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EFFECT OF INTERFACIAL PRESSURE JUMP AND VIRTUAL MASS TERMS ON SOUND WAVE PROPAGATION IN THE TWO-PHASE FLOW

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Virtual mass terms as an interfacial force, taking account of relative acceleration of the bubbles in the liquid phase, have been generally accepted in the two-phase flow models since they conditionally stabilize the numerical scheme. Despite the convincing physical reasoning associated with the bubble flow dynamics, it can be shown that the virtual mass terms unfortunately cause non-physical dispersion in the sound wave propagation. By introducing in the momentum equations new interfacial pressure jump terms based on the surface tension and represented by a function of the fluid bulk moduli, the governing equation system becomes strictly hyperbolic in the present paper with real eigenvalues, regardless of inclusion of the virtual mass terms. It is remarkable that the eigenvalues give realistic speeds of sound when the objective virtual mass terms are reduced more and more until they vanish. On the occasion that the virtual mass terms have to be kept with the interfacial pressure jump terms in the wave-dominant two-phase flow problems, we recommend that the non-physical wave dispersion due to the virtual mass terms should be appropriately controlled.

1. INTRODUCTION

In this paper, we will analyze the effect of interfacial pressure jump terms employed simultaneously with the virtual mass terms in the momentum equations of a two-fluid, two-phase flow. It is well known that the coefficient matrices of the time and the spatial derivative terms in the conservation equation system determine the system eigenvalues by which the speeds of sound propagation are evaluated. We can show that each of the interfacial pressure jump terms and the virtual mass terms retained in the momentum equations help to improve the numerical stability of the initial value problems, extending the parameter range in which the equation system yields real eigenvalues. It is also well known that the system eigenvalues are influenced not only by the rate of change of the conservation variables but also by the source terms if they are not of algebraic form like the virtual mass terms.

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Most of the conventional two-fluid models claiming continuous pressure across the phasic interface suffer from the numerical instability of the ill-posed initial value problem or due to the complex eigenvalues as Lyczkowski [1], Ramsaw and Trapp [2], and Stewart [3] have indicated. From the mechanical point of view, it seems more reasonable to assume that the pressure non-equilibrium across the interface is due to the unequal flow velocities between the gas and the liquid phases. Therefore, the two-fluid model with the pressure jump terms based on the unequal phase velocities should give more realistic results for the two-phase flow.

It is well known that wave speed of the small-amplitude, short-wavelength disturbances are dictated by the real part of the eigenvalues, while the complex part does amplify the wave amplitude with the consequence of short-wavelength numerical instability. If the eigenvalues are all real and distinct, the initial value problem is well posed in the sense that the system is stable against the small-amplitude, short-wavelength disturbances.

The virtual mass as a momentum source term plays some positive roles as has been reported in the literature: see references [4, 5]. For example, the virtual mass terms are made approximately proportional to the unsteady acceleration of the bubbles in the liquid medium so that the conservation equations with these terms are capable of modelling bubble dynamics of the two-phase flow better than the one without this virtual mass force. However, it is unfortunate that the proportionality factor or the so-called virtual mass coefficient has still to be determined empirically without any theoretical reasoning.

It is also true that the numerical computation shows more stability with the virtual mass terms. However, it should be made clear that the virtual mass coefficient has definitely an admissible range of values for numerical stability: see reference [6]. Lahey *et al.* [4] reported that inclusion of the virtual mass terms dramatically improved the computational efficiency. They indicated that for the flow condition of high spatial acceleration such as a critical two-phase flow, the virtual mass was of prime importance and could not be neglected. Drew and Lahey [7] have realized the principle of objectivity for the virtual mass terms by calculating exactly the force imposed on a single sphere in an inviscid, incompressible fluid. It has been argued that this effect is brought about by the system eigenvalues altered by the virtual mass terms: see reference [8].

In this paper, we will show that the conventional two-phase, two-fluid model with the objective form of virtual mass terms produces not only the complex eigenvalues but also unrealistic wave dispersion. Derivation of a quantitative criterion in which the virtual mass coefficient gives reasonable speed of sound is unfortunately impossible because the system of equations with the virtual mass terms only, not together with the interfacial pressure jump terms, has complex eigenvalues.

