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A NOTE ON THE STABILITY OF PIPES CONVEYING FLUID

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In their paper [1] Lundgren, Sethna and Bajaj have made a theoretical and experimental study of self-induced non-planar vibrations of a flexible tube conveying a fluid. The tube was fixed at one end and the fluid issued from a nozzle inclined to the axis of the tube at the free end. In order to study the stability boundaries for flow-induced motions of the tube, when the nozzle angle is small, authors of the paper [1] have used the following, represented in operator form, equations for small in-plane (u) and out-of-plane (v) motions

$$\Lambda(\rho, \theta_j)\psi = 0 \tag{1}$$

with boundary conditions

$$\psi = \partial\psi/\partial\xi = 0 \quad \text{at } \xi = 0, \qquad \partial^2\psi/\partial\xi^2 = 0 \quad \text{at } \xi = 1, \qquad B(\rho, \theta_j)\psi = 0 \quad \text{at } \xi = 1, \tag{2}$$

where ψ is either u or v, $\rho = \sqrt{(MU^2L^2/EI)(A/A_j)}$, M is the fluid mass per unit length, U is the flow rate, L is the tube length, EI is the flexural rigidity, A_j, A are the cross-sectional areas of the nozzle and the tube, respectively, θ_j is an angle, with the z-axis of a Cartesian co-ordinate system, which is being made by a nozzle when the tube being in its straight undeformed state extends along the z-axis, $\xi = s/L$, where s is the arc length along the deformed tube. Since the tube is inextensible s is used as a material variable. Λ and B are linear differential operators in τ and ξ with ρ and θ_j as parameters, $\tau = t/\sqrt{(m+M)/EIL^2}$, τ being a dimensionless time, and m is the tube mass per unit length. Both Λ and B have been expanded by Lundgren *et al.* in powers of θ_j as follows:

$$\Lambda(\rho, \theta_j) = \Lambda_0(\rho) + \theta_j^2\Lambda_1(\rho) + \cdots, \tag{3}$$

where

$$\Lambda_0(\rho) \equiv \partial^4/\partial\xi^4 + \rho^2(\partial^2/\partial\xi^2) + \partial^2/\partial\tau^2 + 2\rho\beta^{1/2}(\partial^2/\partial\tau\partial\xi), \tag{4}$$

$$B(\rho, \theta_j) = \partial^3/\partial\xi^3 + \theta_j^2 B_1(\rho), \tag{5}$$

$$\beta = \{M/(m+M)\}(A_j/A). \tag{6}$$

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The linear operators Λ_1 and B_1 are different for the u and v problems. Those for u take the form

$$\Lambda_{1,u}(\rho) = \frac{\partial}{\partial\xi}\left\{-\frac{1}{2}\rho^2\frac{\partial}{\partial\xi} + \frac{3}{2}\left(\frac{d\Phi_0}{d\xi}\right)^2\frac{\partial}{\partial\xi} + 3\Phi_0\frac{d\Phi_0}{d\xi}\frac{\partial^2}{\partial\xi^2} + \Phi_0\int_\xi^1\left[\int_0^\xi\frac{d\Phi_0}{d\xi}\frac{\partial^2(\,)}{\partial\tau^2}d\xi\right]d\xi\right\}, \tag{7}$$

$$B_{1,u}(\rho) = -\rho^2(1-\cos\rho)(\partial/\partial\xi), \tag{8}$$

while those for v take the form

$$\Lambda_{1,v}(\rho) = \frac{\partial}{\partial\xi}\left\{-\frac{1}{2}\rho^2\frac{\partial}{\partial\xi} + \frac{3}{2}\left(\frac{d\Phi_0}{d\xi}\right)^2\frac{\partial}{\partial\xi}\right\}, \tag{9}$$

$$B_{1,v}(\rho) = \rho^2\int_0^1\frac{d\Phi_0}{d\xi}\frac{\partial}{\partial\xi}(\,)\,d\xi, \tag{10}$$

where $\Phi_0(\xi) = \cos\rho - \cos\rho(1-\xi)$.

It is immediately apparent that equations (1) cannot be simply solved. They can be used, however, to calculate the critical ρ by using, for example, Galerkin's method. In the analysis presented here Galerkin's method is used with the following shape functions, which satisfy the boundary conditions (2):

$$\phi_{n,u} = \phi^0_{n,u} + \theta_j^2\phi^1_{n,u} = \frac{1}{2^{n-1}}\left\{\sin\rho(1-\xi) + \rho\xi\cos\rho + \frac{\rho^3}{(n\pi)^3}[n\pi\xi - \sin(n\pi\xi)] - \sin\rho + C_u\theta_j^2[n\pi\xi - \sin(n\pi\xi)]\right\}, \tag{11}$$

$$\phi_{n,v} = \phi^0_{n,v} + \theta_j^2\phi^1_{n,v} = \frac{1}{2^{n-1}}\left\{\sin\rho(1-\xi) + \rho\xi\cos\rho + \frac{\rho^3}{(n\pi)^3}[n\pi\xi - \sin(n\pi\xi)] - \sin\rho + C_v\theta_j^2[n\pi\xi - \sin(n\pi\xi)]\right\}, \tag{12}$$

