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THEORETICAL AND EXPERIMENTAL IDENTIFICATION OF A NON-LINEAR BEAM

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The identification of the dynamic characteristics of linear systems is now widely used and interest in non-linear systems has increased. The objective of this paper is to demonstrate the performance of the restoring force surface method as far as the identification of non-linear systems is concerned. The vibrations of a clamped beam are investigated for two different kinds of non-linearity. Firstly, the beam shows a non-linear behaviour characterized by a piecewise linear stiffness and secondly, the non-linearity comes from a bilinear stiffness. Both numerical and experimental results are presented.

1. INTRODUCTION

The aim of the identification is to generate a mathematical model of a system. Generally speaking, it requires the knowledge of the applied force and of the response of the system. Once the model parameters are identified, the model may be used afterwards to predict the behaviour of the system.

For modal analysis, most mechanical structures are approximated by a linear model. However, when these structures are subject to large displacement amplitudes, non-linear effects may become important and the linear model consequently fails. Even when the amplitudes remain restricted, some non-linear distortions may occur due to dry friction for instance. Both reasons demonstrate why interest in non-linear identification is increasing.

Identification of non-linear systems ranges from methods which simply detect the presence or type of a non-linearity to those which seek to quantify the dynamic behaviour through a mathematical model. In this latter category lies the non-parametric scheme called the restoring force surface method.

The restoring force surface method offers an efficient and reliable identification of non-linear systems. Masri *et al.* laid down the foundations of the method [1, 2] and significant improvements were brought about since the original papers. Worden and Tomlinson [3] presented an effective way of identifying the mass. Duym and Schoukens [4] designed optimized excitation signals in order to guarantee the quality of the fit by uniformly covering the phase plane. They also used a local non-parametric identification of the non-linear force [5].

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The present paper applies the restoring force surface method to two different cases:

- a symmetrical system consisting of a beam characterized by a piecewise linear stiffness;
- an asymmetrical system consisting of the same beam but characterized by a bilinear stiffness.

The authors know of, only one previous paper [6] which tried to identify such non-linearities using the restoring force surface method and poor instrumentation forced bad results. Even if the beam is a multi-degree-of-freedom system, the study is focussed on identification of single-degree-of-freedom (s.d.o.f.) systems. It is important to note that it is possible to reduce the beam to an s.d.o.f. system while keeping its piecewise or bilinear characteristics completely.

The paper is organized as follows. In the next section, the restoring force surface method is briefly introduced. Then, the data processing associated with the method is discussed. Sections 4 and 5 consider the numerical simulation of a system with piecewise linear and bilinear stiffness respectively. In sections 6 and 7, the results of the experimental applications are presented.

2. THEORETICAL BACKGROUND

The restoring force surface method is based on Newton's second law:

$$m\ddot{x}(t) + f(x(t), \dot{x}(t)) = p(t), \tag{1}$$

where $p(t)$ is the applied force and $f(x, \dot{x})$ is the restoring force, i.e., a non-linear function of the displacement and velocity. The time histories of the displacement and its derivatives, and of the applied force are assumed to be measured. In practice, the data must be sampled simultaneously at regular intervals. From equation (1), it is possible to find the restoring force defined as $f_i = p_i - m\ddot{x}_i$ where subscript i refers to the ith sampled value. Thus, for each sampling instant a triplet $(x_i, \dot{x}_i, f_i)$ is found, i.e., the value of the restoring force is known for each point in the phase plane $(x_i, \dot{x}_i)$.

It is important to describe the system by a mathematical model. The usual way is to fit to the data a model of the form:

$$f(x, \dot{x}) = \sum_{i=0}^{m} \sum_{j=0}^{n} \alpha_{ij} x^i \dot{x}^j. \tag{2}$$

Least-squares parameter estimation can be used to obtain the values of the coefficients α_{ij}. To have a measure of the error between the measured value x_i and the predicted value $\hat{x}_i$, the mean-square error (MSE) indicator is defined as

$$MSE(x) = \frac{100}{N\sigma_x^2} \sum_i (x_i - \hat{x}_i)^2, \tag{3}$$

where N is the total number of samples and σ_x^2 the variance of the measured input. Experience shows that an MSE value of less than 5% indicates good agreement while a value of less than 1% reflects an excellent fit. To determine which terms are significant and which terms can be safely discarded in equation (2), the significance factor [7] is used:

$$s_\theta = 100\,\frac{\sigma_\theta^2}{\sigma_x^2}, \tag{4}$$

where σ_x^2 corresponds to the variance of the sum of all the terms of the model and σ_θ^2 is the variance of the considered term. Roughly speaking, the significance factor represents the percentage of the contribution of the term to the model variance.

3. DATA PROCESSING

From the foregoing developments, it appears that the method requires the measurement of displacement, velocity, acceleration and force time histories at each degree of freedom. A pragmatic approach to the procedure demands that only one signal should be measured and the other two should be estimated from it. Numerical integration and/or differentiation may be adopted.

The differentiation can be carried out in the time domain or in the frequency domain. A polynomial can be fitted to N data points such that the point at which the derivative is required is at the centre. The analytic derivative of the fitted polynomial is then computed. This illustrates a possible way of differentiation in the time domain. However, it can be shown that numerical differentiation leads to an inaccurate estimation of the acceleration. Considerably more detailed discussion is available in reference [8].

The practical solution is to measure the acceleration and numerically integrate it to find velocity and displacement. Various methods for achieving integration exist: trapezium rule, Simpson's rule, integration in the frequency domain, and so forth. There are two main problems associated with the integration, the introduction of low- and high-frequency components. The trapezium rule only suffers from the introduction of low-frequency components and does not require the use of a low-pass filter. Furthermore, it is the simplest integration process and offers considerable saving of time. For these reasons, the trapezium rule is considered throughout the paper.

It can be argued [9] that the transfer function of the trapezium rule is

$$H(\omega) = \frac{FT(estimated\ \ results)}{FT(true\ \ results)} = \cos(\omega/2)\,\frac{\omega/2}{\sin(\omega/2)}, \tag{5}$$

where FT is the Fourier transform and ω the normalized frequency, i.e., the frequency of interest divided by the sampling frequency. Equation (5) means that the trapezium rule only integrates constant signals without error and underestimates the integral at all other frequencies. Therefore, the sampling frequency must be chosen to be high enough in order that the highest frequency of interest is characterized by a low normalized frequency. A sampling frequency 10 or 20 times higher than the highest frequency of interest seems to be a reasonable choice.

