Journal of Sound and Vibration (2001) **245**(1), 113–132
doi:10.1006/jsvi.2000.3481, available online at http://www.idealibrary.com on IDEAL®

MODAL OPTIMAL CONTROL PROCEDURE FOR NEAR DEFECTIVE SYSTEMS

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(*Received* 15 *May* 2000, *and in final form* 7 *December* 2000)

This study discusses a modal optimal control procedure for defective systems with repeated eigenvalues. From the view point of mathematics, although near defective close eigenvalues are distinct, the characteristic of the system is also defective. Therefore, we have to transform near defective systems into the defective one, and then modal optimal control procedure for the defective systems can be extended to deal with the corresponding problems for near defective systems with close eigenvalues. Because of the defective characteristic of the system, we have to use an invariant sub-space recursive method with numerical stability to calculate the generalized modes of the defective and near defective systems. The Potter's approach is extended to solve the Riccati equations in the generalized model subspace of the defective system. Because the order of the Jordan block matrix of the defective eigenvalues, m, is much smaller than that of the state matrix, n, i.e., $m \ll n$, the present modal optimal control procedure is very simple and reduces the computing effort for the complex system with large number of degrees of freedom. A numerical example is given to illustrate and verify the validity of the procedure.

1. INTRODUCTION

In vibration optimal control, the continuous Riccati equation plays a fundamental and important role. Many methods for solving the Riccati equation have been proposed. The main algorithm includes the matrix transformation [1, 2], the eigenvector method [2, 3], the Schur method [1] and the iterative algorithm [4], and so on. For the matrix transformation method, the price for replacing the solution of a set of linear equations is to double the order of the set; for the eigenvector method, it needs to solve all the eigensolutions of the matrix with order $2n$ which may be impossible if the order n is very large; for the iterative algorithm, it is simple, but its results depend on the selection of the initial value.

Recently, reference [5] presents a new block simultaneous iterative algorithm for solving the Riccati equation using its special feature in optimal shape control. Because in this algorithm the three equations are solved simultaneously and some items are common ones, it can both save the computer memory and raise the computing efficiency.

The standard modal control theory can be found in reference [8]. However, the above discussions on the modal control mainly involve the control problem of the non-defective system, which has the complete eigenvectors to span the eigenspace, i.e., the state matrix **A** can be diagonalized. However, in actual engineering problems, such as general damping systems, flutter analysis of aeroelasticity, and so on, the system which is called a defective system has no set of complete eigenvectors to space the eigenspace [6]. In these special cases, the state matrix **A** cannot be diagonalized. It is well known that the defective system with repeated eigenvalues is ill-conditioned because the dynamic characteristic is very

0022-460X/01/310113 + 20 $35.00/0

sensitive to the changes of parameters of the defective system with repeated eigenvalues, and it can be changed into a near defective system with close eigenvalues [7]. Therefore, the difficulty arises for designing the modal control of the defective or near defective system. The major difficulty is that the generalized right and left modal matrices, **U** and **V**, cannot be obtained with the standard methods for extracting the modal matrix, and that from the view point of mathematics, the close eigenvalues of near defective systems are distinct, but the dynamic characteristic is still defective. For these reasons, the standard methods for obtaining the feedback matrix presented by references [1–5] cannot be directly used to deal with the modal control problems of the defective and near defective systems.

This study will present the modal control procedure for the defective system with repeated eigenvalues based on the modal control equations. For a near defective system, we first transform it into a defective one, and then apply the same method to deal with the near defective system. The theory is illustrated by a numerical example to prove the validity.

We start with a brief review of the generalized modal theory of the defective system with repeated eigenvalues [6], and then give a procedure for extracting the generalized right and left modal matrices, **U** and **V**. Finally, the Potter's algorithm for solving the Riccati equation is extended to deal with the defective and near defective system.

2. GENERALIZED MODAL THEORY OF THE DEFECTIVE SYSTEMS

Consider linear vibrational equation

$$\mathbf{M}\ddot{\mathbf{x}} + (\mathbf{D} + \mathbf{G})\dot{\mathbf{x}} + (\mathbf{K} + \mathbf{H})\mathbf{x} = 0, \tag{1}$$

where it is assumed that **M**, **D**, **G**, **K**, and **H** are real matrices, **M**, **D**, **K** are symmetric matrices, $\mathbf{M} = \mathbf{M}^{\mathrm{T}}$, $\mathbf{D} = \mathbf{D}^{\mathrm{T}}$, $\mathbf{K} = \mathbf{K}^{\mathrm{T}}$, corresponding to mass, damping, and stiffness, and **G**, and **H** are skew-symmetric matrices, $\mathbf{G}^{\mathrm{T}} = -\mathbf{G}$, $\mathbf{H}^{\mathrm{T}} = -\mathbf{H}$, corresponding to gyroscopic and circulatory (or non-conservative positional) forces, and **M** is assumed to be positive definite. The eigenvalue problem is as follows:

$$(\mathbf{M}\lambda^2 + (\mathbf{D} + \mathbf{G})\lambda + (\mathbf{K} + \mathbf{H}))\mathbf{x} = 0. \tag{2}$$

Using the state vector

$$\mathbf{u} = \begin{bmatrix} \lambda\mathbf{x} \\ \mathbf{x} \end{bmatrix} \tag{3}$$

one has

$$\mathbf{A}\mathbf{u} = \lambda\mathbf{u}, \tag{4}$$

where

$$\mathbf{A} = \begin{bmatrix} -\mathbf{M}^{-1}(\mathbf{D} + \mathbf{G}) & -\mathbf{M}^{-1}(\mathbf{K} + \mathbf{H}) \\ \mathbf{I} & 0 \end{bmatrix}. \tag{5}$$

In equation (5) **I** is the unit matrix and **O** is the zero matrix of the same order.

