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AN EXACT SERIES SOLUTION FOR CALCULATING THE EIGENFREQUENCIES OF ORTHOTROPIC PLATES WITH COMPLETELY FREE BOUNDARY†

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An analysis for solving boundary value problems in structural mechanics which was proposed by Wang and Lin (1996 *Journal of Sound and Vibration* **196**, 285–293; 1999 *Journal of Applied Mechanics* **66**, 380–387) [1, 2] has been extended to the calculation of the eigenfrequencies of an orthotropic plate under all free boundary conditions. The convergence of the series solution is assured and the procedure leads to pointwise exact solutions. The calculated eigenfrequencies have been verified by a different approach and indicate that the present method is simple and effective.

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1. INTRODUCTION

Vibrations of plates have been frequently studied. While Chladni [3] pioneered experimental investigations, Navier and Levy obtained analytical solutions for special boundary conditions [4]. Unfortunately, there exists no closed-form solution for the case of a rectangular plate with all free boundary conditions, but several approximate methods have been proposed. Warburton [5] used characteristic beam vibration functions in Rayleigh's method [6] to obtain a useful and simple approximate expression for the natural frequencies of vibration of thin, isotropic plates. His work was extended by Hearmon [7] and applied to specially orthotropic plates, and by Dickinson [8], who included the effect of uniform direct, in-plane loads. Warburton's expression and its generalizations, together with a table permit the straightforward calculation of the natural frequencies of plates having any combination of free, clamped, or simply supported edges. However, should one or more free edges exist then the accuracy of the frequencies can be significantly diminished. Kim and Dickinson [9] provided an improved approximate expression, where they use Rayleigh's method but in combination with the minimum potential energy theorem. Iguchi [10] gave solutions of an isotropic rectangular plate. However, the determination is limited

†Dipl.-Ing. S. Hurlebaus and Professor L. Gaul dedicate this article to their esteemed co-author, teacher and friend, Professor J. T.-S. Wang, who passed away on 24 September 2000.

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to square plates only. Rajalingham *et al.* [11] reduced the plate vibration equation to simultaneous ordinary differential equations. However, only the results for plate characteristic function parameters for clamped rectangular isotropic plates are given. Liew and Lam [12] determined the vibration analysis of point-supported rectangular plates, based on the Rayleigh–Ritz approach and a Gram–Schmidt set of orthogonal plate functions. An approximate method was proposed, which is applicable to a wide range of arbitrarily distributed point-supported plate problems with any combination of classical boundary conditions. Leissa [13] presented comprehensive and accurate analytical results for the free vibration of rectangular plates. Leissa applied the Ritz method [14] and compared the results with the method of Warburton [5], although his work is limited to isotropic plates. Leissa's work was extended by Deobald and Gibson [15], who also applied the Rayleigh–Ritz method to orthotropic plates. Gorman [16, 17] solved the differential equation for isotropic as well as orthotropic plates using a superposition method, which allows the boundary conditions to be met with desired accuracy.

Wang and Lin [2] presented a systematic analysis for solving boundary value problems in structural mechanics, where a weighted residual form of the differential equations is used with sinusoidal weighting functions. An overview of techniques of weighted residuals is given in reference [18].

The approach of the present paper generalizes the proposed method by Wang and Lin [2] by treating the vibration of orthotropic plates. The exact series solution is derived and the results are applied to an orthotropic plate with specific material and geometrical properties. A discussion of the results is given.

2. EXACT SERIES SOLUTION

Consider the classical Kirchhoff plate theory. The governing equation of motion for an unloaded plate with thickness h is

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2\,\frac{\partial^2 M_{xy}}{\partial x\,\partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + \rho h\,\frac{\partial^2 w}{\partial t^2} = 0, \tag{1}$$

where M_{xx}, M_{yy} and M_{xy} are stress couples, w the out-of-plane displacement and ρ the plate density. Weighting with $\cos\alpha_m x \cos\gamma_n y$, where $\alpha_m = m\pi/a$ and $\gamma_n = n\pi/b$ and integration with respect to the plate area $a \times b$ gives

$$\int_0^a \int_0^b \left(\frac{\partial^2 M_{xx}}{\partial x^2} + 2\,\frac{\partial^2 M_{xy}}{\partial x\,\partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + \rho h\,\frac{\partial^2 w}{\partial t^2}\right)\cos\alpha_m x \cos\gamma_n y\,\mathrm{d}y\,\mathrm{d}x = 0, \tag{2}$$

where the first term of the integral can be integrated by parts as

$$\int_0^a \int_0^b \frac{\partial^2 M_{xx}}{\partial x^2}\cos\alpha_m x \cos\gamma_n y\,\mathrm{d}y\,\mathrm{d}x = -\,\alpha_m^2 \int_0^b \int_0^a M_{xx}\cos\alpha_m x \cos\gamma_n y\,\mathrm{d}x\,\mathrm{d}y$$
$$+ \int_0^b \cos\gamma_n y\left[(-1)^m \left.\frac{\partial M_{xx}}{\partial x}\right|_{x=a} - \left.\frac{\partial M_{xx}}{\partial x}\right|_{x=0}\right]\mathrm{d}y \tag{3}$$

and the third term of the integral as

$$\int_0^a \int_0^b \frac{\partial^2 M_{yy}}{\partial y^2} \cos \alpha_m x \cos \gamma_n y \,\mathrm{d}y \,\mathrm{d}x = -\gamma_n^2 \int_0^b \int_0^a M_{yy} \cos \alpha_m x \cos \gamma_n y \,\mathrm{d}x \,\mathrm{d}y + \int_0^a \cos \alpha_m x \left[(-1)^n \left.\frac{\partial M_{yy}}{\partial y}\right|_{y=b} - \left.\frac{\partial M_{yy}}{\partial y}\right|_{y=0} \right] \mathrm{d}x. \quad (4)$$

