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LETTERS TO THE EDITOR

COMMENTS ON THE FREE VIBRATIONS OF BEAMS WITH A SINGLE-EDGE CRACK

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1. INTRODUCTION

It has been long recognized that the dynamic behavior of structural members is affected by local changes in physical properties such as stiffness, mass and damping. This has motivated numerous investigations in the past few years with the objective of relating changes in vibrational responses to the presence of damaged sites in a structure. A thorough review of such methods may be found in reference [1], where the author stresses the importance of the utilization of a mathematical model in the diagnostic method when not only detection, but also location and quantification of the damage are intended.

Of particular interest is the identification problem when damage has the form of a fatigue crack. The development of reliable vibration-based crack diagnostic is yet to be accomplished, but many advances have been reported in the past few years. Modelling techniques for the vibrations of cracked structures have been the subject of a very detailed literature review found in reference [2]. Recently, some researchers have directed their efforts to the development of a family of continuous models of cracked bars and beams. Christides and Barr [3] published the first model of this family with the investigation of double-edge, symmetric cracks. They used a stationary variational principle and cleverly chosen kinematic assumptions to derive a partial differential equation of motion for a cracked slender beam. Numerous publications applying those ideas have appeared in recent years [4–6]. In particular, Shen and Pierre [6] published a very interesting extension of the methodology to the case of single-edge cracks. This paper aims to report and discuss some results obtained when implementing Shen's single-edge crack model for investigations of a damage detection methodology. The basic equations are reviewed and the terms believed to cause numerical problems are analyzed.

2. REVIEW OF THE MATHEMATICAL MODEL

2.1. THE HU–WASHIZU–BARR VARIATIONAL PRINCIPLE

The equations of motion of a cracked beam-like structure are derived through the Hu–Washizu–Barr [7] variational method, which can be viewed as an extension of the

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Hellinger–Reissner stationary principle [8] and allows independent kinematic assumptions on the displacement, velocities, strain and stress fields in elastodynamic problems. The variational method states that for independent variations of displacements u_i, strains ε_{ij}, stresses σ_{ij}, and momentum p_i, one must have

$$\int_V \{(\sigma_{ij,j} + B_i - \rho\dot{p}_i)\,\delta u_i + (\sigma_{ij} - W(\varepsilon_{ij})_{,\varepsilon_{ij}})\,\delta\varepsilon_{ij} + [\varepsilon_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i})]\,\delta\sigma_{ij} + [\rho\dot{u}_i - \hat{T}(p_i)_{,p_i}]\,\delta p_i\}\,\mathrm{d}V + \int_{S_1} (\bar{g}_i - g_i)\,\delta u_i\,\mathrm{d}S_1 + \int_{S_2} (u_i - \bar{u}_i)\,\delta g_i\,\mathrm{d}S_2 = 0, \qquad (1)$$

where $W(\varepsilon_{ij})$ is the strain energy density, $\hat{T}(p_i)$ the kinetic energy density, B_i the body forces, and g_i the surface tractions. The overbarred quantities $\bar{g}_i$ and $\bar{u}_i$ are the prescribed surface tractions and displacements acting over surfaces S_1 and S_2 respectively. Equation (1) is the starting point for the derivation of an approximate model of a cracked beam including shear deformations.

2.2. KINEMATIC ASSUMPTIONS

The reduction of the 3-D elastic problem to a simplified, beam-like unidimensional system is achieved by the imposition of convenient kinematic assumptions to the stress, strain and displacement fields. The stress and strain fields in the axial direction are assumed to be modified, in reference [6], as

$$\sigma_{xx}(x, z, t) = (-z + f(x, z))\,T(x, t), \qquad (2)$$

and

$$\varepsilon_{xx}(x, z, t) = (-z + f(x, z))\,S(x, t), \qquad (3)$$

where $T(x, t)$ and $S(x, t)$ are the unknown unidimensional stress and strain fields respectively. The function $f(x, z)$ is the so-called crack disturbance function and is chosen such that the above fields can represent the effect of stress concentration in the vicinity of the cracked region. For a cracked beam with the geometry depicted in Figure 1, that