Here

$$C_u = \frac{\rho^3(1-\cos\rho)(1-\cos\rho-(2\rho^2/(n\pi)^2))}{(n\pi)^3[1+(2\rho^2\theta_j^2/(n\pi)^2)(1-\cos\rho)]}, \tag{13}$$

$$C_v = \frac{\rho^3}{2(n\pi)}\frac{\sin^2\rho - 2(1-\cos\rho)(\rho^2/(n\pi)^2+\cos\rho) + (2\rho^4/(n\pi)^2[(n\pi)^2-\rho^2])(1+\cos\rho)}{(n\pi)^2+\rho^2\theta_j^2[1-\cos\rho-(\rho^2/(n\pi)^2-\rho^2)(1+\cos\rho)]}, \tag{14}$$

n is an odd integer.

One can check by substitution that the shape functions (11) satisfy the boundary conditions (2). Substitution and differentiation shows that at $\xi = 0$

$$\Phi_{n,u}(\xi = 0) = \frac{1}{2^{n-1}}\left\{\sin\rho(1-\xi) + \rho\xi\cos\rho + \frac{\rho^3}{(n\pi)^3}[n\pi\xi - \sin(n\pi\xi)] - \sin\rho + C_u\theta_j^2[n\pi\xi - \sin(n\pi\xi)]\right\}$$

$$= \frac{1}{2^{n-1}}\left\{\sin\rho(1-0) + \rho 0\cos\rho + \frac{\rho^3}{(n\pi)^3}[n\pi 0 - \sin(n\pi 0)] - \sin\rho + C_u\theta_j^2[n\pi 0 - \sin(n\pi 0)]\right\}$$

$$= \frac{1}{2^{n-1}}\{\sin\rho - \sin\rho\} = 0,$$

$$\frac{d\Phi_{n,u}}{d\xi} = \frac{1}{2^{n-1}}\left\{-\rho\cos\rho(1-\xi) + \rho\cos\rho + \frac{\rho^3}{(n\pi)^3}[n\pi - n\pi\cos(n\pi\xi)] + C_u\theta_j^2[n\pi - n\pi\cos(n\pi\xi)]\right\}$$

$$= \frac{1}{2^{n-1}}\left\{-\rho\cos\rho(1-0) + \rho\cos\rho + \frac{\rho^3}{(n\pi)^3}[n\pi - n\pi\cos(n\pi 0)] + C_u\theta_j^2[n\pi - n\pi\cos(n\pi 0)]\right\}$$

$$= \frac{1}{2^{n-1}}\left\{-\rho\cos\rho + \rho\cos\rho + \frac{\rho^3}{(n\pi)^3}[n\pi - n\pi] + C_u\theta_j^2[n\pi - n\pi]\right\} = 0.$$

Substitution and differentiation shows that at $\xi = 1$

$$\frac{d^2\Phi_{n,u}}{d\xi^2} = \frac{1}{2^{n-1}}\left\{-\rho^2\sin\rho(1-\xi) + \frac{\rho^3(n\pi)^2}{(n\pi)^3}[\sin(n\pi\xi)] + C_u\theta_j^2[(n\pi)^2\sin(n\pi\xi)]\right\}$$

$$= \frac{1}{2^{n-1}}\left\{-\rho^2\sin\rho(1-1) + \frac{\rho^3(n\pi)^2}{(n\pi)^3}[\sin(n\pi 1)] + C_u\theta_j^2[(n\pi)^2\sin(n\pi 1)]\right\} = 0.$$

Further, substituting equation (11) into equation $B(\rho, \theta_j)\Phi_{n,u} = \partial^3\Phi_{n,u}/\partial\xi^3 + \theta_j^2 B_{1,u}(\rho)\Phi_{n,u} = 0$ and performing the elementary differentiation at $\xi = 1$ gives

$$B(\rho, \theta_j)\Phi_{n,u} = \partial^3\Phi_{n,u}/\partial\xi^3 + \theta_j^2 B_{1,u}(\rho)\Phi_{n,u} = \frac{d^3\Phi_{n,u}}{d\xi^3} - \theta_j^2\rho^2(1-\cos\rho)\frac{d\Phi_{n,u}}{d\xi}$$

$$= \frac{1}{2^{n-1}}\left\{\rho^3\cos\rho(1-\xi) + \frac{\rho^3(n\pi)^3}{(n\pi)^3}[\cos(n\pi\xi)] + C_u\theta_j^2[(n\pi)^3\cos(n\pi\xi)]\right\}$$

$$- \theta_j^2\rho^2(1-\cos\rho)\frac{1}{2^{n-1}}\left\{-\rho\cos\rho(1-\xi) + \rho\cos\rho\right.$$

$$\left. + \frac{\rho^3}{(n\pi)^3}[n\pi - n\pi\cos(n\pi\xi)] + C_u\theta_j^2[n\pi - n\pi\cos(n\pi\xi)]\right\}$$