Recently, a promising approach to improve mathematical property of the two-phase, two-fluid equation system has been proposed by the present authors. New interfacial pressure jump terms based on the surface tension were added to the two-fluid momentum equations: see references [9–11]. It has been shown that the system of equations produces real eigenvalues in all realistic ranges of the bubbly, slug, and annular flows. Further, it is noteworthy that the analytically obtained eigenvalues yield speeds of sound wave propagating in the two-phase flow well agreeing with the measured data, even with the conventional virtual mass terms totally dropped from the momentum equations.

On the occasion when the virtual mass terms were included to investigate the unsteady two-phase flow, propagation of the small-amplitude disturbance was predicted slightly better but only when the virtual mass coefficient was taken small. We will show in this paper how much the speed of sound is deviated from the measured data by inclusion of the virtual mass terms, with the virtual mass coefficient as a parameter, in the relatively low void fraction range.

2. GOVERNING EQUATIONS

The basic conservation laws for the area-averaged phasic variables are, for one-dimensional unsteady two-phase flows, as follows:

Continuity:

$$\frac{\partial(\alpha_k \rho_k)}{\partial t} + \frac{\partial(\alpha_k \rho_k v_k)}{\partial x} = \phi_{c,k}, \tag{1}$$

Momentum:

$$\frac{\partial(\alpha_k \rho_k v_k)}{\partial t} + \frac{\partial(\alpha_k \rho_k v_k^2)}{\partial x} + \alpha_k \frac{\partial p_k}{\partial x} + (p_k - p_i)\frac{\partial \alpha_k}{\partial x} = \phi_{m,k}. \tag{2}$$

Here, α_k, ρ_k, p_k, and v_k denote void fraction, density, pressure, and flow velocity of phase k respectively. We assign $k = g$ for the gas and $k = l$ for the liquid. The terms $\phi_{c,k}$ and $\phi_{m,k}$ represent, respectively, the mass source and the momentum source terms including the virtual mass. We set here the mass source neglected.

2.1. INTERFACIAL PRESSURE JUMP TERMS

Young and Laplace proposed the well-known surface tension equation, for a bubble of radius R,

$$p_g - p_l = \frac{2\sigma}{R}. \tag{3}$$

We assume an infinitesimal surface thickness δ between two radii R_g and R_l as in Figure 1. It may represent the hypothetical interfacial thickness introduced earlier in the statistical mechanics: see references [12–14]. Brackbill *et al.* [15] have presented a numerical method, by interpreting the surface tension as a continuous, three-dimensional effect across the interface of finite film thickness, that alleviates the interface topology constraints. It is called the continuum surface force (CSF) model. In the limit that the width of transition in the direction normal to the interface thickness goes to zero, the volume force becomes the integral of surface tension multiplied by a delta function.

On an imaginary sphere inside the film at the average radius R_i, $R_g + R_l = 2R_i$, we assume that equation (3) still holds true, namely,

$$p_g - p_l = \frac{2\delta}{R_g + \delta/2}\left(\frac{\sigma}{\delta}\right) = \frac{2\delta}{R_l - \delta/2}\left(\frac{\sigma}{\delta}\right). \tag{4}$$

By a small radial increment, ΔR_i, the surface area of the bubble is increased by the amount ΔA_i and the inner and the outer spherical volumes by ΔV_g and ΔV_l respectively. We can readily show that these variables satisfy the following relations:

$$\frac{R_g}{2}\left(\frac{\Delta A_i/V}{\Delta V_g/V}\right) = \left(1 - \frac{\delta/2}{R_g + \delta/2}\right), \tag{5}$$

$$\frac{R_l}{2}\left(\frac{\Delta A_i/V}{\Delta V_l/V}\right) = -\left(1 + \frac{\delta/2}{R_l - \delta/2}\right). \tag{6}$$

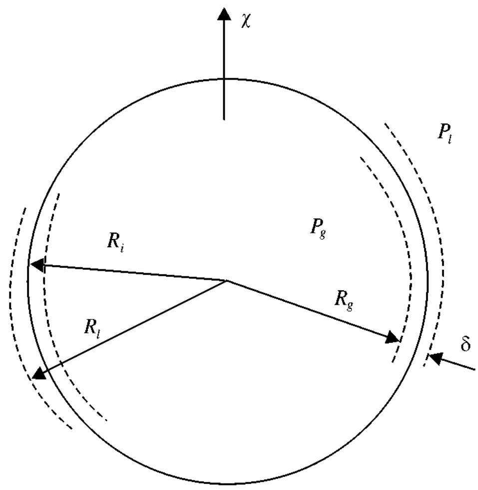

Figure 1. Hypothetical midsphere at R_i (the solid circle) in the film of thickness δ.