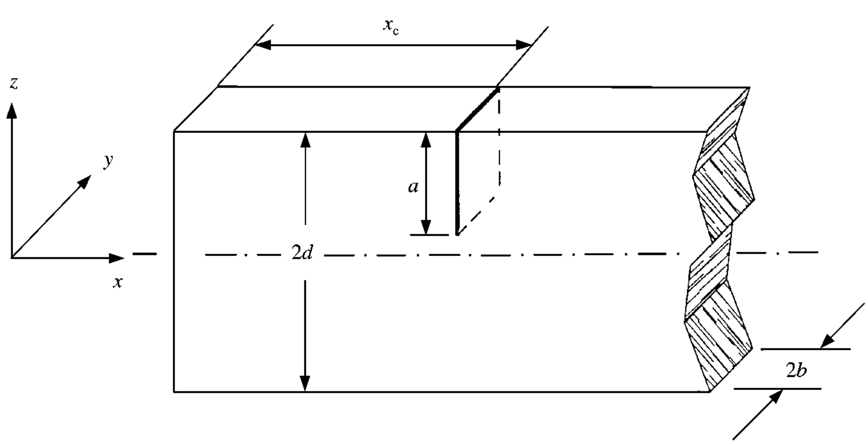

Figure 1. Typical geometry of a cracked beam.

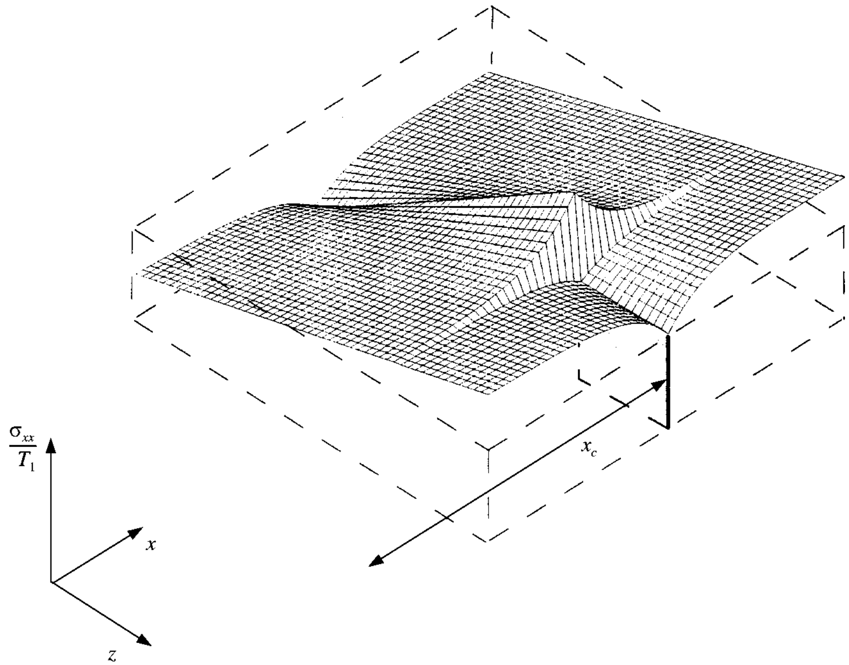

Figure 2. Crack function for normal stress disturbance f_1.

function is defined as

$$f(x,z) = \left[z - m\left(z + \frac{a}{2}\right)H(d - a - z)\right]\exp\left(-\alpha\frac{|x - x_c|}{d}\right). \tag{4}$$

The localized nature of the stress concentration effect is represented by the exponential decay function $H(z)$ is the unit step function and is included to represent the zero stress condition on the crack faces. The parameter m is the slope of the linear stress distribution at the cracked section and is calculated by assuming continuity of the bending moment at the crack site, as discussed later. The parameter α is the rate of stress decay in the x direction and has to be estimated from experimental results [7] or from a detailed finite element model [6]. Figure 2 shows a plot of the crack function in the vicinity of the crack tip. In the absence of a crack the stress and strain fields will reduce to a linear function in the z direction (along the cross-section), which is the usual mathematical representation of the assumption that cross-sections remain plane and perpendicular to the neutral line after deformation, typical of the Euler–Bernoulli (E–B) beam model.