Since the trapezium rule basically acts as an amplifier of the low-frequency components, the integrated signals are to be high-pass filtered. High-pass filtering with cut-off $n/2N\Delta t$ is equivalent to a polynomial trend removal of order n where N is the number of points and Δt the sampling interval [8]. Accordingly, choosing a cut-off frequency higher than 0 Hz immediately imposes the filtered signals to be of zero mean since a polynomial trend of order 0, i.e., a constant, is removed. This leads to an inaccurate estimation of velocity and displacement of asymmetrical systems and particularly for the bilinear case.

4. NUMERICAL SIMULATION OF A SYSTEM WITH PIECEWISE LINEAR STIFFNESS

The experimental beam is mounted vertically with a clamped end and a free end as shown in Figure 1. If the amplitude of the transverse motion exceeds a fixed limit, the beam makes

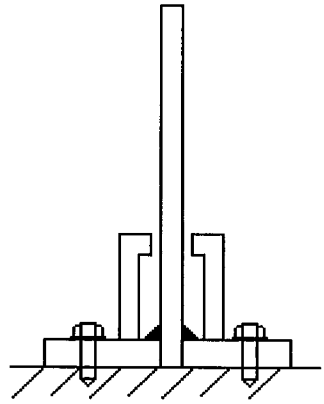

Figure 1. The piecewise linear beam.

contact with a steel bush. Thus, the beam possesses a piecewise linear stiffness. The system is first simulated. For this purpose, the vibrations of the first mode of the beam are analyzed and an s.d.o.f. system with a piecewise linear stiffness is chosen to model the experimental beam. This system is described by the following equations:

$$25\dot{x} + 15\dot{x} + 330000x = p(t) \quad \text{if } |x| < 0{\cdot}0004, \tag{6}$$

$$25\ddot{x} + 15\dot{x} + 1500000x = p(t) \quad \text{if } |x| \geqslant 0{\cdot}0004, \tag{7}$$

where $p(t)$ is a white noise sequence band-limited into the 10–25 Hz range. It can be noted that equation (7) takes the increase in stiffness into account as the beam makes contact with a bush.

The system is simulated using a Runge–Kutta procedure. The sampling frequency is set to 1000 Hz and white Gaussian noise is added to the data in such a way that the noise contributes to 5% of the signal r.m.s. value. The sampling frequency may appear too high while the frequencies of interest are below 100 Hz. It is worth recalling that the sampling frequency chosen should be 10 or 20 times higher than the frequencies of interest in order to give an accurate integration procedure (see section 3). For the sake of simplicity, all signals are assumed to be known during the identification procedure. The acceleration and its power spectral density (*PSD*) are presented in Figure 2. The presence of a sequence of harmonics in the *PSD* at $n\omega$ where $n = 3$, 5, 7, etc. is the sign of the non-linearity of the system.

A major problem usually encountered in identification is the choice of a model which has to represent the system. Since the stiffness is piecewise linear, it is easily understandable that a polynomial model will not fit the behaviour of the beam perfectly. Hence, it is worth comparing the results obtained via a polynomial model for the restoring force, i.e., $f(x, \dot{x}) = \sum_{i=1}^{m} \alpha_i x^i + \sum_{j=1}^{n} \beta_j \dot{x}^j$, with those given by a non-polynomial model which should obviously be the following:

$$f(x, \dot{x}) = k_- x + c_- \dot{x} \quad \text{if } x \leqslant -d,$$

$$f(x, \dot{x}) = k_+ x + c_+ \dot{x} \quad \text{if } x \geqslant d,$$

$$f(x, \dot{x}) = kx + c\dot{x} \qquad \text{else},$$

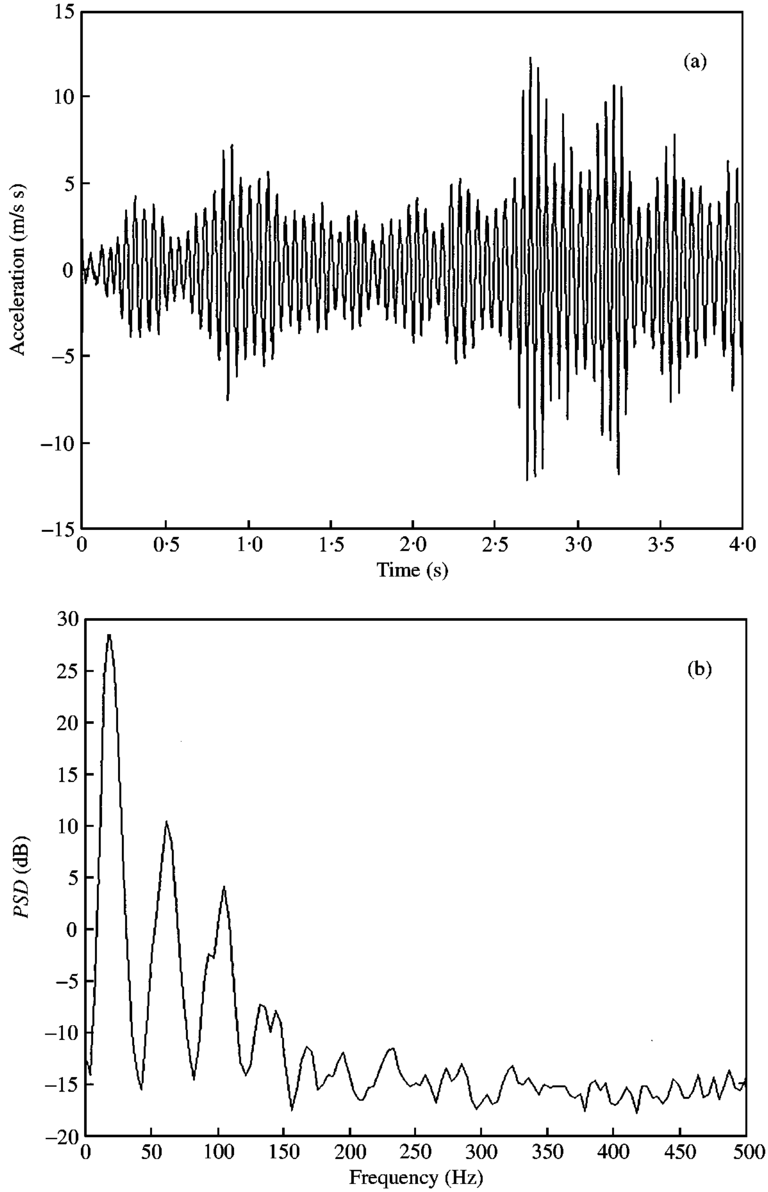

Figure 2. (a) Acceleration time history and (b) its *PSD*.

where d is the clearance value. If there is no *a priori* knowledge about the non-linearity, plotting the stiffness curve (Figure 3), i.e., the measured restoring force versus the displacement, clearly indicates that the stiffness is piecewise linear.