$$= \frac{1}{2^{n-1}}\left\{\rho^3\cos\rho(1-1) + \frac{\rho^3(n\pi)^3}{(n\pi)^3}[\cos(n\pi 1)] + C_u\theta_j^2[(n\pi)^3\cos(n\pi 1)]\right\}$$

$$- \theta_j^2\rho^2(1-\cos\rho)\frac{1}{2^{n-1}}\left\{-\rho\cos\rho(1-1) + \rho\cos\rho\right.$$

$$\left. + \frac{\rho^3}{(n\pi)^3}[n\pi - n\pi\cos(n\pi 1)] + C_u\theta_j^2[n\pi - n\pi\cos(n\pi 1)]\right\}$$

$$= \frac{1}{2^{n-1}}\{-C_u\theta_j^2(n\pi)^3\} - \theta_j^2\rho^2(1-\cos\rho)$$

$$\times\frac{1}{2^{n-1}}\left\{-\rho + \rho\cos\rho + \frac{2(n\pi)\rho^3}{(n\pi)^3} + C_u\theta_j^2 2(n\pi)\right\}$$

$$= \frac{1}{2^{n-1}}\{-C_u\theta_j^2(n\pi)^3\} - \theta_j^2\rho^2(1-\cos\rho)\frac{1}{2^{n-1}}\left\{-\rho + \rho\cos\rho\right.$$

$$\left. + \frac{2(n\pi)\rho^3}{(n\pi)^3} + C_u\theta_j^2 2(n\pi)\right\}$$

$$= -\frac{1}{2^{n-1}}\theta_j^2(n\pi)^3 C_u\left[1 + \frac{2\rho^2\theta_j^2}{(n\pi)^2}(1-\cos\rho)\right]$$

$$+ \frac{1}{2^{n-1}}\theta_j^2\rho^3(1-\cos\rho)\left(1 - \cos\rho - \frac{2\rho^2}{(n\pi)^2}\right) = 0.$$

That is to say, the shape functions (11) satisfy the boundary conditions (2). Similarly, it can be shown that the shape functions (12) also satisfy the boundary conditions (2). In particular, substituting equation (12) into equations

$$B(\rho,\theta_j)\Phi_{n,v} = \partial^3\Phi_{n,v}/\partial\xi^3 + \theta_j^2\rho^2\int_0^1 \frac{\mathrm{d}\Phi_0}{\mathrm{d}\xi}\frac{\partial\Phi_{n,v}}{\partial\xi}\,\mathrm{d}\xi = 0$$

and performing the elementary integrations gives

$$\begin{aligned}
B(\rho,\theta_j)\Phi_{n,v} &= \partial^3\Phi_{n,v}/\partial\xi^3 + \theta_j^2\rho^2\int_0^1 \frac{\mathrm{d}\Phi_0}{\mathrm{d}\xi}\frac{\partial\Phi_{n,v}}{\partial\xi}\,\mathrm{d}\xi \\
&= \frac{1}{2^{n-1}}\left\{\rho^3\cos\rho(1-\xi) + \frac{\rho^3}{(n\pi)^3}(n\pi)^3\cos n\pi\xi + C_v\theta_j^2(n\pi)^3\cos n\pi\xi\right\} \\
&\quad - \frac{1}{2^{n-1}}\theta_j^2\rho^3\int_0^1 \sin\rho(1-\xi)\Bigg(\rho\cos\rho - \rho\cos\rho(1-\xi) \\
&\quad + \frac{\rho^3}{(n\pi)^2} - \frac{\rho^3}{(n\pi)^2}\cos n\pi\xi + C_v\theta_j^2 n\pi - C_v\theta_j^2 n\pi\cos n\pi\xi\Bigg)\mathrm{d}\xi \\
&= \frac{1}{2^{n-1}}\left[\rho^3\cos\rho(1-1) + \frac{\rho^3}{(n\pi)^3}(n\pi)^3\cos n\pi 1 + C_v\theta_j^2(n\pi)^3\cos n\pi 1\right] \\
&\quad - \theta_j^2\frac{1}{2^{n-1}}\Bigg(\rho^4\cos\rho\int_0^1\sin\rho(1-\xi)\,\mathrm{d}\xi - \rho^4\int_0^1\sin\rho(1-\xi)\cos\rho(1-\xi)\,\mathrm{d}\xi \\
&\quad + \rho^3\left(\frac{\rho^3}{(n\pi)^2} + C_v\theta_j^2 n\pi\right)\int_0^1\sin\rho(1-\xi)\,\mathrm{d}\xi\Bigg) \\
&\quad + \frac{1}{2^{n-1}}\theta_j^2\rho^3\left(\frac{\rho^3}{(n\pi)^2} + C_v\theta_j^2 n\pi\right)\int_0^1\sin\rho(1-\xi)\cos n\pi\xi\,\mathrm{d}\xi \\
&= -\frac{1}{2^{n-1}}C_v\theta_j^2(n\pi)^3 - \frac{1}{2^{n-1}}\theta_j^2\rho^3\cos\rho(1-\cos\rho) + \frac{1}{2^{n-1}}\theta_j^2\frac{\rho^3}{2}\sin^2\rho \\
&\quad - \frac{1}{2^{n-1}}\theta_j^2\rho^2\left(\frac{\rho^3}{(n\pi)^2} + C_v\theta_j^2 n\pi\right)(1-\cos\rho) \\
&\quad + \frac{1}{2^{n-1}}\theta_j^2\frac{\rho^4}{(n\pi)^2-\rho^2}\left(\frac{\rho^3}{(n\pi)^2} + C_v\theta_j^2 n\pi\right)(1+\cos\rho)
\end{aligned}$$