The second term is split into two parts. The first part considers the partial derivative with respect to x first

$$\int_0^a \int_0^b \frac{\partial^2 M_{xy}}{\partial x \,\partial y} \cos \alpha_m x \cos \gamma_n y \,\mathrm{d}y \,\mathrm{d}x = \gamma_n \alpha_m \int_0^a \int_0^b M_{xy} \sin \alpha_m x \sin \gamma_n y \,\mathrm{d}x \,\mathrm{d}y + \int_0^a \cos \alpha_m x \left[(-1)^n \left.\frac{\partial M_{xy}}{\partial x}\right|_{y=b} - \left.\frac{\partial M_{xy}}{\partial x}\right|_{y=0} \right] \mathrm{d}x + \gamma_n \int_0^b \sin \gamma_n y \left[(-1)^n M_{xy}|_{x=a} - M_{xy}|_{x=0} \right] \mathrm{d}y, \quad (5)$$

while the second part considers the partial derivative with respect to y first

$$\int_0^a \int_0^b \frac{\partial^2 M_{xy}}{\partial x \,\partial y} \cos \alpha_m x \cos \gamma_n y \,\mathrm{d}y \,\mathrm{d}x = \gamma_n \alpha_m \int_0^a \int_0^b M_{xy} \sin \alpha_m x \sin \gamma_n y \,\mathrm{d}x \,\mathrm{d}y + \int_0^b \cos \gamma_n y \left[(-1)^m \left.\frac{\partial M_{xy}}{\partial y}\right|_{x=a} - \left.\frac{\partial M_{xy}}{\partial y}\right|_{x=0} \right] \mathrm{d}y + \alpha_m \int_0^a \sin \alpha_m x \left[(-1)^n M_{xy}|_{y=b} - M_{xy}|_{y=0} \right] \mathrm{d}x. \quad (6)$$

The boundary conditions of a free plate are

$$Q_x = \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} = 0 \quad \text{at } x = 0 \quad \text{and} \quad a, \quad (7)$$

$$Q_y = \frac{\partial M_{yy}}{\partial y} + \frac{\partial M_{xy}}{\partial x} = 0 \quad \text{at } y = 0 \quad \text{and} \quad b, \quad (8)$$

$$M_{xy} = 0 \quad \text{at } x = 0 \quad \text{and} \quad a, \quad (9)$$

$$M_{xy} = 0 \quad \text{at } y = 0 \quad \text{and} \quad b \quad (10)$$

and the separation of variables $w(x, y, t) = W(x, y) \exp(\mathrm{i}\omega t)$ for steady state vibration with circular frequency ω leads to

$$2\gamma_n \alpha_m \int_0^a \int_0^b M_{xy} \sin \alpha_m x \sin \gamma_n y \,\mathrm{d}y \,\mathrm{d}x + \rho h \omega^2 \int_0^a \int_0^b W \cos \alpha_m x \cos \gamma_n y \,\mathrm{d}y \,\mathrm{d}x - \alpha_m^2 \int_0^b \int_0^a M_{xx} \cos \alpha_m x \cos \gamma_n y \,\mathrm{d}x \,\mathrm{d}y - \gamma_n^2 \int_0^b \int_0^a M_{yy} \cos \alpha_m x \cos \gamma_n y \,\mathrm{d}x \,\mathrm{d}y = 0. \quad (11)$$

Using the constitutive equations for orthotropic plates, where the principal directions of orthotropy are parallel to the plate edges, leads to the moment curvature equations

$$M_{yy} = -D_y \left\{ \frac{\partial^2 W}{\partial y^2} + v_x \frac{\partial^2 W}{\partial x^2} \right\}, \tag{12}$$

$$M_{xx} = -D_x \left\{ \frac{\partial^2 W}{\partial x^2} + v_y \frac{\partial^2 W}{\partial y^2} \right\}, \tag{13}$$

$$M_{xy} = -2D_{xy} \frac{\partial^2 W}{\partial x \, \partial y}. \tag{14}$$

Equation (11) is given in terms of the deflection shape W:

$$\begin{aligned} 0 = {} & (D_x \alpha_m^2 + v_x D_y \gamma_n^2) \int_0^b \int_0^a \frac{\partial^2 W}{\partial x^2} \cos \alpha_m x \cos \gamma_n y \, \mathrm{d}x \, \mathrm{d}y \\ & + (D_y \gamma_n^2 + v_y D_x \alpha_m^2) \int_0^b \int_0^a \frac{\partial^2 W}{\partial y^2} \cos \alpha_m x \cos \gamma_n y \, \mathrm{d}x \, \mathrm{d}y \\ & - 4\alpha_m \gamma_n D_{xy} \int_0^a \int_0^b \frac{\partial^2 W}{\partial x \, \partial y} \sin \alpha_m x \sin \gamma_n y \, \mathrm{d}y \, \mathrm{d}x \\ & + \int_0^a \int_0^b \rho h \omega^2 \cos \alpha_m x \cos \gamma_n y \, \mathrm{d}y \, \mathrm{d}x, \end{aligned} \tag{15}$$

