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LETTERS TO THE EDITOR

COMMENTS ON "NON-LINEAR DYNAMICS OF GEAR-PAIR SYSTEMS WITH PERIODIC STIFFNESS AND BACKLASH"

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In a highly interesting paper [1] Theodossiades and Natsiavas studied the non-linear dynamics of a gear-pair system with backlash, periodic mesh stiffness and external excitation due to torsional moments and errors of the teeth geometry. The equation of motion is written in the normalized form

$$\ddot{u} + 2\varepsilon\mu\,\dot{u} + [1 + 2\varepsilon\cos\Omega t]\,g(u) = f_0 + \varepsilon f_1\cos(\Omega t + \theta), \tag{1}$$

where the backlash is represented by the function

$$g(u) = \begin{cases} u - 1, & u \geqslant 1, \\ 0, & -1 < u < 1, \\ u + 1, & u \leqslant -1. \end{cases} \tag{2}$$

The parameter ε is small. The parameters f_0 and f_1 are related to the amplitudes of the static and the periodic excitations. Ω and θ denote the frequency and the phase of the periodic excitation. In the case of simultaneous fundamental parametric resonance and principal external resonance, four types of periodic steady state solutions are determined by techniques applicable to piecewise linear systems and to systems with time-periodic coefficients. In addition, an appropriate stability analysis is established. In the second part of the paper [1] the efficiency of these analytical techniques is illustrated by comparing with the results obtained by direct numerical integration of the equation of motion. For some choices of the parameters complex behavior is found including boundary crises and intermittent chaos. With $\varepsilon = 0{\cdot}03$, $\mu = 0{\cdot}1$, $f_0 = 0$, $f_1 = 2{\cdot}5$, $\theta = 0$, a period-doubling cascade with respect to the parameter Ω was detected. In their Figure 10 the authors provide response histories at $\Omega = 0{\cdot}226$ ($1P$ solution having the same period as the excitation), 0·22195 ($2P$ solution), 0·2215 ($4P$ solution) and 0·221314 (intermittent chaos).

I would like to make the following comment on the paper. It is to be emphasized that there exists another period-doubling cascade in the same range of the forcing frequency, which has not been mentioned in reference [1]. In order to illustrate the existence of both cascades, equation (1) is integrated by the use of the Runge–Kutta–Hůta method of order six [2, 3], which is a very accurate scheme. The transient regime is deleted and the Poincaré section point for u at $t = 0$ with the sampling period $P = 2\pi/\Omega$ is plotted in terms of the parameter Ω. Figure 1 represents the bifurcation diagram for the cascade mentioned in

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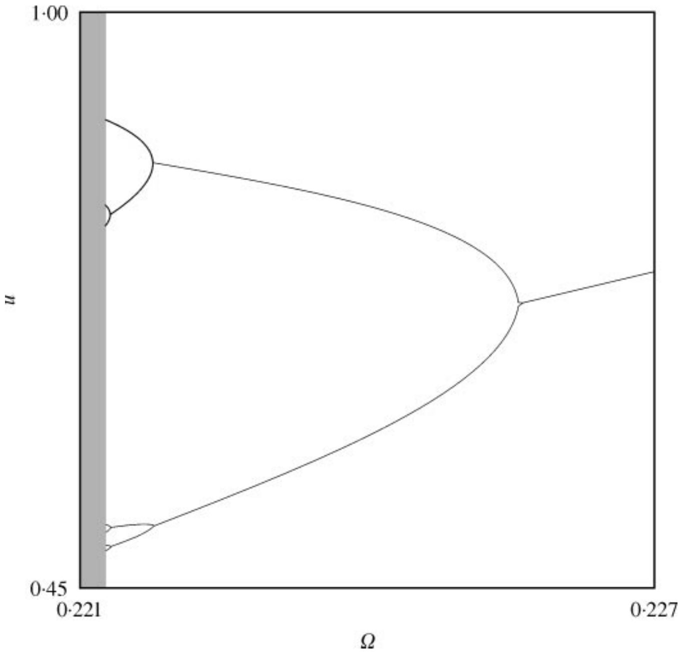

Figure 1. First cascade of period-doubling bifurcations (Type A) in the range $0{\cdot}221 \leqslant \Omega \leqslant 0{\cdot}227$.

reference [1] (called here as Type A). It has been established by continuation with respect to the parameter Ω starting at $\Omega = 0{\cdot}227$ for which a $1P$ solution is found with $u = 0{\cdot}75088$, $\dot{u} = 0{\cdot}32439$ as the co-ordinates of the Poincaré section point corresponding to $t = 0$ in the phase plane. Three transitions are discerned in this figure occurring at $\Omega \approx 0{\cdot}2256$ ($1P \rightarrow 2P$), 0·22175 ($2P \rightarrow 4P$) and 0·22133 ($4P \rightarrow 8P$) respectively. Periodic motion with period $8P$ is readily seen. In the limit chaotic motion sets in. In the zone of chaos the Poincaré section points extend to the region $-1 \leqslant u \leqslant 1{\cdot}8$.

