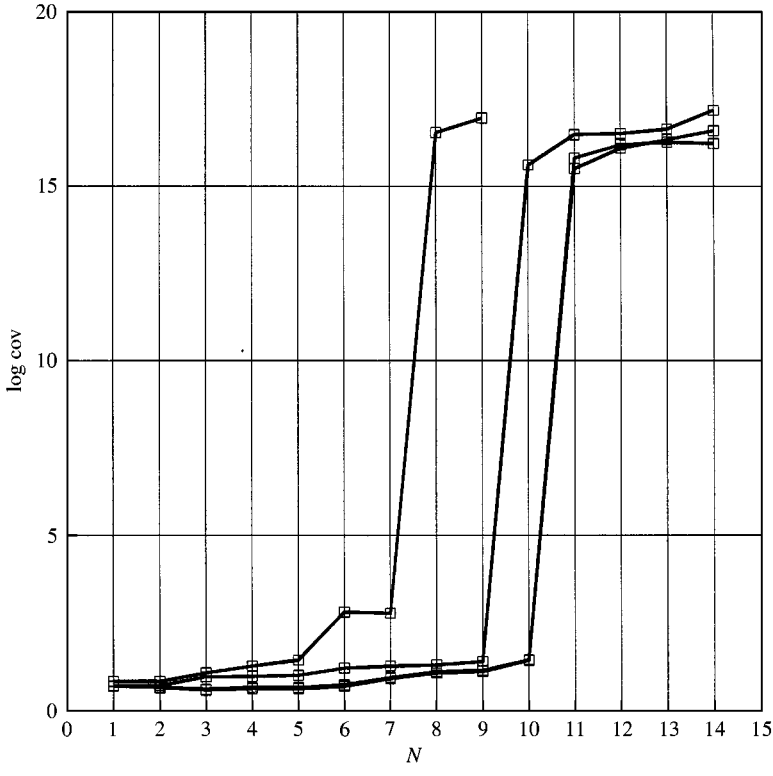

Figure 2. Number of maximum identifiable parameters by using exact Fox and Kapoor eigenvector derivatives.

TABLE 2

Number of maximum identifiable parameters, N, versus the number M_e of experimental modes for the three cases examined

	Case (1), (2) $N < \frac{M_e}{2}[2G_a - (M_e + 1)]$	Case (3) $N < \frac{M_e}{2}[2M_a - (M_e + 1)]$
M_e	N	N
1	4	3
2	7	5
3	9	6
4	10	6
5	10	—

Cases (1) and (2) are governed by equation (22); case (3) is governed by equation (29). The maximum number of d.o.f.s is $M_{e_{max}} = G_a = M_{a_{max}} = 5$. Table 2 shows the values of the transition value of N according to the number of experimental modes M_e.

Figure 2 shows the logarithm of the covariance trace scaled to the number of parameters (31) for the case (1). The transition point corresponds exactly to the theoretical limit predicted in Table 2.

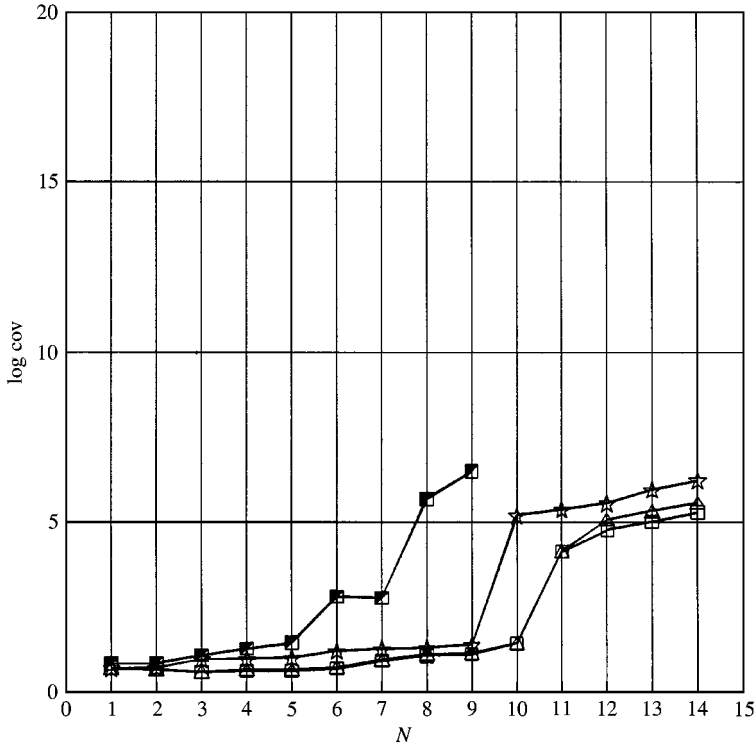

Figure 3. Number of maximum identifiable parameters by using exact Nelson eigenvector derivatives.