We can then rewrite equation (4) as

$$p_g - p_l = \frac{4\sigma}{\delta}\left(1 - \frac{R_g}{2}\frac{\Delta A_i}{\Delta V_g}\right) = -\frac{4\sigma}{\delta}\left(1 + \frac{R_l}{2}\frac{\Delta A_i}{\Delta V_l}\right). \tag{7}$$

In equation (7), the coefficient $4\sigma/\delta$ plays the role of *Lagrangian multiplier* introduced by Aubin and Ekeland [16]; the quantity in parentheses may be recognized as a *slack variable* used in the non-linear analysis, see reference [17]. We recall here the relation, $L = c\sigma/\delta$, used in physical chemistry and statistical mechanics, where L is the bulk modulus and c is a constant. Then we can assume that the ambiguous surface tension stress $4\sigma/\delta$ can be given in terms of the more definitive quantity, the bulk moduli of the two phases. That is

$$\frac{4\sigma}{\delta} = L_g + L_l \quad \text{or} \quad \delta = \frac{4\sigma}{L_g + L_l}. \tag{8}$$

Using the phasic speed of sound, $c_k^2 = (\partial p_k/\partial \rho_k)_s$, we get the bulk modulus $L_k = \rho_k c_k^2$.

It is noteworthy that the surface thickness δ in equation (8) complies with the relation, $l \approx \sigma\chi/2c$, which is derived by Van Stralen [18] from the Van der Waals and Cahn–Hilliard equation, where l is the interfacial thickness, of order $O(10^{-10}\ \text{m})$ for liquids, χ is isothermal compressibility, and c is a constant.

Equations (7) and (8) lead to

$$p_g - p_l = L_g\left(1 - \frac{R_g}{2}\frac{\Delta A_i}{\Delta V_g}\right) - L_l\left(1 + \frac{R_l}{2}\frac{\Delta A_i}{\Delta V_l}\right). \tag{9}$$

An identity,

$$p_g - p_l = (p_g - p_i) + (p_i - p_l), \tag{10}$$

is used where p_i is a hypothetical interfacial pressure represented on the sphere of radius R_i. By assuming that the phasic contributions are matching between the two equations (9) and (10), we take

$$(p_g - p_i) = L_g\left(1 - \frac{R_g}{2}\frac{\Delta A_i}{\Delta V_g}\right), \tag{11}$$

$$(p_i - p_l) = -\,L_l\left(1 + \frac{R_l}{2}\frac{\Delta A_i}{\Delta V_l}\right). \tag{12}$$

Multiplying the factor $\partial\alpha_g/\partial x$ with equation (11) and $\partial\alpha_l/\partial x$ with equation (12), and taking the limit $\Delta R_i \to 0$, we get

$$(p_g - p_i)\frac{\partial\alpha_g}{\partial x} = L_g\left(\frac{\partial\alpha_g}{\partial x} - \frac{R_g}{2}\frac{\partial a_i}{\partial x}\right), \tag{13}$$

$$(p_i - p_l)\frac{\partial\alpha_l}{\partial x} = -\,L_l\left(\frac{\partial\alpha_l}{\partial x} + \frac{R_l}{2}\frac{\partial a_i}{\partial x}\right), \tag{14}$$

where a_i is the interfacial area density. The interfacial pressure jump terms in equation (2), the last term on the left-hand side, have been derived in the following form using the original surface tension modelling, [9–11]:

$$(p_g - p_i)\frac{\partial\alpha_g}{\partial x} = L_g\left(1 - \frac{R_g}{2}\frac{\partial a_i}{\partial\alpha_g}\right)\frac{\partial\alpha_g}{\partial x} = C_i L_g\frac{\partial\alpha_g}{\partial x}, \tag{15}$$

$$(p_l - p_i)\frac{\partial\alpha_l}{\partial x} = -\,L_l\left(1 + \frac{R_l}{2}\frac{\partial a_i}{\partial\alpha_l}\right)\frac{\partial\alpha_l}{\partial x} = -\,C_i L_l\frac{\partial\alpha_l}{\partial x}. \tag{16}$$

We use for the bubbly flow the interfacial area density relation, $a_i = 3{\cdot}6\alpha_g/D_{ave}$, which is suggested by Ishii and Mishima [19]. The averaged bubble diameter D_{ave} is generally obtained by using the Weber number, $\mathrm{We} \equiv 2D_{ave}\rho_l(v_g - v_l)^2/\sigma$. However, if we simply assume that the two radii R_g and R_l are equal to half of the averaged bubble diameter D_{ave}, then the coefficient of interfacial pressure jump C_i is a constant having an order of magnitude $O(10^{-1})$.