In the non-polynomial model, another problem arises: the clearance d is unknown *a priori*. Inspection of the stiffness curve points out that the change in stiffness occurs around 0·0004 m. To increase the accuracy, the MSE is computed for a hundred values of d regularly spaced between 0·0003 and 0·0005 m. The evolution of the MSE with the clearance is presented in Figure 4. It turns out that the optimum value for d is 0·000401 m. This value is almost identical to the exact one while 5% of noise is added to the data.

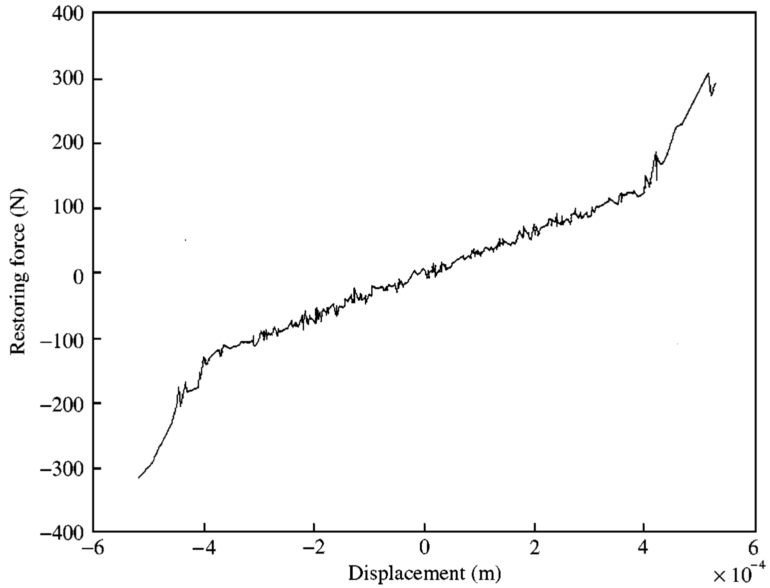

Figure 3. Stiffness curve.

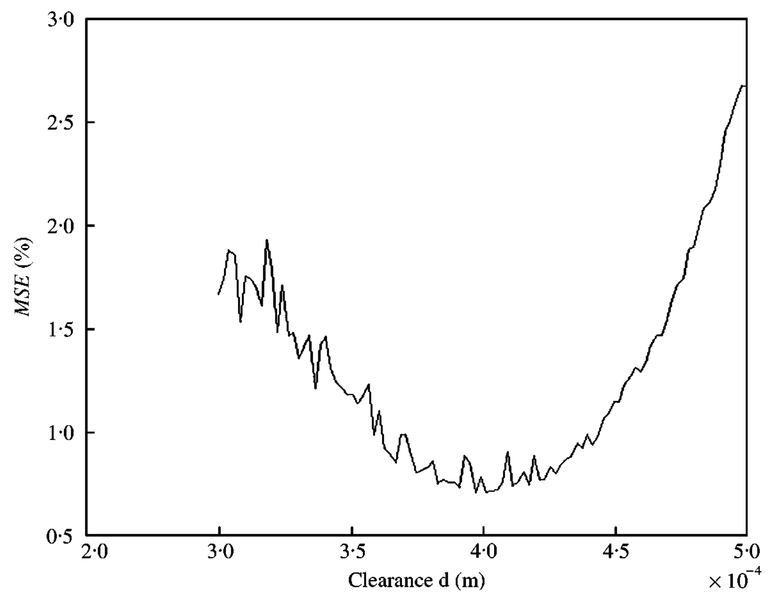

Figure 4. *MSE* versus clearance.

Having chosen a value for the clearance, the results obtained with the polynomial model can now be compared with those given by the non-polynomial model. Tables 1 and 2 present the identified parameters and their significance factors for the polynomial and non-polynomial models respectively.

- For the polynomial model, model orders higher than 3 have no more influence on the *MSE* which is 1·79%.

TABLE 1

Identification results for the polynomial model (piecewise linear case)

	Order 1	Order 2	Order 3
Displacement: parameters	278 003	$-$ 206 354	$6{\cdot}93 \times 10^{11}$
Displacement: significance factors	**60·19**	$2{\cdot}02 \times 10^{-6}$	**6·19**
Velocity: parameters	22·02	$-$ 63·17	$-$ 1169·3
Velocity: significance factors	$5{\cdot}41 \times 10^{-3}$	$4{\cdot}12 \times 10^{-5}$	$5{\cdot}69 \times 10^{-5}$

TABLE 2

Identification results for the non-polynomial model (piecewise linear case)

	k_-	k	k_+	c_-	c	c_+
Parameters	$1{\cdot}46 \times 10^6$	330 459	$1{\cdot}45 \times 10^6$	95·37	18·48	50·43
Significance factors	**99·53**	**99·99**	**99·91**	0·17	$5{\cdot}65 \times 10^{-3}$	$4{\cdot}59 \times 10^{-2}$

- For the non-polynomial model, it is not necessary to include higher order terms than 1. In this case, the *MSE* is equal to 0·71%.

Analysis of these tables is straightforward. Both models provide a reliable identification since the mean square errors are around 1%. Nevertheless, the non-polynomial model is more accurate which is expected since the stiffness is piecewise linear; also it is preferred as the polynomial model is input dependent. Finally, it should be noted that the damping coefficients are badly estimated although damping should actually be in the model. The reason is probably because damping in aluminium is low. Hopefully, this is not a problem since the corresponding significance factors are negligible.

5. NUMERICAL SIMULATION OF A SYSTEM WITH BILINEAR STIFFNESS

It is of interest to study the case of a bilinear stiffness because it is not an odd non-linearity. The beam is the same as for the piecewise linear case except that only one bush is present and that the clearance value is different. Again, it was decided to concentrate attention on the vibrations of the first mode of the beam. An s.d.o.f. system with a bilinear stiffness models the experimental beam. Before investigating the experimental data, first simulate the following bilinear system:

$$25\ddot{x} + 15\dot{x} + 330000x = p(t) \quad \text{if } x < 0{\cdot}00072, \tag{8}$$

$$25\ddot{x} + 15\dot{x} + 930000x = (t) \quad \text{if } x \geqslant 0{\cdot}00072, \tag{9}$$

where $p(t)$ is a white noise sequence band-limited into the 10–25 Hz range. Figure 5 illustrates the PSD of the acceleration obtained through the simulation process. The bilinear characteristic of the system introduces a sequence of harmonics at $n\omega$ where $n = 2$, 3, 4, etc.