$$= -\frac{1}{2^{n-1}}\theta_j^2 C_v(n\pi)\left[(n\pi)^2 + \rho^2\theta_j^2\left[1 - \cos\rho - \frac{\rho^2}{(n\pi)^2 - \rho^2}(1 + \cos\rho)\right]\right]$$

$$+ \frac{1}{2^{n-1}}\frac{\theta_j^2\rho^3}{2}\left[\sin^2\rho - 2(1 - \cos\rho)\left[\frac{\rho^2}{(n\pi)^2} + \cos\rho\right]\right]$$

$$+ \frac{1}{2^{n-1}}\frac{\theta_j^2\rho^3}{2}\left[\frac{2\rho^4}{(n\pi)^2[(n\pi)^2 - \rho^2]}(1 + \cos\rho)\right] = 0.$$

That is to say, the shape functions (12) also satisfy the boundary conditions (2). Note that in deriving the above equations, expressions (13) and (14) for C_u and C_v have been used.

The partial differential equation (1) can be solved to the required approximation by Galerkin's method. Then, by developing $\psi(\xi, \tau)$ on the functions $\Phi_n(\xi)$, which satisfy the boundary conditions, Galerkin's conditions can be expressed as

$$\int_0^1 \Lambda(\rho, \theta_j)\psi_n(\xi, \tau)\Phi_m(\xi)\,\mathrm{d}\xi = 0, \quad m = 1, 3, 5, \ldots, N, \tag{15}$$

with

$$\psi(\xi, \tau) = \sum_{n=2k-1}^{N} \psi_n(\xi, \tau) = \sum_{n=2k-1}^{N} r_n(\tau)\Phi_n(\xi), \quad k = 1, 2, 3, \ldots, \tag{16}$$

where $r_n(\tau)$ are the time-dependent amplitude coefficients to be obtained from the ordinary differential equations (15) by a suitable integration technique, $\Phi_n(\xi)$ are the shape functions (11) or (12).

At the outset one must make the major hypothesis that equations (1) have possible solutions having the form (16) which converge on the true solutions as $N \to \infty$.

Following Lundgren *et al.* the stability boundary is found by seeking a solution of the form

$$\psi(\xi, \tau) = \sum_{n=2k-1}^{N} \psi_n(\xi, \tau) = \sum_{n=2k-1}^{N} C_n\Phi_n(\xi)\mathrm{e}^{\lambda\tau}, \quad k = 1, 2, 3, \ldots \tag{17}$$

and establishing the parameter values for which $\operatorname{Re}\lambda = 0$. Here C_n, the coefficients, form a finite set of generalized co-ordinates, to be determined. Before solving the problem, note that $\Phi_n(\xi)$, $\Phi_m(\xi)$ are real functions and ρ, β are real non-negative parameters.

First consider the case $\theta_j = 0$. Substituting equation (17) into equation (15), one obtains

$$\mathrm{e}^{\lambda\tau}\sum_{n=2k-1}^{N} C_n \int_0^1 \left[\frac{\mathrm{d}^4\Phi_n}{\mathrm{d}\xi^4} + \rho^2\frac{\mathrm{d}^2\Phi_n}{\mathrm{d}\xi^2} + \lambda^2\Phi_n + 2\rho\beta^{1/2}\lambda\frac{\mathrm{d}\Phi_n}{\mathrm{d}\xi}\right]\Phi_m\,\mathrm{d}\xi$$

$$= \mathrm{e}^{\lambda\tau}\sum_{n=2k-1}^{N} C_n \int_0^1 \left[\frac{\mathrm{d}^4\Phi_n}{\mathrm{d}\xi^4} + \rho^2\frac{\mathrm{d}^2\Phi_n}{\mathrm{d}\xi^2} + (\lambda_R^2 - \lambda_I^2)\Phi_n + 2\rho\beta^{1/2}\lambda_R\frac{\mathrm{d}\Phi_n}{\mathrm{d}\xi}\right]\Phi_m\,\mathrm{d}\xi$$

$$+ \mathrm{i}2\lambda_I\mathrm{e}^{\lambda\tau}\sum_{n=2k-1}^{N} C_n \int_0^1 \left[\lambda_R\Phi_n + \rho\beta^{1/2}\frac{\mathrm{d}\Phi_n}{\mathrm{d}\xi}\right]\Phi_m\,\mathrm{d}\xi = 0, \quad m = 1, 3, 5, 7\ldots \tag{18}$$

where $\lambda_R = \mathrm{Re}\,\lambda$, $\lambda_I = \mathrm{Im}\,\lambda$, λ_I is the dimensionless circular frequency, $\sqrt{(m+M)/EI}\,L^2\Omega$, Ω is the circular frequency of oscillation.