Figure 3 shows the same function in the case of Nelson's eigenvector derivatives. The transition point occurs at the same value of the previous case.

Obviously cases (1) and (2) present the same limit because the eigenvector derivatives are exact both in the complete modal method or in Nelson's algebraic approach. In the present work both the methods are taken into account to underline that the only difference is a numerical one. The difference in the entity of the transition jump is due to the different method in the calculation of the eigenvector derivatives. The importance of the demonstration is not the numerical sensitivity of the method but the evidence of a transition jump. The goal of the work is to evaluate the necessary condition for the model updating in the modal scheme.

The upper limit of the cases (1) and (2) is given, in this lumped parameters example, by equation (31).

In the real case of a large structure the modes are extracted through numerical methods via subspace iteration, and usually $M_a \ll G_a$. If the modal method is employed, the eigenvector derivatives are truncated by using the extracted subset of analytical modes. The theoretical limit to the number of identifiable parameters is given in this case by equation (29). An example is obtained by using only four analytical modes of the lumped structure as reported in column (3) of Table 2.

Figure 4 shows the weighted covariance trace versus the number of parameters to be identified in the case of the approximate eigenvector derivative calculated with the truncated modal method. The transition points occur at the same values predicted in Table 2, thus proving the validity of equation (29).

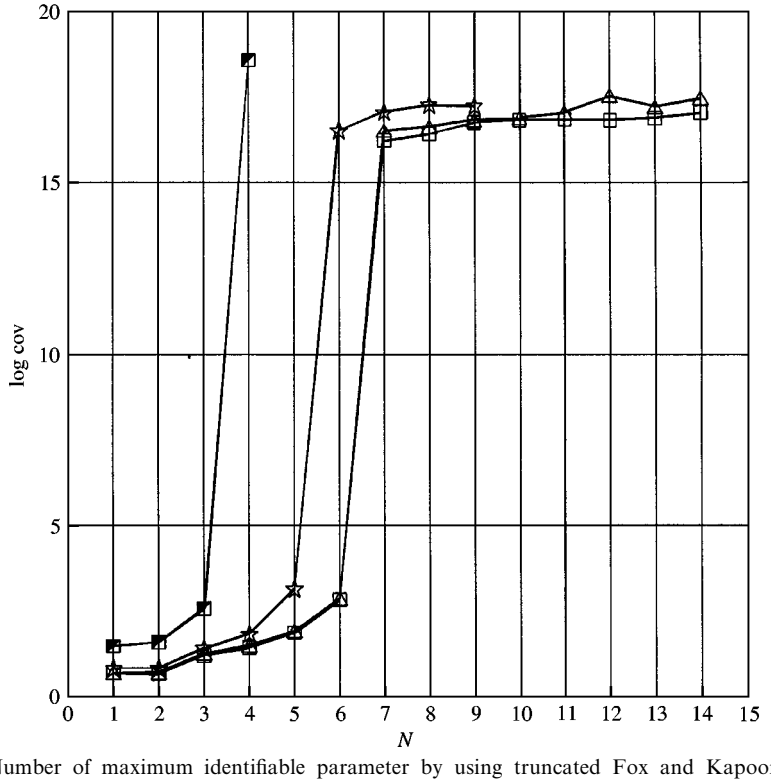

Figure 4. Number of maximum identifiable parameter by using truncated Fox and Kapoor eigenvector derivatives.

8. CONCLUSIONS

(1) The method provides a limit on the parameters to be identified, which is useful, although must be recognized that this is the maximum.
(2) It was shown theoretically that the number of identifiable stiffness parameters of a structure has an upper limit that depends on the structure of the calculation of the eigenvector derivatives. Two possibilities have been examined, i.e., the "modal" method of Fox and Kapoor and the "algebraic" method of Nelson. A number of numerical examples regarding a simulated structure give results which are in accord with the above theoretical predictions.
(3) With the "modal" method, usually only a part of the total eigenvectors of the structure is calculated, and therefore the sensitivity formula is truncated. It was shown that in this case the matrix of least-squares convergence of numerical data to experimental data has a rank lower than the number of independent measurements, accordingly limiting the number of theoretically identifiable parameters.
(4) This finding gives a better foundation to the number of eigenvectors to be considered in the truncated Fox and Kapoor formula, as compared to the usual empirical recommendation of bringing the double number of experimental modes into play. This also allows a basic planning of the experiment, i.e., the choice of the number of experimental mode shapes and the number of experimental d.o.f. where measurements are taken.
(5) In the case of real large structure the present method gives a basic number of necessary conditions that must be proved to be sufficient for the updating. The

selection of modal data and experimental grid proposed in the present paper has to be improved mainly due to the incompleteness and the noise of the experimental data.