In addition to the changes in the axial stress and strain, the displacement field is also modified to account for the local change in the neutral axis caused by the presence of a crack on one edge only. The assumption is

$$u'_x(x,z,t) = (-z + g(x,z))w''(x,t). \tag{5}$$

The function $g(x, z)$ modified the first derivative of the axial displacement, and has a form similar to the stress/strain crack functions

$$g(x,z) = \left[z - \left(z + \frac{a}{2}\right)H\left(d - a - z\right)\exp\left(-2\beta\frac{|x - x_c|}{d}\right)\right]. \tag{6}$$

Here the decay parameter β is obtained by a curve fitting of $g(x, z)$ to the function $1/\sqrt{r}$, where r is the distance to the crack tip, in order to reproduce the effect of the stress/strain singularity, well known from linear elastic fracture mechanics results. In summary, the kinematic assumptions are

$$u'_x = (-z + g(x, z))w''(x, t), \qquad u_y = 0, \qquad u_z = w(x, t),$$
$$\varepsilon_{xx}(x, z, t) = (-z + f(x, z))S(x, t), \qquad \varepsilon_{yy} = \varepsilon_{zz} = -\nu\varepsilon_{xx}, \qquad \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{xz} = 0, \quad (7)$$
$$\sigma_{xx}(x, z, t) = (-z + f(x, z))\,T(x, t), \qquad \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yz} = \sigma_{xz} = 0,$$
$$p_z = P_z(x, t), \qquad p_x = p_y = 0, \qquad B_x = B_y = B_z = 0.$$

2.3. EQUATIONS OF MOTION

Introducing equation (7) in the functional in equation (1) and considering the variations δT, δS, δP_z, δw are independent and arbitrary, Shen and Pierre derived the differential equation for the free vibrations of a cracked beam

$$E(I_0 - I_1 - I_6 + I_7)Q_1 w^{(iv)} + E[2(I_0 - I_1 - I_6 + I_7)Q'_1 + (2I_{10} + I_8 - I'_6 - 2I'_1)Q_1]\,w'''$$
$$+ E[(I_0 - I_1 - I_6 + I_7)Q''_1 + (2I_{10} + I_8 - I'_6 - 2I'_1)Q'_1 + (I'_{10} - I''_1)\;Q_1]\,w'' + \rho A\ddot{w} = 0, \quad (8)$$

where the coefficients are defined in Appendix A. Details of this derivation may be found in reference [6].

3. DISCUSSION

3.1. SELF-ADJOINTNESS OF THE STIFFNESS OPERATOR

The eigenvalue problem associated with equation (33) may be rewritten in simplified notation as

$$p_4(x)W^{(iv)}(x) + p_3(x)W'''(x) + p_2(x)W''(x) = \Omega^2\frac{\rho A}{E}W. \quad (9)$$

The left-hand side of equation (9) is the stiffness operator $\mathscr{L}$ applied to the eigenvectors W of the problem. A necessary condition for a fourth order differential operator to be self-adjoint is that it can be written in the general form

$$\mathscr{L}W = (q_2(x)W''(x))'' + (q_1(x)\,W'(x))' + q_0(x)W. \quad (10)$$

In equation (9) there are no terms in W' or W, which implies

$$q'_1 = q_0 = 0, \qquad q_2 = p_4,$$

and the following relations should be verified:

$$p_3 = 2p'_4, \qquad p''_4 - p_2 = \text{const.} \quad (11)$$

Both expressions in equation (11) hold only when $g(x, z)$ is identically zero. Therefore, the stiffness operator is in general not self-adjoint and may produce a stiffness matrix that is not symmetric and consequently yields complex eigenvalues and eigenvectors, which are inconsistent with the physics of the problem. It is believed that this is caused by the use of the following approximation in the derivation:

$$\delta u_x(x,z,t) = (-z + g(x,z))\,\delta w'(x,t), \tag{12}$$

which would be true if

$$u_x(x,z,t) = (-z + g(x,z))w'(x,t), \tag{13}$$

yielding

$$u'_x(x,z,t) = (-z + g(x,z))w''(x,t) + g'(x,z)\,w'(x,t), \tag{14}$$

which in turn is inconsistent with equation (5). To overcome this limitation, a slight modification in the kinematic assumptions is proposed here as

$$u_x(x,z,t) = (-z + g(x,z))\,w'(x,t), \tag{15}$$

with all the corresponding derivatives and variations. Rederiving the equations of motion, we get