In practice, only the acceleration is measured and the displacement and velocity have to be estimated. Thus, it is meaningful to compare the estimated signals with those resulting

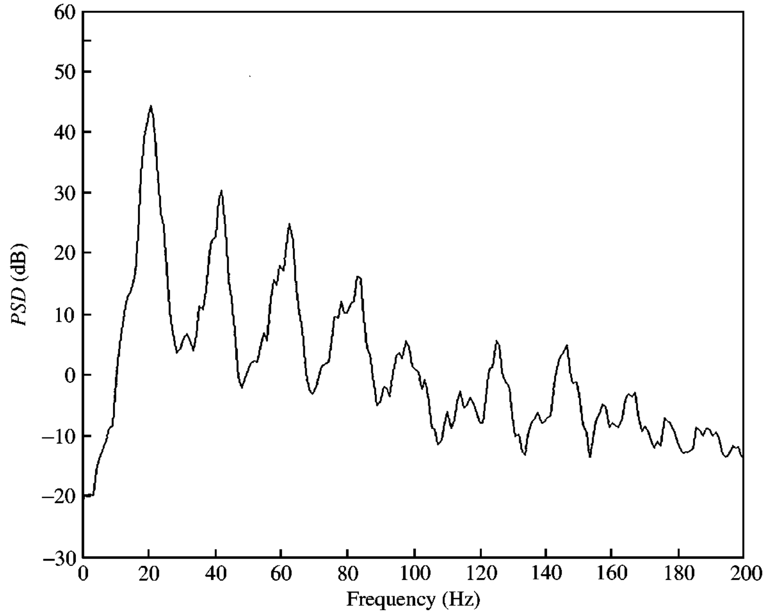

Figure 5. *PSD* of the acceleration.

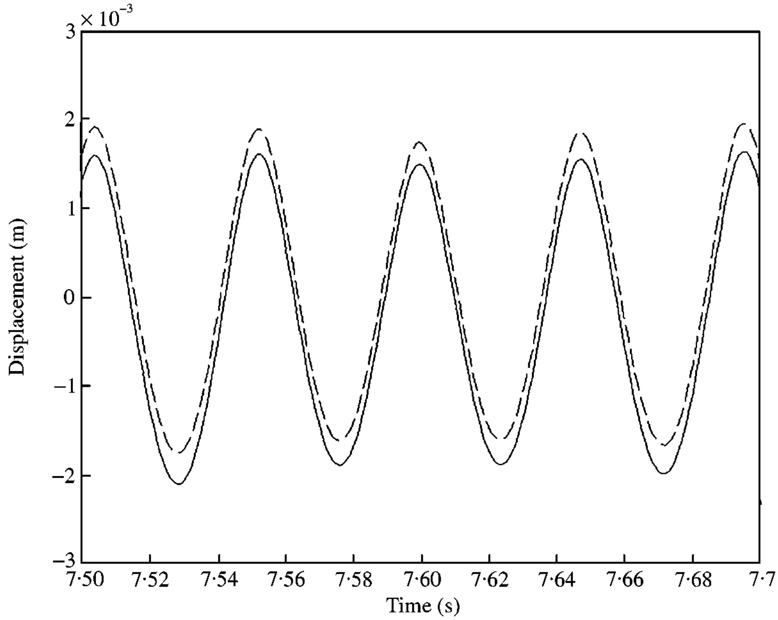

Figure 6. Exact and estimated displacements: ——, exact; – – –, estimated.

from the simulation. If the integrated and filtered velocity is nearly identical to the exact velocity, it is not the case for the displacement represented in Figure 6. The distortions between both displacements are due to the high-pass filtering procedure as explained in section 3.

The identification is realized using the estimated signals and a bilinear model for the restoring force:

$$f(x, \dot{x}) = k_+ x + c_+ \dot{x} \quad \text{if } x \geqslant d,$$

$$f(x, \dot{x}) = kx + c\dot{x} \qquad \text{else},$$

where d is the clearance value and needs to be estimated. To this end, the same procedure as the one used for the piecewise linear model can be exploited i.e., plotting the evolution of the *MSE* as a function of the clearance. However, inspection of the estimated stiffness curve illustrated in Figure 7 reveals that the change in stiffness occurs at around 0·00075 m. This value is almost identical to the exact one (0·00072 m) even if the displacement is badly estimated.

The second row of Table 3 displays the identification results. The results are not as bad as expected. Indeed, the identified parameters are not so different from the exact ones. This is confirmed by the *MSE* equal to 1·46%. Nevertheless, the results can be substantially improved. Since the estimated displacement mainly differs from the exact displacement by

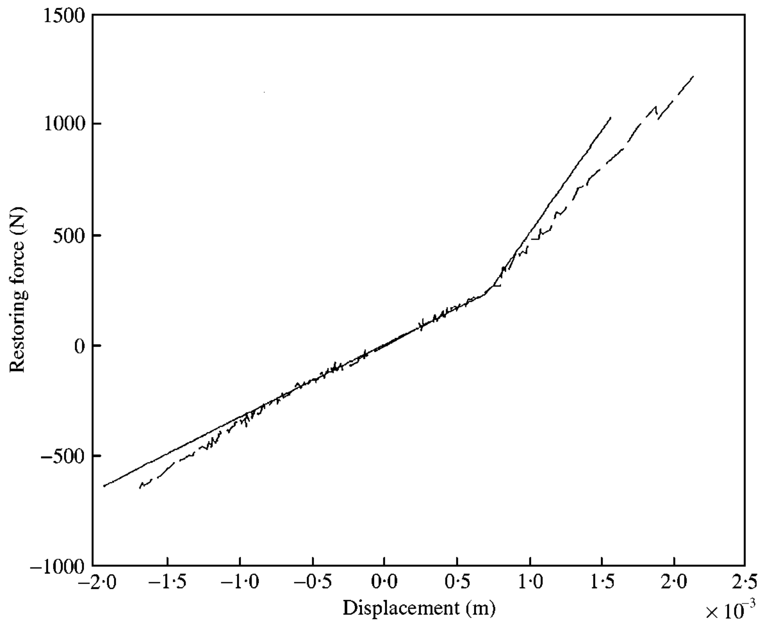

Figure 7. Exact and estimated stiffness curves: ——, exact; – – -, estimated.