where the plate stiffnesses are

$$D_x = \frac{E_x h^3}{12(1 - v_x v_y)}, \quad D_y = \frac{E_y h^3}{12(1 - v_x v_y)} \quad \text{and} \quad D_{xy} = \frac{G_{xy} h^3}{12}, \tag{16}$$

in which E_x and E_y are Young's moduli in the x and y directions respectively, G_{xy} is the shear modulus, and v_x and v_y are the Poisson ratios. Substituting equations (12–14) in conjunction with equations (3–6) into equation (15) leads to

$$\begin{aligned} 0 = {} & -(D_x \alpha_m^2 + v_x D_y \gamma_n^2) \int_0^b \cos \gamma_n y \left[(-1)^m \left. \frac{\partial W}{\partial x} \right|_{x=a} - \left. \frac{\partial W}{\partial x} \right|_{x=0} \right] \mathrm{d}y \\ & + (D_x \alpha_m^4 + 2H \alpha_m^2 \gamma_n^2 + D_y \gamma_n^4 - \rho h \omega^2) \int_0^b \int_0^a W \cos \alpha_m x \cos \gamma_n y \, \mathrm{d}x \, \mathrm{d}y \\ & - (D_y \gamma_n^2 + v_y D_x \alpha_m^2) \int_0^a \cos \alpha_m x \left[(-1)^n \left. \frac{\partial W}{\partial y} \right|_{y=b} - \left. \frac{\partial W}{\partial y} \right|_{y=0} \right] \mathrm{d}x, \end{aligned} \tag{17}$$

where $H = v_x D_y + 2D_{xy}$. Defining the partial derivatives of the displacement at the boundary

$$\frac{\partial W}{\partial x} = \sum_{n=0}^{\infty} W'_{an} \cos \gamma_n y \quad \text{along } x = a, \tag{18}$$

$$\frac{\partial W}{\partial x} = \sum_{n=0}^{\infty} W'_{0n} \cos \gamma_n y \quad \text{along } x = 0, \tag{19}$$

$$\frac{\partial W}{\partial y} = \sum_{m=0}^{\infty} W^{\star}_{mb} \cos \alpha_m x \quad \text{along } y = b, \tag{20}$$

$$\frac{\partial W}{\partial y} = \sum_{m=0}^{\infty} W^{\star}_{m0} \cos \alpha_m x \quad \text{along } y = 0, \tag{21}$$

where prime and asterix denote those coefficients associated to the partial derivative of W with respect to x and y respectively. Assuming the general solution in the following form so that rigid-body motions are included

$$W = \frac{F_{00}}{4} + \sum_{m=1}^{\infty} \frac{F_{m0}}{2} \cos \alpha_m x + \sum_{n=1}^{\infty} \frac{F_{0n}}{2} \cos \gamma_n y + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} F_{mn} \cos \alpha_m x \cos \gamma_n y \tag{22}$$

and computing the double integral

$$\int_0^b \int_0^a W \cos \alpha_m x \cos \gamma_n y \, \mathrm{d}x \, \mathrm{d}y = \int_0^b \int_0^a \left(\frac{F_{00}}{4} + \sum_{m=1}^{\infty} \frac{F_{m0}}{2} \cos \alpha_m x + \sum_{n=1}^{\infty} \frac{F_{0n}}{2} \cos \gamma_n y + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} F_{mn} \cos \alpha_m x \cos \gamma_n y \right) \cos \alpha_m x \cos \gamma_n y \, \mathrm{d}x \, \mathrm{d}y$$
$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} F_{mn} \frac{ab}{4} \tag{23}$$

to be used in equation (17) gives

$$F_{mn} = \frac{4}{ab(D_x \alpha_m^4 + 2H \alpha_m^2 \gamma_n^2 + D_y \gamma_n^4 - \rho h \omega^2)} \times \left((D_x \alpha_m^2 + v_x D_y \gamma_n^2) \int_0^b \cos \gamma_n y \left[(-1)^m W'_{an} \cos \gamma_n y - W'_{0n} \cos \gamma_n y \right] \mathrm{d}y + (D_y \gamma_n^2 + v_y D_x \alpha_m^2) \int_0^a \cos \alpha_m x \left[(-1)^n W^{\star}_{mb} \cos \alpha_m x - W^{\star}_{m0} \cos \alpha_m x \right] \mathrm{d}x \right). \tag{24}$$

Computing the integrals in equation (24) leads to

$$F_{00} = 0, \tag{25}$$

$$F_{m0} = \frac{4 D_x \alpha_m^2}{ab(D_x \alpha_m^4 - \rho h \omega^2)} \left(b \left[(-1)^m W'_{a0} - W'_{00} \right] + v_y \frac{a}{2} \left[W^{\star}_{mb} - W^{\star}_{m0} \right] \right), \tag{26}$$

$$F_{0n} = \frac{4 D_y \gamma_n^2}{ab(D_y \gamma_n^4 - \rho h \omega^2)} \left(v_x \frac{b}{2} \left[W'_{an} - W'_{0n} \right] + a \left[(-1)^n W^{\star}_{0b} - W^{\star}_{00} \right] \right), \tag{27}$$

$$F_{mn} = \frac{2}{(D_x\alpha_m^4 + 2H\alpha_m^2\gamma_n^2 + D_y\gamma_n^4 - \rho h\omega^2)} \times \left((D_x\,\alpha_m^2 + \nu_x D_y\gamma_n^2)\frac{1}{a}[(-1)^m W'_{an} - W'_{0n}] + (D_y\gamma_n^2 + \nu_y D_x\alpha_m^2)\frac{1}{b}[(-1)^n W^\star_{mb} - W^\star_{m0}] \right) \tag{28}$$