Since the interface of one phase fluid can be regarded as the elastic boundary of the other, the sound speed of one phase fluid should show a dependency upon the bulk modulus of the other fluid. Here, the speed of sound decreases with increasing elasticity of the surrounding fluid. The interfacial pressure jump represented by the two fluid bulk moduli accordingly gives appropriate influence to the wave propagation speed in the mixture. Further, because the two-phase fluids have drastically different bulk moduli, different in the order of magnitude, a slight increase in the void fraction of the mixture would result in a significant reduction of the speed of sound in the two-phase flow. For this reason, the interfacial pressure jump terms in equations (15) and (16) contribute physically to the realistic speed of sound in the two-phase mixture and mathematically to the real eigenvalues of the equation system.

2.2. VIRTUAL MASS TERMS

The virtual mass, also known as added mass or apparent mass, is associated with the force required to accelerate the fluid surrounding a moving body of different phase. It has the effect of liquid retarding, interpreted as inertia force acting on the accelerating bubble.

The most general objective form of the virtual mass is

$$\phi_{m,k} = \pm\alpha_g\rho_l C\left\{\frac{\partial v_g}{\partial t} - \frac{\partial v_l}{\partial t} + v_g\frac{\partial(v_g - v_l)}{\partial x} + (v_g - v_l)\left[(\varepsilon - 2)\frac{\partial v_g}{\partial x} + (1-\varepsilon)\frac{\partial v_l}{\partial x}\right]\right\}, \quad (17)$$

where $(-)$ is for the gas and $(+)$ is for the liquid, C is the coefficient of virtual mass and ε is a function of void fraction. With $\varepsilon = 2$ for the low void fraction limit and $C_v \equiv \alpha_g\rho_l C$, equation (17) becomes the objective formulation proposed by Drew and Lahey [7]:

$$\phi_{m,k} = \mu C_v\left\{\frac{\partial v_g}{\partial t} + v_g\frac{\partial v_g}{\partial x} - \frac{\partial v_l}{\partial t} - v_l\frac{\partial v_l}{\partial x}\right\}. \quad (18)$$

The coefficient C can be taken as 0·5 for a spherical bubble.

2.3. MATRIX FORM OF THE GOVERNING EQUATIONS

Using the identity $\partial p_g/\partial x = \partial p_l/\partial x$ obtained by differentiating the Young and Laplace equation (3) and the speed of sound defined previously, we can explicitly write the governing equations as follows.

Continuity:

$$\rho_g\frac{\partial\alpha_g}{\partial t} + \frac{\alpha_g}{c_g^2}\frac{\partial p_g}{\partial t} + \rho_g v_g\frac{\partial\alpha_g}{\partial x} + \frac{\alpha_g v_g}{c_g^2}\frac{\partial p_g}{\partial x} + \rho_g\alpha_g\frac{\partial v_g}{\partial x} = 0, \quad (19)$$

$$\rho_l\frac{\partial\alpha_l}{\partial t} + \frac{\alpha_l}{c_l^2}\frac{\partial p_g}{\partial t} + \rho_l v_l\frac{\partial\alpha_l}{\partial x} + \frac{\alpha_l v_l}{c_l^2}\frac{\partial p_g}{\partial x} + \rho_l\alpha_l\frac{\partial v_l}{\partial x} = 0, \quad (20)$$

Momentum:

$$\alpha_g\rho_g\frac{\partial v_g}{\partial t} + \alpha_g\frac{\partial p_g}{\partial x} + \alpha_g\rho_g v_g\frac{\partial v_g}{\partial x} + C_i L_m\frac{\partial\alpha_g}{\partial x} = -C_v\left\{\frac{\partial(v_g - v_l)}{\partial t} + v_g\frac{\partial v_g}{\partial x} - v_l\frac{\partial v_l}{\partial x}\right\}, \quad (21)$$

$$\alpha_l\rho_l\frac{\partial v_l}{\partial t} + \alpha_l\frac{\partial p_l}{\partial x} + \alpha_l\rho_l v_l\frac{\partial v_l}{\partial x} - C_i L_m\frac{\partial\alpha_l}{\partial x} = C_v\left\{\frac{\partial(v_g - v_l)}{\partial t} + v_g\frac{\partial v_g}{\partial x} - v_l\frac{\partial v_l}{\partial x}\right\}. \quad (22)$$