TABLE 3

Identification results for bilinear case

	k	k_+	c	c_+
Exact parameters	330 000	930 000	15	15
Identified parameters (estimated displacement)	384 864	773 958	12·16	48·65
Identified parameters (improved displacement)	334 553	817 966	15·96	− 27·67

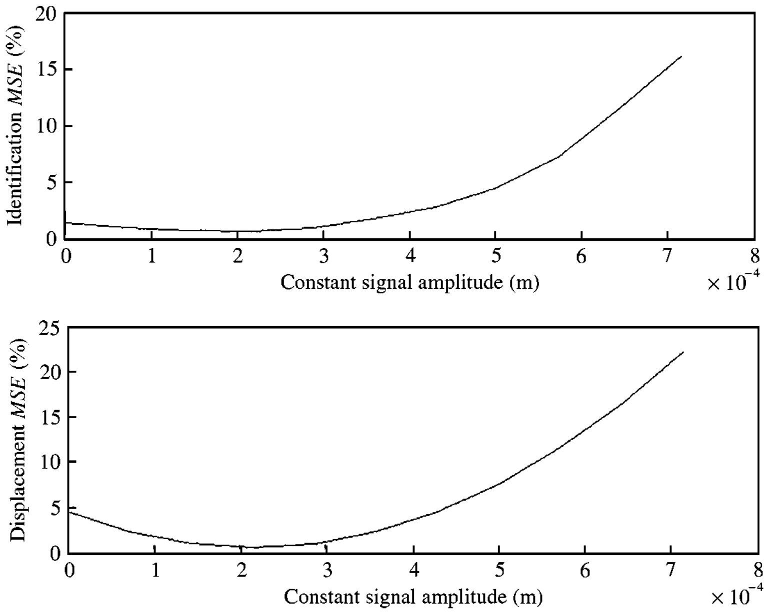

Figure 8. Evolution of the identification and displacement *MSE*s as a function of the constant signal amplitude.

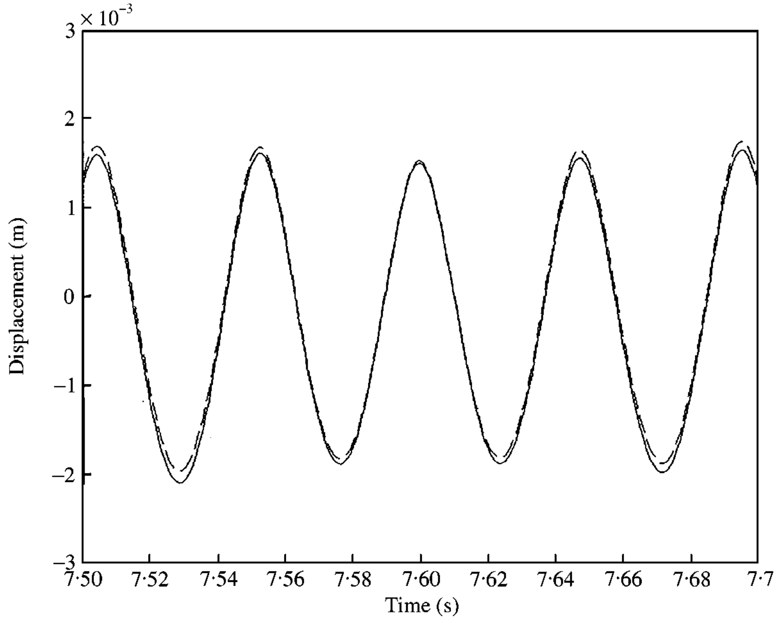

Figure 9. Exact and "improved" displacements: ——, exact; - - -, improved.

a constant signal, it is of interest to add this kind of signal to the estimated displacement and to look at evolution of the *MSE* as a function of the amplitude of the signal. Figure 8 shows that the *MSE* for the identification procedure is minimum for a value of the constant signal amplitude equal to $2{\cdot}15 \times 10^{-4}$ m. Moreover, this value almost corresponds to the minimum of the *MSE* between the exact displacement and the estimated displacement improved by the constant signal. Figure 9 compares both displacements and confirms that the "improved" displacement is close to the exact displacement. The third row of Table 3 presents the identification results when the "improved" displacement is used. Much better results are obtained and the *MSE* is now equal to 0·65%.

In conclusion, attention should be paid to the estimation of the displacement. Even if significant improvements are obtained when a constant signal is added to the estimated displacement, it will always be preferred in practice to measure displacement as well as acceleration signals.

6. EXPERIMENTAL STUDY OF A PIECEWISE LINEAR BEAM

The aim of this paragraph is to study the vibrations of the first mode of the experimental beam described in section 4. The first mode is located at around 18 Hz. If a band-limited white noise centered on this first natural frequency is used, one may reasonably expect the beam to behave as an s.d.o.f. system. With this assumption in mind, a white noise sequence band-limited in the 10–25 Hz range [Figure 10(a)] is produced using the LMS 3.4.08 software and a DIFA SCADAS II. The signal is amplified using a Gear and Watson amplifier. The shaker is attached to the beam by a rigid link and the input force is measured using a PCB 208B force transducer. The response [Figure 10(b)] is measured with a PCB 338M12 accelerometer. Finally, the data are acquired on an HP9000 Unix machine with a sampling frequency set to 1000 Hz.

The acceleration is integrated once to give the velocity and twice to obtain the displacement. The integrated signals are filtered using a high-pass Butterworth filter with a cut-off at 10 Hz. Nevertheless, this is not sufficient. Indeed, the use of a low-pass filter is also necessary, as shown in Figure 11(a). This plot illustrates the evolution of the *MSE* with the cut-off frequency of the low-pass filter. It can be noticed that the *MSE* greatly increases in the frequency interval from 100 to 150 Hz. This observation means that the second mode (around 115 Hz) participates significantly in the response of the beam. Therefore, despite the band-limited excitation signal, it is difficult to excite only the first mode. Thus, the contribution of the second mode is to be discarded but keeping in mind that it is important to capture as many harmonics as possible. The choice of a 70 Hz cut-off frequency is found to be the best compromise. Figure 11(b) compares the *PSD* of the acceleration before and after filtering. The contribution of the second mode is well rejected while the first harmonic (around 70 Hz) is still present.