Equations (18) yield a system of linear equations in C_n of the form

$$AC = 0, \tag{19}$$

where A is the matrix with the elements

$$a_{nm} = \int_0^1 \left[\frac{\mathrm{d}^4\Phi_n}{\mathrm{d}\xi^4} + \rho^2 \frac{\mathrm{d}^2\Phi_n}{\mathrm{d}\xi^2} + (\lambda_R^2 - \lambda_I^2)\Phi_n + \lambda_R 2\rho\beta^{1/2} \frac{\mathrm{d}\Phi_n}{\mathrm{d}\xi} \right] \Phi_m \,\mathrm{d}\xi$$
$$+ \mathrm{i}2\lambda_I \int_0^1 \left[\lambda_R \Phi_n + \rho\beta^{1/2} \frac{\mathrm{d}\Phi_n}{\mathrm{d}\xi} \right] \Phi_m \,\mathrm{d}\xi \tag{20}$$

and C is the vector with the components C_n. It is in general possible to determine values of ρ and β which reduce the real part of λ to zero. Such values of ρ and β follow from the equation

$$\det A = 0, \quad \text{with } \lambda_R = 0. \tag{21}$$

For simplicity, one can consider a single mode approximating equation (15) for the u problem: i.e., case $k = 2$, $n = m = N = 3$, $\Phi_n = \Phi_{3,u}$. Then, from equation (21) by separating into real and imaginary parts, one obtains

$$\int_0^1 \left[\frac{\mathrm{d}^4\Phi_{3,u}}{\mathrm{d}\xi^4} + \rho^2 \frac{\mathrm{d}\Phi_{3,u}}{\mathrm{d}\xi^2} + (\lambda_R^2 - \lambda_I^2)\Phi_{3,u} + \lambda_R 2\rho\beta^{1/2} \frac{\mathrm{d}\Phi_{3,u}}{\mathrm{d}\xi} \right] \Phi_{3,u} \,\mathrm{d}\xi = 0, \tag{22}$$

$$2\lambda_I \int_0^1 \left[\lambda_R \Phi_{3,u} + \rho\beta^{1/2} \frac{\mathrm{d}\Phi_{3,u}}{\mathrm{d}\xi} \right] \Phi_{3,u} \,\mathrm{d}\xi = 0. \tag{23}$$

From equation (23), it is obvious that in the case $\lambda_I \neq 0$, the factor $2\int_0^1 [\lambda_R \Phi_{3,u} + \rho\beta^{1/2}(\mathrm{d}\Phi_{3,u}/\mathrm{d}\xi)]\,\Phi_{3,u}\,\mathrm{d}\xi$ must be equal to zero: i.e.,

$$2\int_0^1 \left[\lambda_R \Phi_{3,u} + \rho\beta^{1/2} \frac{\mathrm{d}\Phi_{3,u}}{\mathrm{d}\xi} \right] \Phi_{3,u} \,\mathrm{d}\xi = 2\lambda_R \int_0^1 (\Phi_{3,u})^2 \,\mathrm{d}\xi + \int_0^1 \left[\rho\beta^{1/2} \frac{\mathrm{d}(\Phi_{3,u})^2}{\mathrm{d}\xi} \right] \mathrm{d}\xi = 0. \tag{24}$$

It is immediately clear that for $\beta \neq 0$ and $\int_0^1 (\Phi_{3,u})^2 \,\mathrm{d}\xi \neq 0$ equation (24) can be re-written as

$$\lambda_R = -\left\{ \rho\beta^{1/2} \int_0^1 \left[\frac{\mathrm{d}(\Phi_{3,u})^2}{\mathrm{d}\xi} \right] \mathrm{d}\xi \right\} \Big/ \left\{ 2\int_0^1 (\Phi_{3,u})^2 \,\mathrm{d}\xi \right\}$$
$$= -\,\rho\beta^{1/2} [\Phi_{3,u}(\xi = 1)]^2 \Big/ \left\{ 2\int_0^1 (\Phi_{3,u})^2 \,\mathrm{d}\xi \right\}. \tag{25}$$

Since ρ, β and $\int_0^1 (\Phi_{3,u})^2 \,\mathrm{d}\xi$ remain positive λ_R remains non-positive and hence flutter does not occur. Substituting the appropriate expression for $\Phi_{3,u}$ at $\theta_j = 0$ from equation (11) into equation (25) gives the following condition of neutral stability ($\lambda_R = 0$):

$$\Phi_{3,u}(\xi = 1) = \rho_{0N} \cos \rho_{0N} - \sin \rho_{N0} + \rho_{0N}^3/(3\pi)^2 = 0, \tag{26}$$

where ρ_{0N} is the value of the mass flow parameter, corresponding to the case of neutral stability. From equations (25) and (26), it follows that $\lambda_R = 0$, if

$$\rho_{0N} \approx 4{\cdot}28. \tag{27}$$

In the case where $\beta = 0$ equation (25) reduces to

$$\lambda_R \equiv 0. \tag{28}$$

Thus, the single mode approximation cannot exhibit flutter. This coincides with results in reference [2] and others. With two or more modes, however, the flutter behaviour might be expected to appear.