The author found the second period-doubling cascade (called Type B) in the same frequency range by continuation in terms of the parameter Ω starting with another $1P$ solution at $\Omega = 0{\cdot}227$. This solution is readily obtained by direct numerical integration and its Poincaré section point at $t = 0$ is given by $u = 0{\cdot}83514$, $\dot{u} = -0{\cdot}33332$. Figure 2 shows the second cascade in which the subsequent transitions $1P \rightarrow 2P$, $2P \rightarrow 4P$ and $4P \rightarrow 8P$ take place at $\Omega \approx 0{\cdot}2259$, 0·2220 and 0·22155 respectively. The complete bifurcation diagram in terms of Ω ranging from $\Omega = 0{\cdot}221$ to 0·227 consists of both parts (Figures 1 and 2) superposed.

One of the most reliable criteria for determining the coexistence of periodic or chaotic attractors is to study their basins of attraction. In this technique (see reference [4]) one considers a grid of initial conditions in the phase plane. By integrating system (1) for each set of initial conditions, the periodic and chaotic attractors that the orbit approaches are searched for. A different color is assigned to each initial condition in terms of the relevant attractor that is approached. Figure 3 illustrates the basins of attraction for the case with $\Omega = 0{\cdot}224$. It was constructed using a 200×200 grid of pixels in the domain of the phase plane given by $0 \leqslant u \leqslant 1{\cdot}5$ and $-0{\cdot}6 \leqslant \dot{u} \leqslant 0{\cdot}6$. In this case, two periodic attractors occur both having the period $2P$ with the following co-ordinates $(u, \dot{u})$ of the Poincaré section

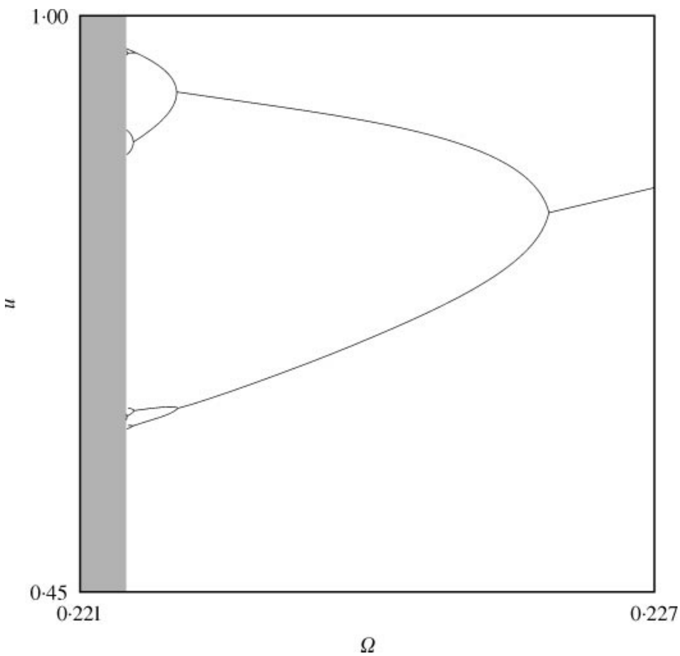

Figure 2. Second cascade of period-doubling bifurcations (Type B) in the range $0{\cdot}221 \leqslant \Omega \leqslant 0{\cdot}227$.

points corresponding to $t = 0$ in the phase plane:

Type A: (0·81359, 0·32082), (0·59534, 0·31810);

Type B: (0·89868, − 0·32770), (0·69057, − 0·32923).

The basins of attraction were constructed by the use of Nusse and Yorke's package DYNAMICS [5]. The basins of attraction are colored in magenta for Type A motion and in light blue for Type B motion. Highly fractal areas in the initial condition space arise indicating an increase of uncertainty in initial conditions. The percentage of pixels for the basin of attraction for periodic motion of Type A amounts to 41% and 59% for Type B motion. It is to be mentioned that the pattern of the basins of attraction shown in Figure 3 is a typical one in the sequences of period-doubling bifurcations represented in Figures 1 and 2. A rather similar pattern is obtained for the case with $\Omega = 0{\cdot}227$, i.e., at the entrance of the cascades.