For the bubbly flow as a perfect mixture, the bulk modulus can be written as

$$L_m = -V\frac{\mathrm{d}p}{\mathrm{d}V} = -V\frac{\mathrm{d}p}{\mathrm{d}V_g + \mathrm{d}V_l} = V\frac{\mathrm{d}p}{V_g\,\mathrm{d}p/L_g + V_l\,\mathrm{d}p/L_l}. \quad (23)$$

Since the fluid bulk modulus is $L_k \equiv \rho_k c_k^2$ and it holds that $L_g \ll L_l$, equation (23) yields

$$C_i L_m \approx C_i L_g/\alpha_g, \quad (24)$$

which holds true in the range $\alpha_g \geqslant L_g/L_l$. We can assume here that the order of magnitude of the mixture bulk modulus is almost equal to that of the gas by taking $\alpha_g \approx O(10^{-1})$ for the bubbly flow, which gives the conclusion $C_i L_m = \rho_g c_g^2$.

As the bubble radius is reduced to zero in an extreme case or when there is no bubble, we put zero on the void fraction. Then, the continuity (1) and the momentum equations (2) of the gas phase disappear. Since the limiting condition of single-phase flow also means that the gradient of void fraction is zero in the flow field, i.e., $\partial \alpha_k / \partial x = 0$, the fourth term of the momentum equation (2) vanishes. The governing equations without the interfacial pressure jump terms are then clearly reduced to those of the single-phase flow.

Equations (19)–(22) are combined in a matrix form

$$A \frac{\partial U}{\partial t} + B \frac{\partial U}{\partial x} = 0, \tag{25}$$

where A and B are the coefficient matrices and U is the state vector consisting of the primitive variables, i.e., $U = (\alpha_g, \rho_g, v_g, v_l)^{\mathrm{T}}$.

3. CHARACTERISTIC ANALYSIS

The eigenvalues of the governing equation system represent propagation speed of the small-amplitude short-wavelength perturbations according to Whitham [20]. For the long-wavelength disturbances, the source terms play an important role whereas for large-amplitude disturbances the non-linear wave interaction has a dominant effect: these waves are, however, not considered in this paper. If the eigenvalues are all real and distinct, the governing equations are hyperbolic and numerically stable in the initial value problem against the short-wavelength disturbances. Characteristic analysis excludes the algebraic source terms which do not affect the system eigenvalues.

3.1. SURFACE TENSION EFFECT

The eigenvalues of the Jacobian matrix $G = A^{-1}B$, without the virtual mass terms in equation (25), are all real as we will derive them analytically using the equation

$$\mathrm{Det}(G - \lambda I) = 0. \tag{26}$$

A fourth order polynomial characteristic equation is obtained,

$$P(\lambda) = (\lambda - v_g)^2(\lambda - v_l)^2 - K_1(\lambda - v_g)^2 - K_2(\lambda - v_l)^2 + K_3 = 0, \tag{27}$$

where K_1, K_2, and K_3 are functions of L_k, α_k, ρ_k, and c_k. Since equation (27) is symmetric with respect to the phasic velocities v_g and v_l, its roots should be obtained as

$$\lambda_{1,2} = v_g \pm c_g, \tag{28}$$

$$\lambda_{3,4} = v_l \pm c_l \sqrt{\frac{\rho_g c_g^2}{\alpha_l \rho_g c_g^2 + \alpha_g \rho_l c_l^2}}. \tag{29}$$

Two eigenvalues λ_1 and λ_3 represent the speed of sound in the gas and the liquid phases respectively. The speed of sound in the mixture is expressed by void fraction weighting as

$$c_m = \frac{\lambda_1 \lambda_3}{\alpha_l \lambda_1 + \alpha_g \lambda_3}. \tag{30}$$

The speed of sound in the two-phase mixture is shown in Figure 2 along with the experimental data given by Henry *et al.* [21]. The present result agrees well with the experimental data up to the void fraction 0·22. Afterwards, the deviation at $\alpha_g = 0{\cdot}3$, for