$$\begin{aligned}
&E(I_0 - I_1 - I_6 + I_7)Q_1 w^{(iv)} + E[(I_0 - I_1 - I_6 + I_7)(2Q'_1 + Q_2) + (2I_{10} + I_8 - I'_6 \\
&\quad - 2I'_1)Q_1]\,w''' + E[(I_0 - I_1 - I_6 + I_7)(Q''_1 + 2Q'_2) + (2I_{10} + I_8 - I'_6 - 2I'_1)(Q'_1 + Q_2) \\
&\quad + (I'_{10} - I''_1)Q_1]w'' + E[(I_0 - I_1 - I_6 + I_7)Q''_2 + (2I_{10} + I_8 - I'_6 - 2I'_1)Q'_2 + (I'_{10} - I''_1) \\
&\quad \times Q_2]\,w' + \rho A\ddot{w} = 0,
\end{aligned} \tag{16}$$

which, with the variationally consistent boundary conditions

$$E(I_0 - I_1 - I_6 + I_7)(Q_1 w'' + Q_2 w') = 0 \quad \text{or} \quad w' = 0. \tag{17}$$

and

$$\begin{aligned}
&E(I_0 - I_1 - I_6 + I_7)(Q_1 w''' + (Q'_1 + Q_2)\,w'' + Q'_2 w') \\
&\quad + E(I_{10} - I'_1)(Q_1 w'' + Q_2 w') = 0 \quad \text{or} \quad w = 0.
\end{aligned} \tag{18}$$

is now self-adjoint, as shown in reference [9].

3.2. EFFECTS OF $g(x,z)$ ON NUMERICAL ACCURACY

The inclusion of the displacement disturbance function proposed in equation (6) can also lead to numerical inconveniences. Due to the steep nature of $g(x,z)$, very small integration steps are needed to capture its effect on the structural matrices. A series of simulations with and without the inclusion of $g(x,z)$ was investigated, using the same cantilever beam example in reference [5]. Natural frequencies of the beam with an open crack were computed for different crack depths and positions. The calculation of the terms in the stiffness matrix were performed using the MATLAB(TM) high order quadrature function QUAD8, with relative error set to its default value, 1e-3, recommended for most cases. This function is basically an adaptive integration routine, which automatically refines the integration step based on precision requirements. The same calculations were then repeated with a more refined precision, now set at 1e-7, and the results for different crack locations are summarized in Figures 3–5. It is apparent that with the new precision settings the results with and without g are identical for any practical purposes. More importantly, they are in very good agreement with the results obtained with the lower precision settings when the displacement functions are not included, which indicates that the only effect of the inclusion of function $g(x,z)$ is the need for a more refined integration procedure, with no apparent