and finally the displacement W can be written as

$$\begin{aligned} W = {} & \frac{2}{a}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(D_x\alpha_m^2 + \nu_x D_y\gamma_n^2)\cos\alpha_m x\cos\gamma_n y}{D_x\alpha_m^4 + 2H\alpha_m^2\gamma_n^2 + D_y\gamma_n^4 - \rho h\omega^2}[(-1)^m W'_{an} - W'_{0n}] \\ & + \frac{2}{b}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(D_y\gamma_n^2 + \nu_y D_x\alpha_m^2)\cos\alpha_m x\cos\gamma_n y}{D_x\alpha_m^4 + 2H\alpha_m^2\gamma_n^2 + D_y\gamma_n^4 - \rho h\omega^2}[(-1)^m W^\star_{mb} - W^\star_{m0}] \\ & + \frac{2}{a}\sum_{m=1}^{\infty}\frac{D_x\alpha_m^2\cos\alpha_m x}{D_x\alpha_m^4 - \rho h\omega^2}[(-1)^m W'_{a0} - W'_{00}] \\ & + \frac{1}{b}\sum_{m=1}^{\infty}\frac{D_x\alpha_m^2\cos\alpha_m x}{D_x\alpha_m^4 - \rho h\omega^2}\nu_y[W^\star_{mb} - W^\star_{m0}] \\ & + \frac{1}{a}\sum_{n=1}^{\infty}\frac{D_y\gamma_n^2\cos\gamma_n y}{D_y\gamma_n^4 - \rho h\omega^2}\nu_x[W'_{an} - W'_{0n}] \\ & + \frac{2}{b}\sum_{n=1}^{\infty}\frac{D_y\gamma_n^2\cos\gamma_n y}{D_y\gamma_n^4 - \rho h\omega^2}[(-1)^n W^\star_{0b} - W^\star_{00}]. \end{aligned} \tag{29}$$

Using the symmetry condition

$$\left.\frac{\partial W}{\partial y}\right|_{y=b} = -c\left.\frac{\partial W}{\partial y}\right|_{y=0} \tag{30}$$

and

$$\left.\frac{\partial W}{\partial x}\right|_{x=a} = -d\left.\frac{\partial W}{\partial x}\right|_{x=0}, \tag{31}$$

where $c = 1$ and -1 correspond to the symmetric and antisymmetric vibration modes about $y = b/2$, respectively, and where $d = 1$ and -1 correspond to the symmetric and antisymmetric modes about $x = a/2$, respectively, the partial derivative of W with respect to x is obtained as

$$\begin{aligned} \frac{\partial W}{\partial x} = {} & -\frac{2}{a}\sum_{m=1}^{\infty}\frac{D_x\alpha_m^3\sin\alpha_m x}{D_x\alpha_m^4 - \rho h\omega^2}[(-1)^m + d]\,W'_{00} \\ & -\frac{1}{b}\sum_{m=1}^{\infty}\frac{D_x\alpha_m^3\sin\alpha_m x}{D_x\alpha_m^4 - \rho h\omega^2}\nu_y[(-1)^0 + c]\,W^\star_{m0} \end{aligned}$$

$$-\frac{2}{a}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(D_x\alpha_m^2+v_xD_y\gamma_n^2)\,\alpha_m\sin\alpha_m x\cos\gamma_n y}{D_x\alpha_m^4+2H\alpha_m^2\gamma_n^2+D_y\gamma_n^4-\rho h\omega^2}[(-1)^m+d]\,W'_{0n}$$
$$-\frac{2}{b}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(D_y\gamma_n^2+v_yD_x\alpha_m^2)\,\alpha_m\sin\alpha_m x\cos\gamma_n y}{D_x\alpha_m^4+2H\alpha_m^2\gamma_n^2+D_y\gamma_n^4-\rho h\omega^2}[(-1)^n+c]\,W^{\star}_{m0} \tag{32}$$

or

$$\frac{\partial W}{\partial x}=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}[-d-(-1)^m]\frac{2}{m\pi}\sin\alpha_m x$$
$$\times\left[1-\frac{2H\alpha_m^2\gamma_n^2+D_y\gamma_n^4-\rho h\omega^2-v_xD_y\gamma_n^2\alpha_m^2}{D_x\alpha_m^4+2H\alpha_m^2\gamma_n^2+D_y\gamma_n^4-\rho h\omega^2}\right]W'_{0n}\cos\gamma_n y$$
$$-\frac{2}{b}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(D_y\gamma_n^2+v_yD_x\alpha_m^2)\alpha_m\sin\alpha_m x\cos\gamma_n y}{D_x\alpha_m^4+2H\alpha_m^2\gamma_n^2+D_y\gamma_n^4-\rho h\omega^2}[(-1)^n+c]\,W^{\star}_{m0}$$
$$-\frac{1}{b}\sum_{m=1}^{\infty}\frac{D_x\alpha_m^3\sin\alpha_m x}{D_x\alpha_m^4-\rho h\omega^2}v_y\,[(-1)^0+c]\,W^{\star}_{m0}. \tag{33}$$