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REFERENCES

1. J. D. COLLINS, G. C. HART, T. K. HASSELMAN and B. KENNEDY 1974 *American Institute of Aeronautics and Astronautics Journal* **12**, 185–190. Statistical identification of structures.
2. A. A. HUCKELBRIDGE and C. LAWRENCE, 1989 *American Society of Mechanical Engineers Journal of Vibration, Acoustic, Stress, Reliability and Design* **111**, 140–147. Identification of structural interface characteristics using component mode synthesis.
3. J. H. HEO and K. F. EHMANN 1991 *American Society of Mechanical Engineers Journal of Vibration, Acoustic, Stress, Reliability and Design* **113**, 201–208. A method for substructural sensitivity synthesis.
4. M. M. GOLA and A. SOMA 1991 *Proceedings of Florence Modal Analysis Conference*, 655–661. Identification of robot joint stiffness using least squares on modal.
5. D. BOTTO, L. GOLGLIO, M. M. GOLA and A. SOMA 1993 *Proceedings of XI International Modal Analysis Conference*, Vol. 1, 703, 709. On the maximum number of stiffness parameters identifiable with modal data.
6. D. BOTTO, M. M. GOLA and A. SOMA 1994 *Proceedings of XII International Modal Analysis Conference*, Vol. 2, 1408–1414. Planning of modal experiments for structural identification with the sensitivity approach.
7. L. ECKERT and B. CAESAR 1991 *Proceedings of Florence Modal Analysis Conference*, 635–643. Model updating under incomplete and noisy modal test data.
8. M. F. ORR 1990 *Proceedings of VIII International Modal Analysis Conference*, Vol. 2, 1397–1402. Eigenvector derivatives for analytical model improvement.
9. M. JUNG and D. J. EWINS 1992 *Proceedings of X International Modal Analysis Conference*, Vol. 2, 992–998. Sensitivity of the inverse eigensensitivity method for model updating.
10. R. L. FOX and M. P. KAPOOR 1968 *American Institute of Aeronautics and Astronautics Journal* **6**, 2426–2429. Rates of change of eigenvalues and eigenvectors.
11. R. B. NELSON 1976 *American Institute of Aeronautics and Astronautics Journal* **14**, 1201–1205. Simplified calculation of eigenvectors derivatives.
12. I. U. OJALVO 1987 *American Institute of Aeronautics and Astronautics Journal* **25**, 1386–1390. Efficient computation of mode-shape derivatives for large dynamic system.
13. K. B. LIM, J. L. JUNKINS and B. P. WANG 1987 *Journal of Guidance and Control* **10**, 581–587. Re-examination of eigenvector derivatives.
14. T. TING 1992 *Communication Application on Numerical Methods* **8**, 61–68. A subspace iteration method for eigenvector derivatives.
15. T. R. SUTTER, C. J. CAMADA, J. L. WALSH and H. M. ADELMAN 1988 *American Institute of Aeronautics and Astronautics Journal* **26**, 1506–1511. Comparison of several method for calculating vibration mode shape derivative.
16. H. GYSIN 1990 *Proceedings of the 15th International Seminar of Modal Analysis*, Vol. 2, 919–937. Expansion, the Achilles heel of FE model updating.
17. N. A. J. LIEVEN and D. J. EWINS 1990 *Proceedings of VIII International Modal Analysis Conference*, Vol. 2, 605–609. Expansion of modal data for correlation.
18. C. LAWSON and R. HANSON 1974 *Solving least squares problems*. New York: Prentice-Hall.
19. D. BOTTO, M. M. GOLA and A. SOMA 1994 *Proceedings of 17th International Seminar on Modal Analysis*, 325–335. Model updating with modal sensistivity: selection of modes and degrees of freedom.
20. J. M. ORTEGA 1987 *Matrix Theory*. New York: Plenum Press.

APPENDIX A: NOMENCLATURE

$[a]$	numerical mode shape matrix
$\{a\}_m$	mth numerical mode shape
$\{b\}$	least-squares column vector
$d^n_{m,j}$	eigenvector derivative coefficient
$\{e\}_m$	mth experimental mode shape
E_a	mode shapes objective function
E_w	frequencies objective function
g	degree of freedom index (d.o.f.)
G_a	number of numerical d.o.f.
G_e	number of measured d.o.f.
$[K]$	stiffness matrix
m	mode index
$[M]$	mass matrix
M_a	subset of chosen numerical eigenvectors
M_e	total number of experimental mode shapes
n	structural parameter index
N	total number of structural parameters
p_n	nth structural parameter
$[S]$	least-squares coefficient matrix
ω_m	mth numerical resonant frequency
Ω_m	mth experimental resonant frequency