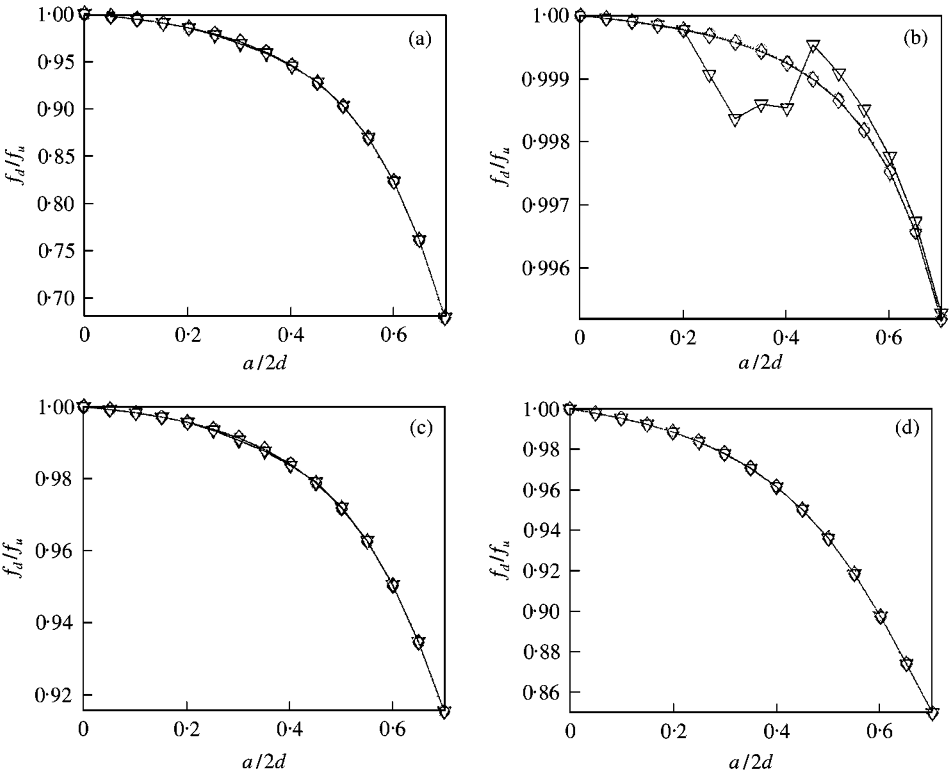

Figure 3. Influence of displacement disturbance function g on the natural frequencies of a simply supported slender beam, $L/2d = 24{\cdot}65$, $x_c/L = 0{\cdot}2$. (a) Natural frequency #1, (b) #2, (c) #3, (d) #4, ∇, with g, default; ○, w/o g, default; ◇, with g, refined.

advantages to the quality of the model. In fact, the more refined precision in the presented simulations increases the integration time by a factor of two or more, which implies an extremely high price to pay to avoid the occasional numerical imprecisions depicted in Figures 3–5. In short, inclusion of $g(x, z)$ as defined introduces very small contributions to the model at an unreasonably high numerical price.

4. CONCLUSION

The continuous model for the vibrations of a beam with a single-edge crack developed by Shen and Pierre was reviewed. A modification to the derivation was proposed in order to overcome the lack of self-adjointness of the resulting stiffness operator and a new version of the equation of motion was presented.

The effect of the crack disturbance function on the natural frequencies of a cantilever cracked beam was investigated. The results were dependent on the refinement of the integration routine, which might require extensive numerical tests before accurate values are obtained. In addition, it was shown that results with and without the inclusion of $g(x, z)$ are virtually identical when the precision of numerical integration is adequate. This ultimately means that the displacement disturbance function can be omitted from the proposed model without compromising the numerical accuracy, providing computational

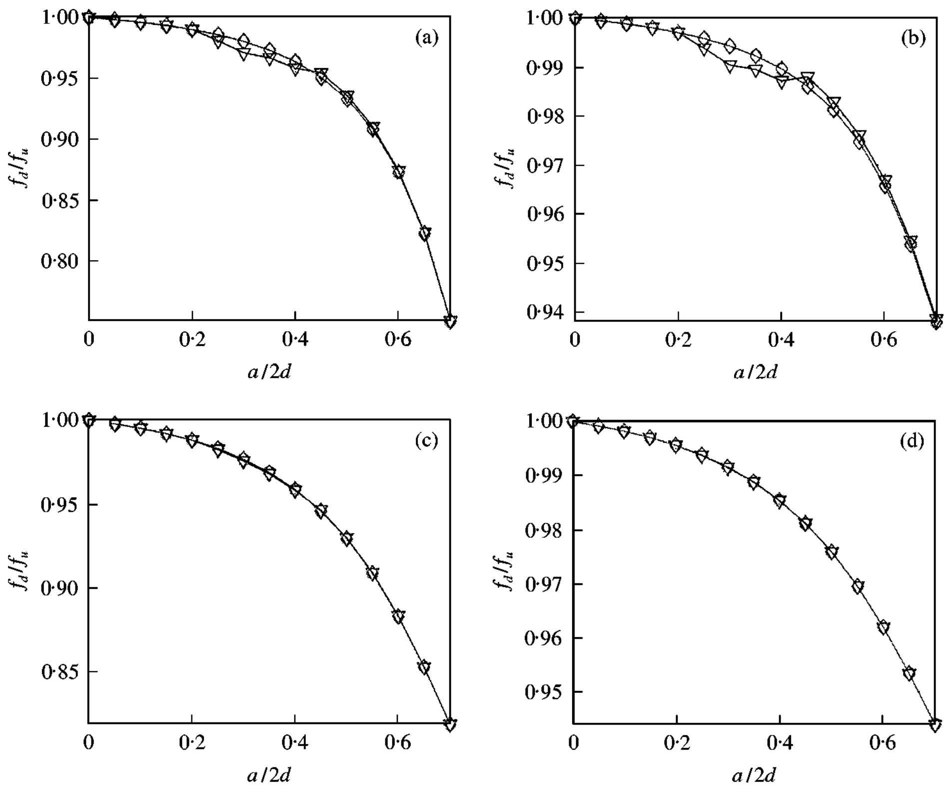

Figure 4. Influence of displacement disturbance function g on the natural frequencies of a simply supported slender beam, $L/2d = 24{\cdot}65$, $x_c/L = 0{\cdot}3$. (a) Natural frequency #1, (b) #2, (c) #3, (d) #4, ∇, with g, default; ○, w/o g, default; ◇, with g, refined.

savings, which is especially important when using the model in iterative model-based crack detection methods.