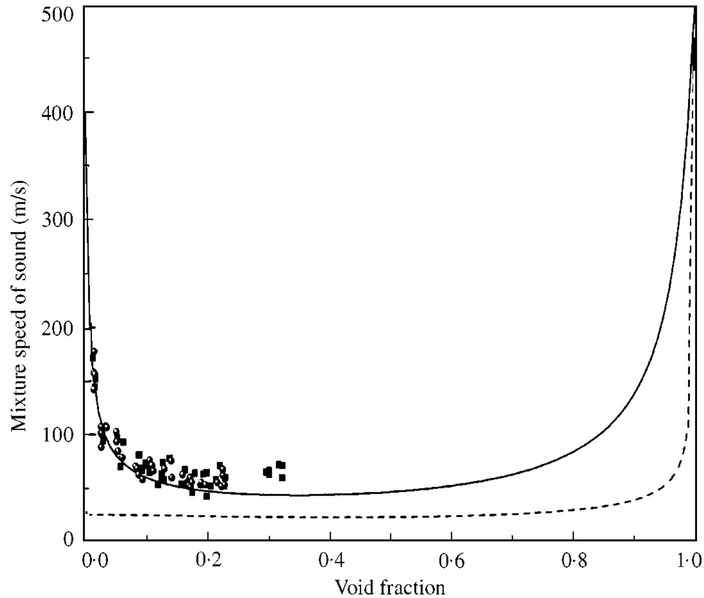

Figure 2. Speed of sound in the mixture with both the interfacial pressure jump and the virtual mass terms included, for low-speed two-phase bubbly flow ($p_g = 283$ kPa). ——, Surface tension terms only; -----, surface tension + virtual mass with $C = 0{\cdot}5$; ■, compression pulse by Henry *et al.*; ●, rarefaction pulse by Henry *et al.*

example, is probably caused by the effect of transition between two-phase flow regimes. For this particular computation, we assumed very low phasic velocities with an interfacial slip ratio $v_g/v_l = 0{\cdot}03/0{\cdot}01 = 3$.

When there is no flow, no virtual mass effect would be found. The eigenvalues (28) and (29) now become the speeds of sound in the stagnant fluids with zero phasic velocities, i.e., $v_g = v_l = 0$:

$$\lambda_{1,2} = \pm\, c_g, \tag{31}$$

$$\lambda_{3,4} = \pm\, c_l \sqrt{\frac{\rho_g c_g^2}{\alpha_l \rho_g c_g^2 + \alpha_g \rho_l c_l^2}}. \tag{32}$$

Then, the speed of sound in the mixture becomes as follows:

$$c_m = \frac{c_g c_l \sqrt{\rho_g c_g^2/(\alpha_l \rho_g c_g^2 + \alpha_g \rho_l c_l^2)}}{\alpha_l c_g + \alpha_g c_l \sqrt{\rho_g c_g^2/(\alpha_l \rho_g c_g^2 + \alpha_g \rho_l c_l^2)}}. \tag{33}$$

The speed of sound in the mixture is now reduced to that of the single-phase fluid in an extreme case of $\alpha_g \to 0$ or $\alpha_l \to 0$. That is,

$$\underset{\alpha_g \to 0}{c_m} = \lim_{\alpha_g \to 0} \frac{c_g c_l \sqrt{\rho_g c_g^2/(\alpha_l \rho_g c_g^2 + \alpha_g \rho_l c_l^2)}}{\alpha_l c_g + \alpha_g c_l \sqrt{\rho_g c_g^2/(\alpha_l \rho_g c_g^2 + \alpha_g \rho_l c_l^2)}} \quad \text{or} \quad \underset{\alpha_l \to 0}{c_m} = c_g. \tag{34}$$

Thus, the result is perfectly consistent with the single-phase flow in the extreme.

3.2. VIRTUAL MASS EFFECT

We first include the virtual mass terms only, not the interfacial pressure jump terms. The eigenvalues of equation (25) shall be obtained as the roots of the fourth order characteristic polynomial equation in a very complicated form. The characteristic polynomial can be reduced, however, to quadratic if we assume that the phasic flow velocities are very low in comparison with the phasic speeds of sound namely,

$$v_g \ll c_g, \qquad v_l \ll c_l. \tag{35}$$

The characteristic equation is then

$$P(\lambda) = a\lambda^2 + b\lambda + c = 0, \tag{36}$$

where

$$a = -\{C_v + (\alpha_g \rho_l + \alpha_l \rho_g)\alpha_g \alpha_l\}\rho_g \rho_l, \tag{37}$$

$$b = 2\{C_v(\alpha_g^2 v_l + \alpha_l^2 v_g + \alpha_g \alpha_l (v_g + v_l)) + \alpha_g \alpha_l (\alpha_g \rho_l v_l + \alpha_l \rho_g v_g)\}\rho_g \rho_l, \tag{38}$$

$$c = -\{C_v(\alpha_g^2 v_l^2 + \alpha_l^2 v_g^2 + \alpha_g \alpha_l (v_g^2 + v_l^2)) + (\alpha_g \rho_l v_l^2 + \alpha_l \rho_g v_g^2)\alpha_g \alpha_l\}\rho_g \rho_l. \tag{39}$$