In a subsequent paper the authors hope to find the critical value of the mass flow parameter for instability of flutter type by using the two-mode approximation.

In the case of instability of divergence type when $\lambda_I = 0$, equation (23) becomes identically zero and equation (22) reduces to

$$\int_0^1 \left[\frac{d^4\Phi_{3,u}}{d\xi^4} + \rho^2 \frac{d^2\Phi_{3,u}}{d\xi^2} + \lambda_R^2 \Phi_{3,u} + \lambda_R 2\rho\beta^{1/2} \frac{d\Phi_{3,u}}{d\xi} \right] \Phi_{3,u}\, d\xi = \lambda_R^2 \int_0^1 (\Phi_{3,u})^2\, d\xi$$

$$+ \lambda_R \rho\beta^{1/2} \int_0^1 \left[\frac{d(\Phi_{3,u})^2}{d\xi} \right] d\xi + \int_0^1 \left[\frac{d^4\Phi_{3,u}}{d\xi^4} + \rho^2 \frac{d^2\Phi_{3,u}}{d\xi^2} \right] \Phi_{3,u}\, d\xi = 0. \tag{29}$$

From equation (29) the condition for the existence of a real pair $\lambda_R = -\, a \pm b$, where

$$a = \rho\beta^{1/2} [\Phi_{3,u}(\xi = 1)]^2 / 2 \int_0^1 (\Phi_{3,u})^2\, d\xi, \tag{30}$$

$$b = \pm \sqrt{\left\{ \frac{\rho\beta^{1/2} [\Phi_{3,u}(\xi = 1)]^2}{2\int_0^1 (\Phi_{3,u})^2\, d\xi} \right\}^2 - \frac{1}{\int_0^1 (\Phi_{3,u})^2\, d\xi} \int_0^1 \left[\frac{d^4\Phi_{3,u}}{d\xi^4} + \rho^2 \frac{d^2\Phi_{3,u}}{d\xi^2} \right] \Phi_{3,u}\, d\xi}, \tag{31}$$

has the form

$$\int_0^1 \left[\frac{d^4\Phi_{3,u}}{d\xi^4} + \rho^2 \frac{d^2\Phi_{3,u}}{d\xi^2} \right] \Phi_{3,u} \quad d\xi \leqslant 0. \tag{32}$$

Obviously, condition (32) for the existence of $\lambda_R > 0$ in the case $\beta \neq 0$ and that corresponding to the case $\beta = 0$ coincide. Substituting the appropriate expression for $\Phi_{3,u}$ at $\theta_j = 0$ from equation (11) into equation (32), integrating from $\xi = 0$ to 1 and taking into account that $\Phi_{3,u} = 0$ at $\xi = 0$, one obtains the following expression for determination of the critical value of the mass flow parameter in the case of an instability of divergence type:

$$\int_0^1 \left[\frac{d^4\Phi_{3,u}}{d\xi^4} + \rho^2 \frac{d^2\Phi_{3,u}}{d\xi^2} \right] \Phi_{3,u}\, d\xi$$

$$= \frac{\rho_{0D}^4}{2} \left[\left(\frac{\rho_{0D}}{3\pi} \right)^2 - 1 \right] \left[\left(\frac{\rho_{0D}}{3\pi} \right)^2 + 2\cos\rho_{0D} \right] - 2\rho_{0D}^3 \sin\rho_{0D} \left[\left(\frac{\rho_{0D}}{3\pi} \right)^2 - \frac{1}{2} \right] = 0. \tag{33}$$

Here ρ_{0D} is the critical value of the mass flow parameter for an instability of divergence type. Equation (33) yields

$$\rho_{0D} \approx 4{\cdot}44. \tag{34}$$

Now, admissible functions (11) and (12) can be used to obtain the expressions for the perturbation of ρ (since $\rho = \rho_0 + \theta_j^2\rho_1 + \cdots$, where $\rho_0 = \rho_{0D} = 4{\cdot}44$ for instability of divergence type) at which instability occurs. One can use equation (6.17) of reference [1]:

$$\frac{\mathrm{d}^4\Phi_1}{\mathrm{d}\xi^4} + \rho_0^2\frac{\mathrm{d}^2\Phi_1}{\mathrm{d}\xi^2} + \lambda_0^2\Phi_1 + 2\rho_0\beta^{1/2}\lambda_0\frac{\mathrm{d}\Phi_1}{\mathrm{d}\xi} = -\rho_1\left[2\rho_0\frac{\mathrm{d}^2\Phi_0}{\mathrm{d}\xi^2} + 2\beta^{1/2}\lambda_0\frac{\mathrm{d}\Phi_0}{\mathrm{d}\xi}\right]$$

$$-\lambda_1\left[2\lambda_0\Phi_0 + 2\rho_0\beta^{1/2}\frac{\mathrm{d}\Phi_0}{\mathrm{d}\xi}\right] - L_1(\rho_0,\lambda_0)\Phi_0(\xi). \tag{35}$$