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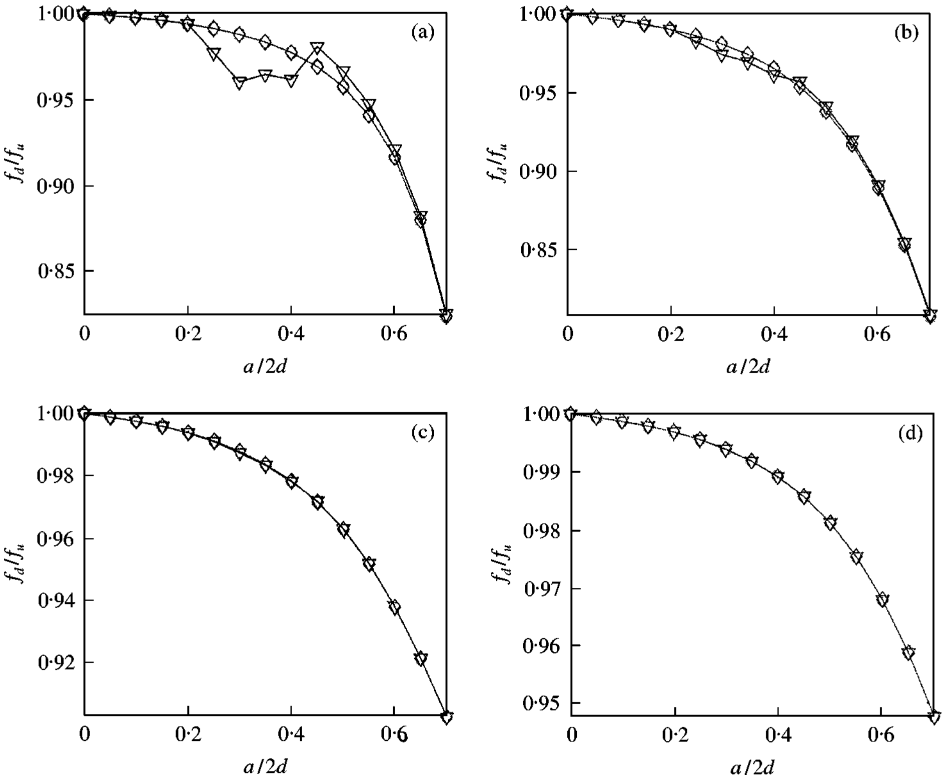

Figure 5. Influence of displacement disturbance function g on the natural frequencies of a simply supported slender beam, $L/2d = 24{\cdot}65$, $x_c/L = 0{\cdot}4$. (a) Natural frequency #1, (b) #2, (c) #3, (d) #4, ∇, with g, default; ○, w/o g, default; ◇, with g, refined.

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APPENDIX A: COEFFICIENT FUNCTIONS IN THE EQUATIONS OF MOTION

The coefficients in equations (8) and (16) are expressed as

$$Q_1(x) = \frac{(I_0 - I_1 - I_6 + I_7)}{(I_0 - 2I_1 + I_2)}, \qquad Q_2(x) = \frac{(I_8 - I_6')}{(I_0 - 2I_1 + I_2)}.$$

The generalized inertia integrals are obtained via integration over the cross-section of different product combinations involving the crack perturbation functions, defined as

$$I_0 = \int_A z^2\,\mathrm{d}A, \qquad I_1 = \int_A zf\,\mathrm{d}A, \qquad I_2 = \int_A f^2\,\mathrm{d}A, \qquad I_6 = \int_A zg\,\mathrm{d}A, \qquad I_7 = \int_A fg\,\mathrm{d}A,$$

$$I_8 = \int_A fg'\,\mathrm{d}A, \qquad I_{10} = \int_A f'g\,\mathrm{d}A.$$