The roots of equation (36) are

$$\lambda_{1,2} = \frac{C_v(\alpha_g^2 v_l + \alpha_l^2 v_g + \alpha_g \alpha_l (v_g + v_l)) + \alpha_g \alpha_l (\alpha_g \rho_l v_l + \alpha_l \rho_g v_g)}{C_v + \alpha_g \alpha_l (\alpha_g \rho_l + \alpha_l \rho_g)} \pm \frac{|v_l - v_g| \sqrt{-\alpha_g \alpha_l (C_v + \alpha_g \alpha_l \rho_g)(C_v + \alpha_g \alpha_l \rho_l)}}{C_v + \alpha_g \alpha_l (\alpha_g \rho_l + \alpha_l \rho_g)}. \tag{40}$$

The above eigenvalues have two imaginary values satisfying the condition

$$b^2 - 4ac = -\alpha_g \alpha_l (C_v + \alpha_g \alpha_l \rho_g)(C_v + \alpha_g \alpha_l \rho_l) < 0, \tag{41}$$

with $C_v \geqslant 0$. Therefore, if there are no surface tension terms, the system of equations having the objective virtual mass terms only always has complex eigenvalues.

3.3. THE COMBINED SURFACE TENSION AND VIRTUAL MASS EFFECT

Next, we investigate the effect of the interfacial pressure jump terms included in addition to the virtual mass terms in the momentum equations. We can show that the system of equations always yields real eigenvalues. Here we use again the previous low-speed flow assumption to obtain the characteristic equation of quadratic form: equation (36) with

$$a = -\{C_v + (\alpha_g \rho_l + \alpha_l \rho_g)\alpha_g \alpha_l\}, \tag{42}$$

$$b = 2\{C_v(\alpha_g^2 v_l + \alpha_l^2 v_g + \alpha_g \alpha_l (v_g + v_l)) + \alpha_g \alpha_l (\alpha_g \rho_l v_l + \alpha_l \rho_g v_g)\}, \tag{43}$$

$$c = -\{C_v(\alpha_g^2 v_l^2 + \alpha_l^2 v_g^2 + \alpha_g \alpha_l (v_g^2 + v_l^2)) + (\alpha_g \rho_l v_l^2 + \alpha_l \rho_g v_g^2 - \rho_g c_g^2)\alpha_g \alpha_l\}. \tag{44}$$

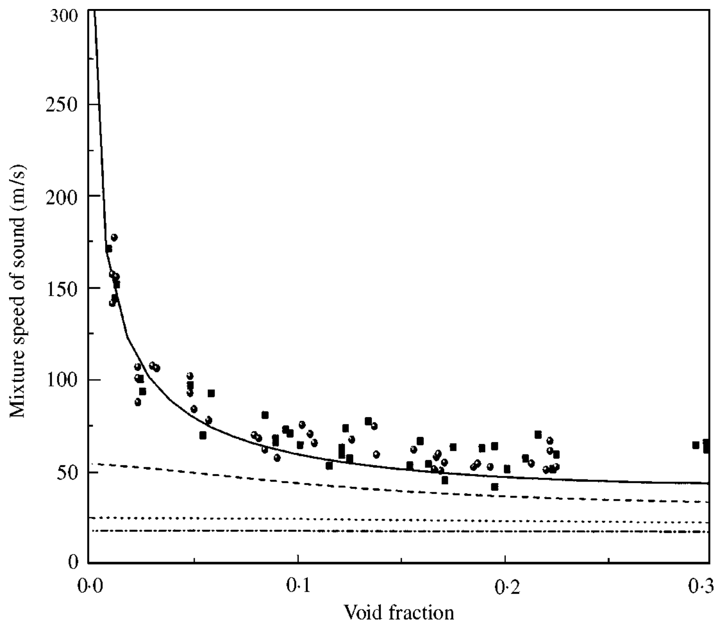

Figure 3. Effect of the virtual mass coefficient on the speed of sound in the mixture. ——, Surface tension + Virtual mass with $C = 0{\cdot}0$; – – – –, surface tension + Virtual mass with $C = 0{\cdot}1$; - - - - -, surface tension + Virtual mass with $C = 0{\cdot}5$; –·–·–·–, surface tension + Virtual mass with $C = 1{\cdot}0$; ■, compression pulse by Henry *et al.*; ●, rarefaction pulse by Henry *et al.*