Then, for example, for the out-of-plane problem with $\mathrm{Re}\,\lambda_0 = 0$ and $\mathrm{Re}\,\lambda_1 = 0$ equation (35) can be written as

$$\frac{\mathrm{d}^4\Phi_{3,v}^1}{\mathrm{d}\xi^4} + \rho_0^2\frac{\mathrm{d}^2\Phi_{3,v}^1}{\mathrm{d}\xi^2} - (\mathrm{Im}\,\lambda_0)^2\Phi_{3,v}^1 + \mathrm{i}2\rho_0\beta^{1/2}(\mathrm{Im}\,\lambda_0)\frac{\mathrm{d}\Phi_{3,v}^1}{\mathrm{d}\xi}$$

$$= -\rho_{1,v}\left[2\rho_0\frac{\mathrm{d}^2\Phi_{3,v}^0}{\mathrm{d}\xi^2} + \mathrm{i}2\beta^{1/2}(\mathrm{Im}\,\lambda_0)\frac{\mathrm{d}\Phi_{3,v}^0}{\mathrm{d}\xi}\right] + 2(\mathrm{Im}\,\lambda_0)(\mathrm{Im}\,\lambda_1)\Phi_{3,v}^0$$

$$-\mathrm{i}2\rho_0\beta^{1/2}(\mathrm{Im}\,\lambda_1)\frac{\mathrm{d}\Phi_{3,v}^0}{\mathrm{d}\xi} - \frac{\mathrm{d}}{\mathrm{d}\xi}\left[-\frac{1}{2}\rho_0^2\frac{\mathrm{d}\Phi_{3,v}^0}{\mathrm{d}\xi} + \frac{3}{2}\left(\frac{\mathrm{d}\Phi_0}{\mathrm{d}\xi}\right)^2\frac{\mathrm{d}\Phi_{3,v}^0}{\mathrm{d}\xi}\right]. \tag{36}$$

Next, separating the real and imaginary parts of equation (36), one obtains

$$\frac{\mathrm{d}^4\Phi_{3,v}^1}{\mathrm{d}\xi^4} + \rho_0^2\frac{\mathrm{d}^2\Phi_{3,v}^1}{\mathrm{d}\xi^2} - (\mathrm{Im}\,\lambda_0)^2\Phi_{3,v}^1 + \rho_{1,v}\left[2\rho_0\frac{\mathrm{d}^2\Phi_{3,v}^0}{\mathrm{d}\xi^2}\right] - 2(\mathrm{Im}\,\lambda_0)(\mathrm{Im}\,\lambda_1)\Phi_{3,v}^0$$

$$+\frac{\mathrm{d}}{\mathrm{d}\xi}\left[-\frac{1}{2}\rho_0^2\frac{\mathrm{d}\Phi_{3,v}^0}{\mathrm{d}\xi} + \frac{3}{2}\left(\frac{\mathrm{d}\Phi_0}{\mathrm{d}\xi}\right)^2\frac{\mathrm{d}\Phi_{3,v}^0}{\mathrm{d}\xi}\right] = 0, \tag{37}$$

$$2\rho_0\beta^{1/2}(\mathrm{Im}\,\lambda_0)\frac{\mathrm{d}\Phi_{3,v}^1}{\mathrm{d}\xi} + \rho_{1,v}\left[2\beta^{1/2}(\mathrm{Im}\,\lambda_0)\frac{\mathrm{d}\Phi_{3,v}^0}{\mathrm{d}\xi}\right] + 2\rho_0\beta^{1/2}(\mathrm{Im}\,\lambda_1)\frac{\mathrm{d}\Phi_{3,v}^0}{\mathrm{d}\xi} = 0. \tag{38}$$

By using appropriate expressions for $\Phi_{3,v}^0$, $\Phi_{3,v}^1$ from equation (12), equation (37) can be re-written as

$$\frac{\mathrm{d}^4\Phi_{3,v}^1}{\mathrm{d}\xi^4} + \rho_0^2\frac{\mathrm{d}^2\Phi_{3,v}^1}{\mathrm{d}\xi^2} - (\mathrm{Im}\,\lambda_0)^2\Phi_{3,v}^1 + \rho_{1,v}\left[2\rho_0\frac{\mathrm{d}^2\Phi_{3,v}^0}{\mathrm{d}\xi^2}\right] - 2(\mathrm{Im}\,\lambda_0)(\mathrm{Im}\,\lambda_1)\Phi_{3,v}^0$$