For the case $d=-1$ and adopting the identity [2]

$$1=\sum_{m=1}^{\infty}[1-(-1)^m]\frac{2}{m\pi}\sin\alpha_m x \tag{34}$$

gives

$$\frac{\partial W}{\partial x}=\sum_{n=0}^{\infty}W'_{0n}\cos\gamma_n y$$
$$-\frac{1}{b}\sum_{m=1}^{\infty}\frac{D_x\alpha_m^3\sin\alpha_m x}{D_x\alpha_m^4-\rho h\omega^2}v_y\,[(-1)^0+c]\,W^{\star}_{m0}$$
$$-\frac{2}{b}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(D_y\gamma_n^2+v_yD_x\alpha_m^2)\alpha_m\sin\alpha_m x\cos\gamma_n y}{D_x\alpha_m^4+2H\alpha_m^2\gamma_n^2+D_y\gamma_n^4-\rho h\omega^2}[(-1)^n+c]\,W^{\star}_{m0}$$
$$+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}[d+(-1)^m]\frac{2}{m\pi}\frac{(2H-v_xD_y)\alpha_m^2\gamma_n^2+D_y\gamma_n^4-\rho h\omega^2}{D_x\alpha_m^4+2H\alpha_m^2\gamma_n^2+D_y\gamma_n^4-\rho h\omega^2}$$
$$\times W'_{0n}\cos\gamma_n y\sin\alpha_m x, \tag{35}$$

while for the case $d=1$ with the relations [19, 2]

$$\sum_{m=1}^{\infty}\frac{\sin\alpha_m x}{\alpha_m}=\frac{a-x}{2}\quad\text{and}\quad 1=\sum_{m=1}^{\infty}[1-(-1)^m]\frac{2}{m\pi}\sin\alpha_m x \tag{36}$$

gives

$$\frac{\partial W}{\partial x}=\frac{2}{a}\sum_{n=0}^{\infty}[-2d]\frac{a-x}{2}\,W'_{0n}\cos\gamma_n y$$
$$+\sum_{n=0}^{\infty}W'_{0n}\cos\gamma_n y$$

$$
\begin{aligned}
&-\frac{2}{b}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(D_y\gamma_n^2+v_yD_x\alpha_m^2)\alpha_m\sin\alpha_m x\cos\gamma_n y}{D_x\alpha_m^4+2H\alpha_m^2\gamma_n^2+D_y\gamma_n^4-\rho h\omega^2}\,[(-1)^n+c]\,W_{m0}^{\star}\\
&-\frac{1}{b}\sum_{m=1}^{\infty}\frac{D_x\alpha_m^3\sin\alpha_m x}{D_x\alpha_m^4-\rho h\omega^2}\,v_y\,[(-1)^0+c]\,W_{m0}^{\star}\\
&+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}[d+(-1)^m]\,\frac{2}{m\pi}\,\frac{(2H-v_xD_y)\alpha_m^2\gamma_n^2+D_y\gamma_n^4-\rho h\omega^2}{D_x\alpha_m^4+2H\alpha_m^2\gamma_n^2+D_y\gamma_n^4-\rho h\omega^2}\\
&\times W_{0n}'\cos\gamma_n y\sin\alpha_m x.
\end{aligned}\tag{37}
$$

The second derivative with respect to x of both cases $d=1$ and -1 can be expressed as

$$
\begin{aligned}
\frac{\partial^2 W}{\partial x^2}=&\frac{2}{a}\sum_{n=0}^{\infty}\left[e+\sum_{m=1}^{\infty}[d+(-1)^m]\right.\\
&\left.\times\frac{(2H-v_xD_y)\alpha_m^2\gamma_n^2+D_y\gamma_n^4-\rho h\omega^2}{D_x\alpha_m^4+2H\alpha_m^2\gamma_n^2+D_y\gamma_n^4-\rho h\omega^2}\cos\alpha_m x\right]W_{0n}'\cos\gamma_n y\\
&-\frac{2}{b}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(D_y\gamma_n^2+v_yD_x\alpha_m^2)\alpha_m^2\cos\alpha_m x\cos\gamma_n y}{D_x\alpha_m^4+2H\alpha_m^2\gamma_n^2+D_y\gamma_n^4-\rho h\omega^2}\,[(-1)^n+c]\,W_{m0}^{\star}\\
&-\frac{1}{b}\sum_{m=1}^{\infty}\frac{D_x\alpha_m^4\cos\alpha_m x}{D_x\alpha_m^4-\rho h\omega^2}\,v_y\,[(-1)^0+c]\,W_{m0}^{\star},
\end{aligned}\tag{38}
$$

where $e=\frac{1}{2}(d+1)$. Similarly, the second derivative of W with respect to y by applying the same steps is given as

$$
\begin{aligned}
\frac{\partial^2 W}{\partial y^2}=&\frac{2}{b}\sum_{m=0}^{\infty}\left[f+\sum_{n=1}^{\infty}[c+(-1)^n]\right.\\
&\left.\times\frac{(2H-v_yD_x)\alpha_m^2\gamma_n^2+D_x\alpha_m^4-\rho h\omega^2}{D_x\alpha_m^4+2H\alpha_m^2\gamma_n^2+D_y\gamma_n^4-\rho h\omega^2}\cos\gamma_n y\right]W_{m0}^{\star}\cos\alpha_m x\\
&-\frac{2}{a}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(D_x\alpha_m^2+v_xD_y\gamma_n^2)\gamma_n^2\cos\alpha_m x\cos\gamma_n y}{D_x\alpha_m^4+2H\alpha_m^2\gamma_n^2+D_y\gamma_n^4-\rho h\omega^2}\,[(-1)^m+d]\,W_{0n}'\\
&-\frac{1}{a}\sum_{n=1}^{\infty}\frac{D_y\gamma_n^4\cos\gamma_n y}{D_y\gamma_n^4-\rho h\omega^2}\,v_x\,[(-1)^0+d]\,W_{0n}',
\end{aligned}\tag{39}
$$