It has been pointed out that the response histories in Figure 10 from reference [1], which corresponds to $1P$, $2P$ and $4P$ motion, are each related to periodic motion of Type A. For comparison, Figure 4 shows the response history of the periodic motion of Type B having the period $1P$ in the case with $\Omega = 0{\cdot}226$. A kind of mirroring occurs with respect to the vertical axis as compared to Figure 10, part (a), in reference [1] for the same value of Ω. Type B motion is characterized by double-sided impacts (a maximum displacement value of u larger than 1 and a minimum value smaller than -1) like the motion of Type A mentioned in reference [1].

In conclusion, the non-linear dynamics of the relevant gear-pair system with periodic stiffness and backlash is more complex than described in reference [1]. In the frequency region determined by $0{\cdot}221 \leqslant \Omega \leqslant 0{\cdot}227$, there exist two cascades of period-doubling bifurcations. The study of the basins of attraction reveals that the two coexisting basins are highly intertwined and are fractal. Therefore, the occurrence of the second cascade of

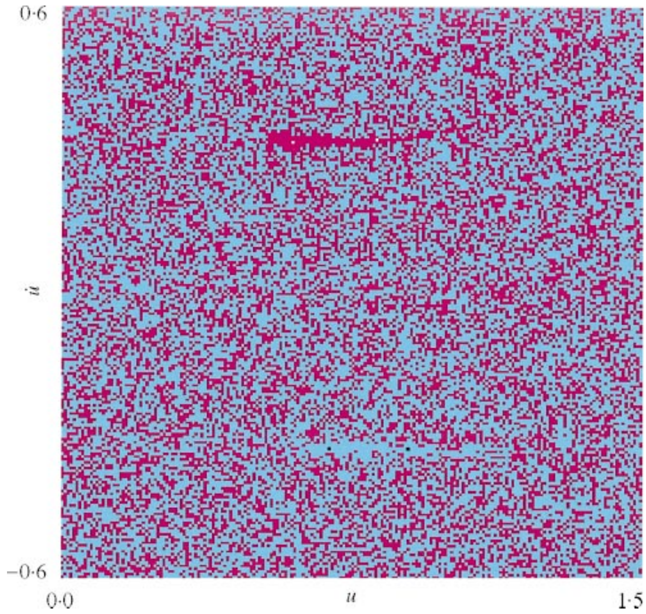

Figure 3. Basins of attraction in the phase-plane for the case $\Omega = 0{\cdot}224$ (two coexisting periodic attractors both having the period $2P$) with $0 \leqslant u \leqslant 1{\cdot}5$ and $-0{\cdot}6 \leqslant \dot{u} \leqslant 0{\cdot}6$. Colors used (gray levels) for basins: dark gray for Type A and light gray for Type B.

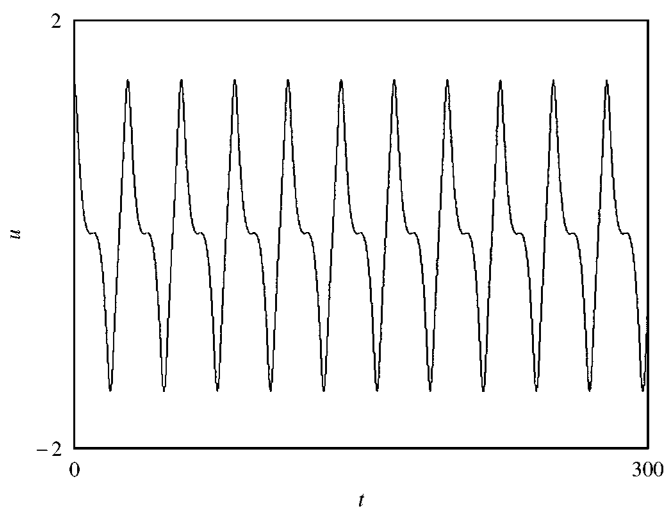

Figure 4. Response history of periodic motion of Type B with period $1P$ at $\Omega = 0{\cdot}226$.

period-doubling bifurcations certainly has to be mentioned in the description of the very complex dynamics of the gear-pair system.

REFERENCES

1. S. THEODOSSIADES and S. NATSIAVAS 2000 *Journal of Sound and Vibration* **229**, 287–310. Non-linear dynamics of gear-pair systems with periodic stiffness and backlash.

2. J. D. LAMBERT 1973 *Computational Methods in Ordinary Differential Equations*. Chichester: John Wiley.
3. R. VAN DOOREN 1996 *Chaos, Solitons and Fractals* **7**, 77–90. Chaos in a pendulum with forced horizontal support motion: a tutorial.
4. C. PEZESHKI and E. H. DOWELL 1987 *Journal of Sound and Vibration* **117**, 219–232. An examination of initial condition maps for the sinusoidally excited buckled beam modeled by the Duffing's equation.
5. H. E. NUSSE and J. A. YORKE 1998 *Dynamics: Numerical Explorations*. New York: Springer; second edition.