Since it is now sure that the response is that of an s.d.o.f. system, the identification can be achieved with the same models as for the numerical simulation. Again, a comparison between the polynomial and non-polynomial models is studied. If the mass was assumed to be known in the numerical example, it is no longer the case for the experimental application. Hence, an estimation of the mass is essential in order to compute the restoring force. At this point, two ways exist for identifying the mass. One solution consists of including the mass in the model and instead of fitting the restoring force, the applied force is fitted. The other solution is to perform a linear test, i.e., to reduce the excitation level in such a way that the beam does not make contact with the bushes. This latter alternative is preferred because it is much better to limit the parameters to be estimated in the nonlinear model.

Therefore, a second test was carried out and the acceleration and the applied force were measured. A polynomial model is used for the identification and a model order higher than one does not improve the *MSE* (0·19%) any more. The mass is found to be equal to 24·96 kg.

The clearance value also needs to be determined. Using the same procedure as for the numerical example, the optimal clearance value is found to be 0·00039 m. The MSE is 1·70

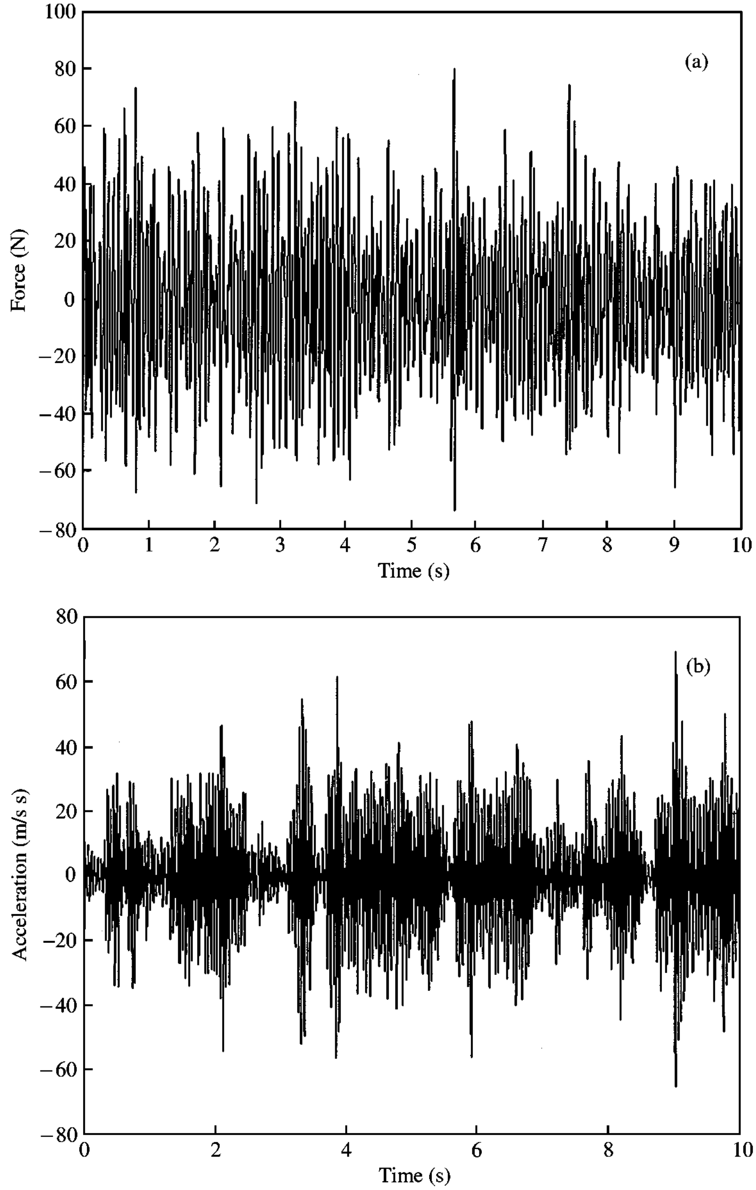

Figure 10. Measured signals. (a) Force time history; (b) acceleration time history.

and 1·80% for the polynomial and non-polynomial models respectively, which is an indication of a good identification. It is rather surprising to note that the polynomial model is superior to the non-polynomial one. However, it might reasonably be expected that the change in stiffness will be smoother in practice than for the numerical example. This may be the reason why a cubic stiffness better matches this change. The identified parameters and their significance factors are listed in Tables 4 and 5. For the non-polynomial model (Table 5), it is worth pointing out that $k_+ = k_-$ as it should be for a symmetrical system. Figure 12 presents the measured and reconstructed stiffness curves. Both reconstructed curves provide a close match to the measured curve, which confirms that the identification has provided good results.