where $f=\frac{1}{2}(c+1)$. Using equations (12) and (13) and incorporating the boundary conditions $M_{xx}=0$ at $x=0$ and a and $M_{yy}=0$ at $y=0$ and b lead to

$$
\begin{aligned}
0=&\frac{2}{a}\sum_{n=0}^{\infty}\left[e-\frac{1}{2}\frac{v_xv_yD_y\gamma_n^4}{D_y\gamma_n^4-\rho h\omega^2}[1+d]\right.\\
&\left.+\sum_{m=1}^{\infty}[d+(-1)^m]\frac{4D_{xy}\alpha_m^2\gamma_n^2+(1-v_yv_x)D_y\gamma_n^4-\rho h\omega^2}{D_x\alpha_m^4+2H\alpha_m^2\gamma_n^2+D_y\gamma_n^4-\rho h\omega^2}\right]W_{0n}'\cos\gamma_n y
\end{aligned}
$$

$$+\frac{2}{b}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}[c+(-1)^n]\frac{(2\nu_y H-\nu_y^2 D_x-D_y)\alpha_m^2\gamma_n^2-\nu_y\rho h\omega^2}{D_x\alpha_m^4+2H\alpha_m^2\gamma_n^2+D_y\gamma_n^4-\rho h\omega^2}\cos\gamma_n y\, W_{m0}^{\star}$$

$$+\frac{2}{b}\sum_{m=0}^{\infty}\nu_y\left(f-\frac{1}{2}\frac{D_x\alpha_m^4}{D_x\alpha_m^4-\rho h\omega^2}[1+c]\right)W_{m0}^{\star} \tag{40}$$

and

$$0=\frac{2}{b}\sum_{m=0}^{\infty}\left[f-\frac{1}{2}\frac{\nu_y\nu_x D_x\alpha_m^4}{D_x\alpha_m^4-\rho h\omega^2}[1+c]\right.$$

$$\left.+\sum_{n=1}^{\infty}[c+(-1)^n]\frac{4D_{xy}\alpha_m^2\gamma_n^2+(1-\nu_x\nu_y)D_x\alpha_m^4-\rho h\omega^2}{D_x\alpha_m^4+2H\alpha_m^2\gamma_n^2+D_y\gamma_n^4-\rho h\omega^2}\right]W_{m0}^{\star}\cos\alpha_m x$$

$$+\frac{2}{a}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}[d+(-1)^m]\frac{(2\nu_x H-\nu_x^2 D_y-D_x)\alpha_m^2\gamma_n^2-\nu_x\rho h\omega^2}{D_x\alpha_m^4+2H\alpha_m^2\gamma_n^2+D_y\gamma_n^4-\rho h\omega^2}\cos\alpha_m x\, W_{0n}'$$

$$+\frac{2}{a}\sum_{n=0}^{\infty}\nu_x\left(e-\frac{1}{2}\frac{D_y\gamma_n^4}{D_y\gamma_n^4-\rho h\omega^2}[1+d]\right)W_{0n}' \tag{41}$$

or

$$0=\sum_{n=0}^{\infty}\left[\frac{1}{2}A_{0n}+\sum_{m=1}^{\infty}A_{mn}\right]W_{0n}'\cos\gamma_n y+\sum_{m=0}^{\infty}\frac{1}{2}B_{m0}W_{m0}^{\star}$$

$$+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}B_{mn}\cos\gamma_n y\, W_{m0}^{\star} \tag{42}$$

and

$$0=\sum_{m=0}^{\infty}\left[\frac{1}{2}C_{m0}+\sum_{n=1}^{\infty}C_{mn}\right]W_{m0}^{\star}\cos\alpha_m x+\sum_{n=0}^{\infty}\frac{1}{2}D_{0n}W_{0n}'$$

$$+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}D_{mn}\cos\alpha_m x\, W_{0n}', \tag{43}$$

where

$$A_{mn}=\frac{2}{a}[d+(-1)^m]\frac{4D_{xy}\alpha_m^2\gamma_n^2+(1-\nu_y\nu_x)D_y\gamma_n^4-\rho h\omega^2}{D_x\alpha_m^4+2H\alpha_m^2\gamma_n^2+D_y\gamma_n^4-\rho h\omega^2}, \tag{44}$$

$$B_{mn}=\frac{2}{b}[c+(-1)^n]\frac{(2\nu_y H-\nu_y^2 D_x-D_y)\alpha_m^2\gamma_n^2-\nu_y\rho h\omega^2}{D_x\alpha_m^4+2H\alpha_m^2\gamma_n^2+D_y\gamma_n^4-\rho h\omega^2}, \tag{45}$$

$$C_{mn}=\frac{2}{b}[c+(-1)^n]\frac{4D_{xy}\alpha_m^2\gamma_n^2+(1-\nu_x\nu_y)D_x\alpha_m^4-\rho h\omega^2}{D_x\alpha_m^4+2H\alpha_m^2\gamma_n^2+D_y\gamma_n^4-\rho h\omega^2}, \tag{46}$$

$$D_{mn}=\frac{2}{a}[d+(-1)^m]\frac{(2\nu_x H-\nu_x^2 D_y-D_x)\alpha_m^2\gamma_n^2-\nu_x\rho h\omega^2}{D_x\alpha_m^4+2H\alpha_m^2\gamma_n^2+D_y\gamma_n^4-\rho h\omega^2}. \tag{47}$$