The roots are

$$\lambda_{1,2} = \frac{C_v(\alpha_g^2 v_l + \alpha_l^2 v_g + \alpha_g\alpha_l(v_g + v_l)) + \alpha_g\alpha_l(\alpha_g\rho_l v_l + \alpha_l\rho_g v_g)}{C_v + \alpha_g\alpha_l(\alpha_g\rho_l + \alpha_l\rho_g)}$$

$$\pm \sqrt{\frac{\{C_v(\alpha_g^2 v_l + \alpha_l^2 v_g + \alpha_g\alpha_l(v_g + v_l)) + \alpha_g\alpha_l(\alpha_g\rho_l v_l + \alpha_l\rho_g v_g)\}^2}{C_v + \alpha_g\alpha_l(\alpha_g\rho_l + \alpha_l\rho_g)} - \frac{\{C_v + \alpha_g\alpha_l(\alpha_g\rho_l + \alpha_l\rho_g)\}\{C_v(\alpha_g^2 v_l^2 + \alpha_l^2 v_g^2 + \alpha_g\alpha_l(v_g^2 + v_l^2))}{C_v + \alpha_g\alpha_l(\alpha_g\rho_l + \alpha_l\rho_g)} + \frac{(\alpha_g\rho_l v_l^2 + \alpha_l\rho_g v_g^2 - \rho_g c_g^2)\alpha_g\alpha_l\}}{C_v + \alpha_g\alpha_l(\alpha_g\rho_l + \alpha_l\rho_g)}}. \tag{45}$$

The eigenvalues are real and distinct for the virtual mass coefficient $C \geqslant 0$ because we can easily show that the arguments of square root in the above are all positive for the low-speed flow assumption in equation (35). The speed of sound in the mixture is plotted in Figure 2 for the virtual mass coefficient taken as $C = 0{\cdot}5$. Figure 3 shows the speed of sound in the mixture obtained numerically without any assumption of low-speed flow by using several different values on the virtual mass coefficient. The fourth order polynomial is numerically solved to obtain four real roots.

In particular, deviation of the predicted curve from the experimental data is reduced as the virtual mass coefficient is reduced. In the void fraction range between 0·1 and 0·25, comparison is quite improved if the virtual mass coefficient is taken as $C \leqslant 0{\cdot}1$. As the void fraction is raised, the curves become close and nearly horizontal, showing less dependency on the virtual mass coefficient. Therefore, it can be argued that if the virtual mass terms have to be included in the governing equations in order to take the relative acceleration into account, their coefficient must be controlled appropriately so as to not introduce excessive dispersion on the wave propagation.

4. CONCLUDING REMARKS

The new interfacial pressure jump terms based on the surface tension model have contributed to the hyperbolic type of governing equation system for the spherical-bubbly flow. The speed of sound in the mixture evaluated by the real system eigenvalues has shown excellent agreement with the existing experimental data. Although it is true that the virtual mass itself can improve numerical efficiency and has a considerable justification for the unsteady bubbly flow, it can guarantee neither the real eigenvalues of the system nor the correct speed of sound in the two-phase mixture. By adding the interfacial pressure jump terms simultaneously in the momentum equations, numerical computation becomes unconditionally stable regardless of the magnitude of the virtual mass terms: however, the wave suffers from the non-physical wave dispersion. For the particular case of the relatively accelerating flow such as the critical flow, the interfacial pressure jump and the virtual mass terms should be used with appropriate amount of virtual mass coefficient in order not to have a non-physical effect on the propagation of sound waves.

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APPENDIX A: NOMENCLATURE

a	interfacial area density
A	coefficient matrix
B	coefficient matrix
c	speed of sound
C	coefficient
D	bubble diameter
L	fluid bulk modulus
p	pressure
R	radius of curvature
t	time
U	state vector
v	flow velocity
x	space co-ordinate

Greek symbols

α	volumetric phase concentration
λ	system eigenvalue
ρ	fluid density
σ	surface tension

Subscripts and Superscripts

g	gas phase
i	index for interface
k	index for each fluid
l	liquid phase
s	isentropic process
m	index for two-phase mixture
v	index for virtual mass