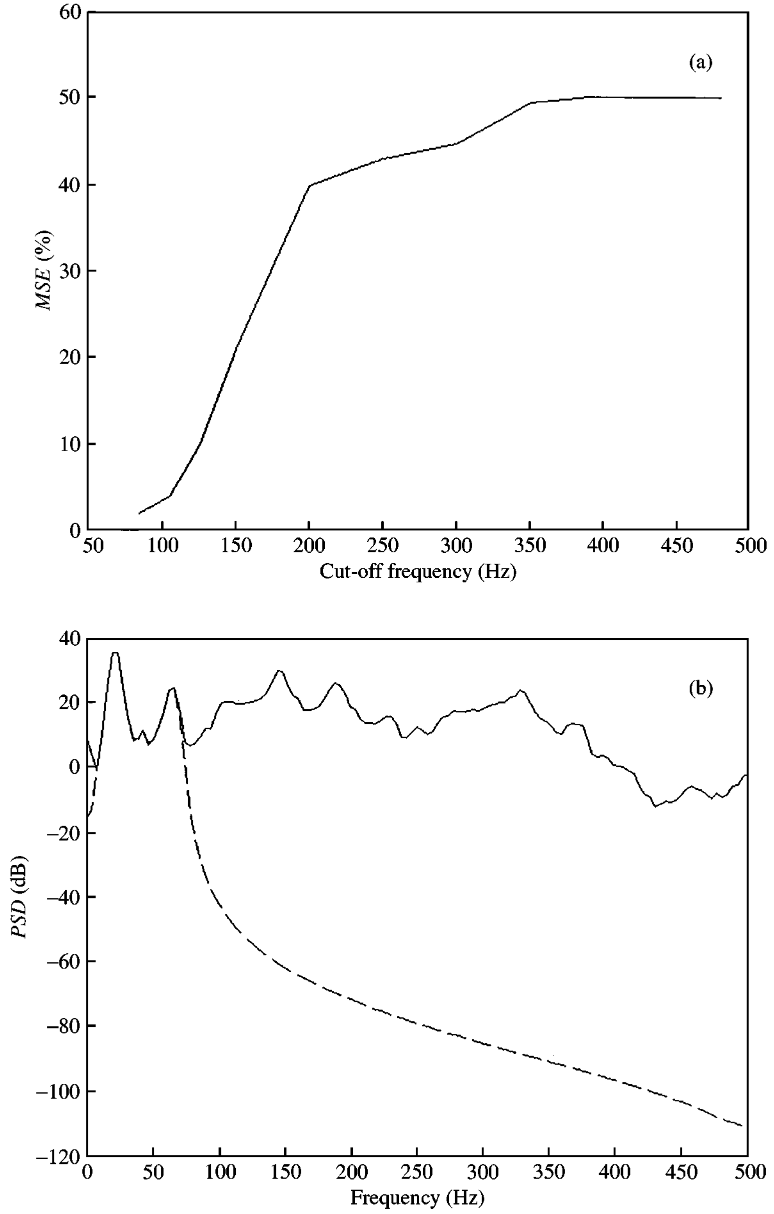

Figure 11. (a) *MSE* versus cut-off frequency; (b) *PSD* of the acceleration before and after filtering: ——, measured; - - -, filtered.

TABLE 4

Identification results for the polynomial model (piecewise linear case)

	Order 1	Order 2	Order 3
Displacement: parameters	248 903	$-2{\cdot}14\times10^{7}$	$5{\cdot}89\times10^{11}$
Displacement: significance factors	**27·89**	0·037	**25·13**
Velocity: parameters	− 32·09	1487	12 949
Velocity: significance factors	$8{\cdot}40\times10^{-3}$	0·052	0·064

TABLE 5

Identification results for the non-polynomial model (piecewise linear case)

	k_-	k	k_+	c_-	c	c_+
Parameters	885 783	281 011	871 787	− 77·99	71·46	11·32
Significance factors	**99·87**	**99·51**	**99·99**	0·13	0·46	$2{\cdot}78\times10^{-3}$

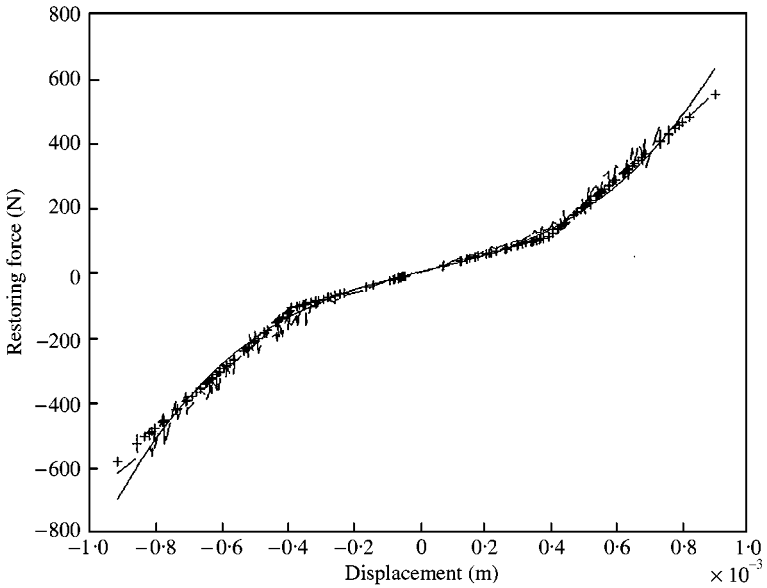

Figure 12. Comparison between the measured and reconstructed stiffness curves: ——, reconstructed (polyn.); +, reconstructed (non-polyn.); - - -, measured.

7. EXPERIMENTAL STUDY OF A BILINEAR BEAM

Section 5 has pointed out that it is always better to measure the displacement when studying an asymmetrical system. Unfortunately, it was not possible to measure the displacement when the measures were acquired and only the acceleration and the input force were measured. This latter consists of a white noise sequence band-limited into the 10–25 Hz range.

The first step of the signal processing was to filter the acceleration in order to keep only the contributions of the first mode. Figure 13 corresponds to the comparison between the PSDs of the measured and filtered accelerations. Afterwards, the acceleration was integrated once and twice to obtain velocity and displacement respectively. Both signals were then filtered using a Butterworth filter with cut-off at 4 Hz. In order to know the clearance value, the stiffness curve was computed and is presented in Figure 14. This graph underlines perfectly the bilinear behaviour of the beam and the clearance is found to be equal to 0·000762 m.

The identification is carried out in two cases. On the one hand, a constant signal is not added to the estimated displacement and on the other hand, this signal is added. The results are illustrated in Table 6. The *MSE* is equal to 2·56% when the displacement is not

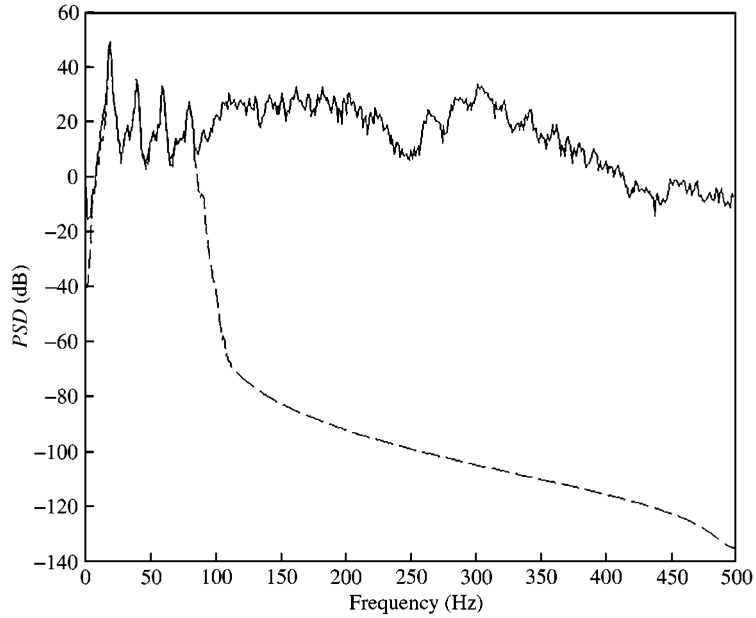

Figure 13. Comparison between the *PSD*s of the measured and filtered accelerations: ——, measured; – – –, filtered.