Each coefficient of the $\cos \alpha_m x$ and $\cos \gamma_n y$ has to vanish. What follows is a system of homogeneous algebraic equations:

$$
\begin{bmatrix}
\frac{1}{2}A_{00} + \sum_{m=1}^{\infty} A_{m0} & 0 & 0 & \cdots & \frac{1}{2}B_{00} & \frac{1}{2}B_{10} & \frac{1}{2}B_{20} & \cdots \\
0 & \frac{1}{2}A_{01} + \sum_{m=1}^{\infty} A_{m1} & 0 & \cdots & B_{01} & B_{11} & B_{21} & \cdots \\
0 & 0 & \frac{1}{2}A_{02} + \sum_{m=1}^{\infty} A_{m2} & \cdots & B_{02} & B_{12} & B_{22} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{1}{2}D_{00} & \frac{1}{2}D_{01} & \frac{1}{2}D_{02} & \cdots & \frac{1}{2}C_{00} + \sum_{n=1}^{\infty} C_{0n} & 0 & 0 & \cdots \\
D_{10} & D_{11} & D_{12} & \cdots & 0 & \frac{1}{2}C_{10} + \sum_{n=1}^{\infty} C_{1n} & 0 & \cdots \\
D_{20} & D_{21} & D_{22} & \cdots & 0 & 0 & \frac{1}{2}C_{20} + \sum_{n=1}^{\infty} C_{2n} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\begin{bmatrix}
W'_{00} \\ W'_{01} \\ W'_{02} \\ \vdots \\ W^{\star}_{00} \\ W^{\star}_{10} \\ W^{\star}_{20} \\ \vdots
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots
\end{bmatrix}.
$$

Non-trivial solution requires the determinant of the coefficient matrix to vanish. From this determinant the eigenfrequencies of the plate are calculated. The associated vibration modes are given by equation (29) after inserting the eigenfrequencies.

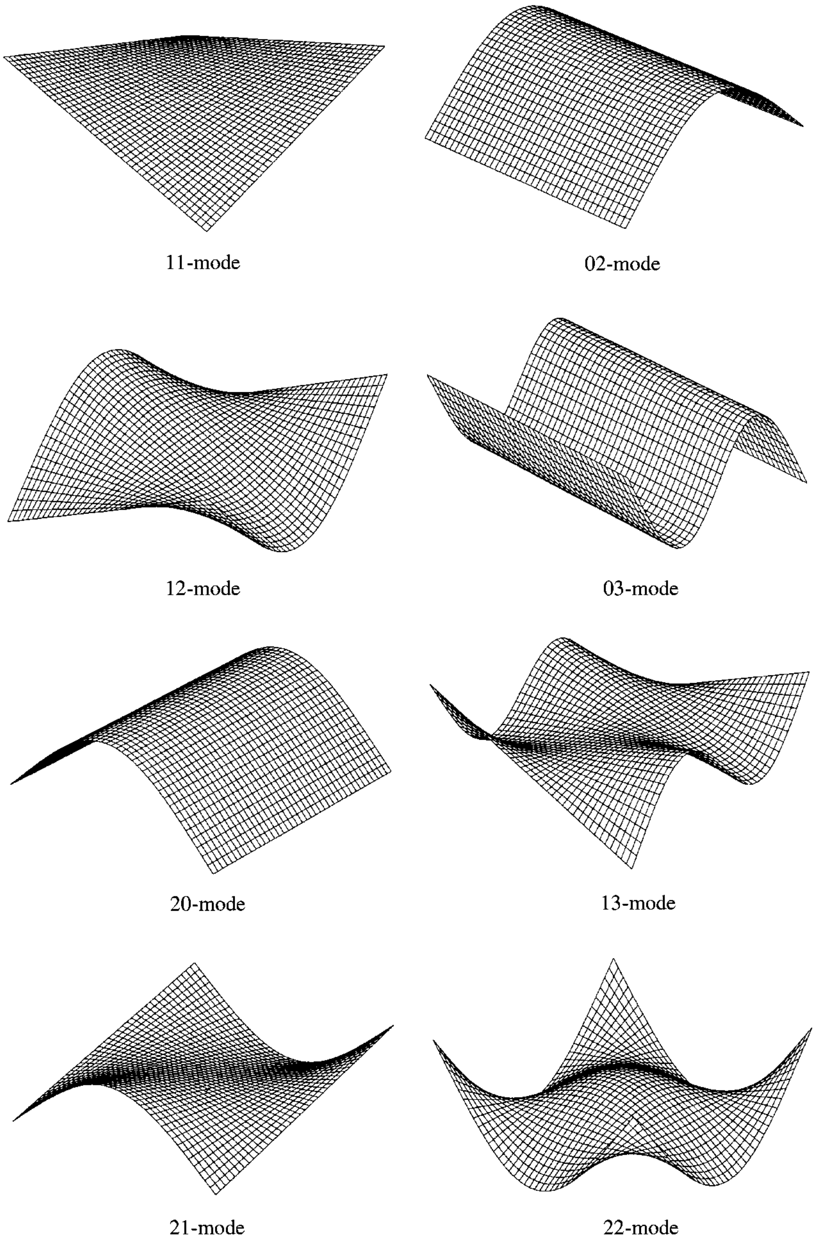

Figure 1. Vibration modes.