$$-\frac{1}{2}\rho_0^2\frac{\mathrm{d}^2\Phi_{3,v}^0}{\mathrm{d}\xi^2} + 3\frac{\mathrm{d}\Phi_0}{\mathrm{d}\xi}\frac{\mathrm{d}^2\Phi_0}{\mathrm{d}\xi^2}\frac{\mathrm{d}\Phi_{3,v}^0}{\mathrm{d}\xi} + \frac{3}{2}\left(\frac{\mathrm{d}\Phi_0}{\mathrm{d}\xi}\right)^2\frac{\mathrm{d}^2\Phi_{3,v}^0}{\mathrm{d}\xi^2}$$

$$
\begin{aligned}
&= - C_v\theta_j^2(3\pi)^4\sin(3\pi\xi) + C_v\theta_j^2(3\pi)^2\rho_0^2\sin(3\pi\xi) \\
&- (\operatorname{Im}\lambda_0)^2 C_v\theta_j^2[3\pi\xi - \sin(3\pi\xi)] + 2\rho_0\rho_1\left[- \rho_0^2\sin\rho_0(1-\xi) + \frac{\rho_0^3}{3\pi}\sin(3\pi\xi)\right] \\
&- 2(\operatorname{Im}\lambda_0)(\operatorname{Im}\lambda_1)\Bigg\{\sin\rho_0(1-\xi) + \rho_0\xi\cos\rho_0 \\
&+ \frac{\rho_0^3}{(3\pi)^3}[3\pi\xi - \sin(3\pi\xi)] - \sin\rho_0\Bigg\} - \frac{\rho_0^2}{2}\left[- \rho_0^2\sin\rho_0(1-\xi) + \frac{\rho_0^3}{3\pi}\sin(3\pi\xi)\right] \\
&- 3\rho_0^3\sin\rho_0(1-\xi)\cos\rho_0(1-\xi)\Bigg\{ - \rho_0\cos\rho_0(1-\xi) + \rho_0\cos\rho_0 \\
&+ \frac{\rho_0^3}{(3\pi)^2}[1 - \cos(3\pi\xi)]\Bigg\} + \frac{3}{2}\rho_0^2\sin^2\rho_0(1-\xi)\Bigg[- \rho_0^2\sin\rho_0(1-\xi) \\
&+ \frac{\rho_0^3}{3\pi}\sin(3\pi\xi)\Bigg] = 0.
\end{aligned}
\tag{39}
$$

Then, for $\xi = 0$, it is easily verified that equation (39) can be written as

$$
\begin{aligned}
&- C_v\theta_j^2(3\pi)^4\sin(3\pi 0) + C_v\theta_j^2(3\pi)^2\rho_0^2\sin(3\pi 0) \\
&- (\operatorname{Im}\lambda_0)^2 C_v\theta_j^2[3\pi 0 - \sin(3\pi 0)] + 2\rho_0\rho_1\left[- \rho_0^2\sin\rho_0(1-0) + \frac{\rho_0^3}{3\pi}\sin(3\pi 0)\right] \\
&- 2(\operatorname{Im}\lambda_0)(\operatorname{Im}\lambda_1)\left\{\sin\rho_0(1-0) + \rho_0 0\cos\rho_0 + \frac{\rho_0^3}{(3\pi)^3}[3\pi 0 - \sin(3\pi 0)] - \sin\rho_0\right\} \\
&- \frac{\rho_0^2}{2}\left[- \rho_0^2\sin\rho_0(1-0) + \frac{\rho_0^3}{3\pi}\sin(3\pi 0)\right] \\
&- 3\rho_0^3\sin\rho_0(1-0)\cos\rho_0(1-0)\left\{ - \rho_0\cos\rho_0(1-0) + \rho_0\cos\rho_0 + \frac{\rho_0^3}{(3\pi)^2}[1-\cos(3\pi 0)]\right\} \\
&+ \frac{3}{2}\rho_0^2\sin^2\rho_0(1-0)\left[- \rho_0^2\sin\rho_0(1-0) + \frac{\rho_0^3}{3\pi}\sin(3\pi 0)\right] = - 2\rho_0^3\rho_{1,v}\sin\rho_0 \\
&+ \frac{\rho_0^4}{2}\sin\rho_0 - \frac{3}{2}\rho_0^4\sin^3\rho_0 = 0.
\end{aligned}
\tag{40}
$$

Next, assuming $\sin\,\rho_0 \neq 0$, one can re-write equation (40) as

$$
\rho_{1,v} = \frac{\rho_0}{4}(1 - 3\sin^2\rho_0).
\tag{41}
$$

Solving this equation for the case of instability of divergence type results in

$$\rho_{1,v}^{D} \approx -1{\cdot}98. \tag{42}$$

By following a similar procedure, the critical value of the perturbation of ρ at which instability occurs with in-plane motions can also be obtained:

$$\rho_{1,u} = \frac{\rho_0}{4}(1 - 9\sin^2 \rho_0). \tag{43}$$

Substituting the value of ρ_0 obtained for the case of an instability of divergence type into equation (43) gives

$$\rho_{1,u}^{D} \approx -\,8{\cdot}16 \tag{44}$$

Thus, since $\rho_{1,u}^{D} < \rho_{1,v}^{D}$ the tube becomes unstable to u perturbations as the flow parameter is increased slowly from zero.

REFERENCES

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