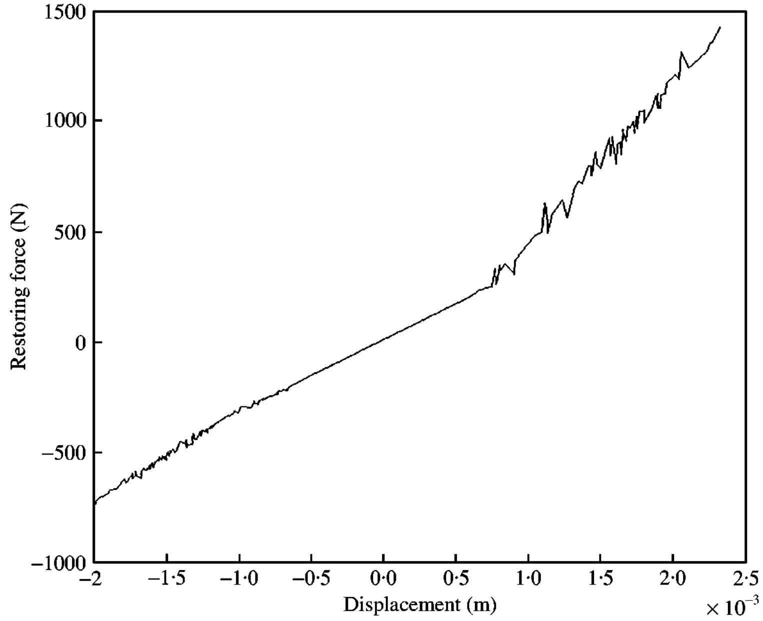

Figure 14. Stiffness curve.

TABLE 6

Identification results for the bilinear case

	k	k_+	c	c_+
Identified parameters (estimated displacement)	356 813	765 733	55·08	40·45
Identified parameters (improved displacement)	312 521	846 963	52·27	− 62·93

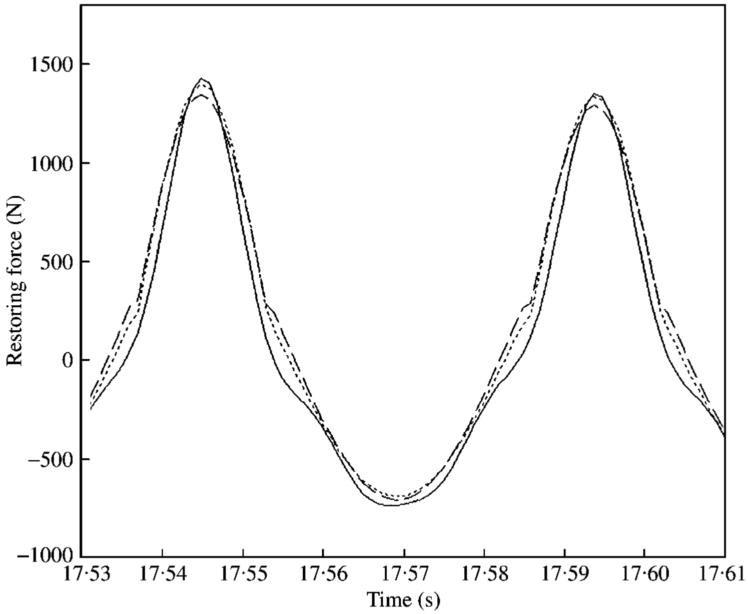

Figure 15. Measured and reconstructed restoring forces: ——, measured; - - - non improved; · · · · ·, improved.

improved. The *MSE* falls to 1·66% when the displacement is improved. Better results obtained with the improved displacement are confirmed in Figure 15 which compares the measured restoring force with the reconstructed restoring forces.

8. CONCLUSIONS

The aim of this study was to compare the numerical and experimental identification of symmetrical and asymmetrical non-linear beams using the restoring force surface method. The beam could be considered as an s.d.o.f. system since the filtering procedure was able to discard all the contributions of the other modes while completely keeping the piecewise or bilinear characteristics.

For the piecewise linear stiffness (symmetrical case), two models were applied to the experimental beam and gave similar results. The polynomial model identified a significant cubic stiffness with an *MSE* of 1·70% and the non-polynomial model led to an *MSE* of 1·80%.

For the bilinear stiffness (asymmetrical case), the *MSE* and the good fit of the restoring force allows it to be concluded that a quite reliable identification of a bilinear beam has been achieved using the restoring force surface method while the displacement was not measured.

In conclusion, the restoring force surface method is confirmed as an efficient tool as far as the identification of s.d.o.f. systems is concerned. However, since it requires the knowledge of acceleration, velocity and displacement signals, great effort has to be spent in processing the data.

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REFERENCES

1. S. F. Masri and T. K. Caughey 1979 *Journal of Applied Mechanics* **46**, 433–447. A nonparametric identification technique for nonlinear dynamic problems.
2. S. F. Masri, H. Sassi and T. K. Caughey 1982 *Journal of Applied Mechanics* **49**, 619–628. A nonparametric identification of nearly arbitrary nonlinear systems.
3. K. Worden and G. R. Tomlinson 1989 *Proceedings of the Seventh International Modal Analysis Conference*, Vol. II, 1347–1355. Application of the restoring force surface method to nonlinear elements.
4. S. Duym and J. Schoukens 1995 *Mechanical Systems and Signal Processing* **9**, 139–158. Design of excitation signals for the restoring force surface method.
5. S. Duym, J. Schoukens and P. Guillaume 1995 *Proceedings of the* 13*th International Modal Analysis Conference*, 1392–1399. A local restoring force surface method.
6. K. Worden and G. R. Tomlinson 1991 *Proceedings of the Nineth International Modal Analysis Conference*, Vol. I, 757–764. An experimental study of a number of nonlinear SDOF systems using the Restoring Force Surface Method.
7. P. Atkins and K. Worden 1997 *Proceedings of the* 15*th International Modal Analysis Conference*, Vol. I, 1023–1028. Identification of a multi degree-of-freedom nonlinear system.
8. K. Worden 1990 *Mechanical Systems and Signal Processing* **4**, 295–319. Data processing and experimental design for the restoring force surface method, part 1: integration and differentiation of measured time data.
9. R. W. Hamming 1989 *Digital Filters*. Englewood Cliffs, NJ: Prentice-Hall; third edition.