3. EXAMPLE

In this section the previously derived expressions are applied to a specific orthotropic plate with the geometrical and material properties shown in Table 1.

The calculated eigenfrequencies of this plate using the present method compared with the values calculated by the superposition method [16, 17] are shown in Table 2. It is obvious that the discrepancies are less than 0·05 per cent indicating excellent agreements. Figure 1

Table 1
Dimensions and material properties of the plate

a, b (m)	h (m)	ρ (kg/m^3)	E_x (GPa)	E_y (GPa)	G_{xy} (GPa)	ν_x
0·254	0·001483	1584	127·9	10·27	7·312	0·22

Table 2
Natural frequencies of free–free orthotropic plate

Mode	11	02	12	03	20	13	21	22
Superposition	51·76	60·17	121·4	165·9	212·6	228·6	236·6	304·9
Exact series	51·74	60·17	121·4	165·9	212·6	228·5	236·5	304·8

shows the corresponding eigenmodes of the orthotropic square plate with completely free boundaries.

4. CONCLUSIONS

The present paper shows that the eigenfrequencies of an orthotropic rectangular plate with completely free boundaries can be calculated by an exact series solution. The method is based on a weighted residual scheme for the classical thin plate equation. The advantages of the method are that the results converge quickly and can be calculated with the desired accuracy; and after finishing the analytical derivation for the frequency determinant the calculation of eigenfrequencies and vibration modes for given material data and geometry becomes straightforward. The analysis approach is very convenient for sensitivity studies. As the analytical solutions for the calculation of eigenfrequencies and modes are of paramount importance for many applications such as the identification of constitutive parameters from experimental plate testing [20], the present method of analysis provides an efficient procedure for accurate results which should be of academic and practical importance.

REFERENCES

1. J. T.-S. Wang and C.-C. Lin 1996 *Journal of Sound and Vibration* **196**, 285–293. Dynamic analysis of generally supported beams using fourier series. doi: 10.1006/jsvi.1996.0484.
2. J. T.-S. Wang and C.-C. Lin 1999 *Journal of Applied Mechanics* **66**, 380–387. A method for exact series solution in structural mechanics.
3. E. E. F. Chladni 1802 *Die Akustik*. Leipzig.
4. K. F. Graff 1991 *Wave Motion in Elastic Solids*. New York: Dover Publications Inc.
5. G. B. Warburton 1954 in *Proceedings of the Institution of Mechanical Engineers* **168**, 371–384. The vibration of rectangular plates.
6. J. W. S. Rayleigh 1945 *The Theory of Sound*. New York: Dover Publications Inc.
7. R. F. S. Hearmon 1959 *Journal of Applied Mechanics* **26,** 537–540. The frequency of flexural vibration of rectangular orthotropic plates with clambed or supported edges.
8. S. M. Dickinson 1978 *Journal of Sound and Vibration* **61**, 1–8. The buckling and frequency of flexural vibration of rectangular isotropic and orthotropic plates using Rayleigh's method.

9. C. S. Kim and S. M. Dickinson 1985 *Journal of Sound and Vibration* **103**, 142–149. Improved approximate expressions for the natural frequencies of isotropic and orthotropic rectangular plates.
10. S. Iguchi 1953 *Ingenieur-Archiv* **21**, 303–322. Die Eigenschwingungen und Klangfiguren der vierseitig freien rechteckigen Platte.
11. C. Rajalingham, R. B. Bhat and G. D. Xistris 1997 *Journal of Sound and Vibration* **203**, 169–180. Vibration of rectangular plates by reduction of the plate partial differential equation into simultaneous ordinary differential equations. doi: 10.1006/jsvi.1996.0814
12. K. M. Liew and K. Y. Lam 1994 *Journal of Sound and Vibration* **174**, 23–36. Effects of arbitrarily distributed elastic point constraints on vibrational behaviour of rectangular plates. doi: 10.1006/jsvi.1994.1259
13. A. W. Leissa 1973 *Journal of Sound and Vibration* **31**, 257–293. The free vibration of rectangular plates.
14. W. Ritz 1909 *Journal für reine und angewandte Mathematik* **135**, 1–61. Über eine neue Methode zur Lösung gewisser Variationsprobleme der mathematischen Physik.
15. L. R. Deobald and R. F. Gibson 1988 *Journal of Sound and Vibration* **124**, 269–283. Determination of elastic constants of orthotropic plates by a modal analysis/Rayleigh–Ritz technique.
16. D. J. Gorman 1993 *Journal of Sound and Vibration* **165**, 409–420. Accurate free vibration analysis of the completely free orthotropic rectangular plate by the method of superposition. doi. 10.1006/jsvi.1993.1267
17. D. J. Gorman 1999 *Vibration Analysis of Plates by the Superposition Method*. Singapore: World Scientific Publishing.
18. L. Gaul and Ch. Fiedler 1996 *Methode der Randelemente in Statik und Dynamik*. Wiesbaden/Braunschweig: Friedrich Vieweg & Sohn.
19. I. N. Bronstein and K. A. Semendjajew 1987 *Taschenbuch der Mathematik*. Frankfurt/Main: Harri Deutsch; 23rd edition.
20. L. Gaul, S. Hurlebaus and K. Willner 1999 in *Proceedings of the International Modal Analysis Conference IMAC* XVII **II**, 1756–1762. Determination of material properties of plates from modal ESPI measurements.