It is assumed that AM is used to denote the algebra multiplicity of the eigenvalue λ in equation (4), and GM is used to denote the number of the linear independent eigenvectors corresponding to λ. If $AM = GM$ for the distinct or repeated eigenvalues, the system is non-defective; if $AM > GM$, the system with repeated eigenvalues is defective.

From the algebra theory for the defective matrix $\mathbf{A}$, there exists non-singular matrix $\mathbf{U}$, such that

$$\mathbf{AU} = \mathbf{UJ}, \tag{6}$$

where $\mathbf{U}$ is the generalized modal matrix of $\mathbf{A}$, $\mathbf{J}$ is the Jordan block of $\mathbf{A}$ given by

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & & & \\ & \mathbf{J}_2 & & \\ & & \ddots & \\ & & & \mathbf{J}_r \end{bmatrix}, \tag{7}$$

$$\mathbf{J}_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}_{m_i \times m_i} \qquad \sum_{i=1}^{r} m_i = n. \tag{8}$$

Equation (6) can be written in the following manner:

$$\begin{aligned} &(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{u}_1^{(i)} = 0 \\ &(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{u}_j^{(i)} = \mathbf{u}_{j-1}^{(i)}, \quad j = 2, 3, \ldots, m_i, \\ &\quad \cdots \cdots \qquad\qquad\qquad i = 1, 2, \ldots, r. \end{aligned} \tag{9}$$

The conjugate and transpose of $\mathbf{A}$ is called adjoint system, i.e., for $\mathbf{A}^{\mathrm{H}}$ the generalized modes satisfy the following equation:

$$\mathbf{A}^{\mathrm{H}}\mathbf{V} = \mathbf{V}\mathbf{J}^{\mathrm{H}}, \tag{10}$$

where $\mathbf{A}^{\mathrm{H}}$ and $\mathbf{J}^{\mathrm{H}}$ are the conjugate and transpose of $\mathbf{A}$ and $\mathbf{J}$, respectively, $\mathbf{V}$ is the generalized modal matrix of the $\mathbf{A}^{\mathrm{H}}$.

Equation (10) can be also written as follows:

$$\begin{aligned} &(\mathbf{A}^{\mathrm{H}} - \tilde{\lambda}_i \mathbf{I})\mathbf{v}_j^{(i)} = \mathbf{v}_{j+1}^{(i)}, \quad j = 1, 2, 3, \ldots, m_{i-1}, \\ &(\mathbf{A}^{\mathrm{H}} - \tilde{\lambda}_i \mathbf{I})\mathbf{v}_{mi}^{(i)} = 0, \qquad i = 1, 2, \ldots, r, \end{aligned} \tag{11}$$

where $\tilde{\lambda}_i$ is the conjugate of λ_i. In general, $\mathbf{u}_i (i = 1, 2, \ldots, r)$ are known as the right eigenvectors, $\mathbf{v}_i (i = 1, 2, \ldots, r)$ are known as the left eigenvectors, $\mathbf{u}_{i+1}, \ldots, \mathbf{u}_{i+m_i-1}$ and $\mathbf{v}_{i+1}, \ldots, \mathbf{v}_{i+m_i-1}$ are the right and left generalized modes corresponding to λ_i respectively.

The right generalized modal matrix $\mathbf{U}$ and the left generalized modal matrix $\mathbf{V}$ satisfy the following orthogonal condition:

$$\mathbf{V}^{\mathrm{H}}\mathbf{U} = \mathbf{I}. \tag{12}$$

3. INVARIANT SUBSPACE RECURSIVE METHOD FOR COMPUTING THE GENERALIZED MODES

It can be seen that if the system is non-defective, the eigenspace can be obtained by using the normal methods such as the Gauss elimination for solving the linear equations; if the system is defective or near defective, i.e., the eigenspace is incomplete or near incomplete, fatal mistakes may occur while computing the generalized modes. Therefore, it is very important to give a reliable method for computing the generalized modes corresponding to the defective eigenvalues λ_i.

In this section, we discuss a reliable method with digital stability, the invariant subspace recursive method, to calculate the generalized modes of defective or near defective system.

It should be noted that equation (9) is equivalent to the following equations:

$$(\mathbf{A} - \lambda_i \mathbf{I}) u_1^{(i)} = 0,$$

$$(\mathbf{A} - \lambda_i \mathbf{I})^2 u_2^{(i)} = 0,$$

$$\cdots\cdots\cdots$$

$$(\mathbf{A} - \lambda_i \mathbf{I})^{(m_i)} u_{m_i}^{(i)} = 0,$$

where the eigenvector $u_1^{(i)}$ is known as the first order generalized mode, $u_2^{(i)}, u_3^{(i)}, \ldots, u_{m_i}^{(i)}$ are known as the second, third, ..., m_ith order generalized modes of λ_i respectively.

According to these definitions, to determine the generalized modal subspace is equivalent to determining the first, second, third, ..., order generalized modes of λ_i to span the generalized modal subspace.

Assume that λ_i is the eigenvalues of $\mathbf{A}$ and $\mathbf{A}$ is the defective or near defective matrix. If the dimension of the null space, $Ker(\mathbf{A} - \lambda_i \mathbf{I})$, is t_1, the t_1 linearly independent vectors can be found from the following equation:

$$(\mathbf{A} - \lambda_i \mathbf{I}) u_1^{(i)} = 0. \tag{13}$$

If the t_1 orthogonal vectors $u_{1,1}^{(i)}, u_{1,2}^{(i)}, \ldots, u_{1,t_1}^{(i)}$, have been found, we use $\mathbf{D}_1$ to denote the invariant subspace spanned by $u_{1,1}^{(i)}, u_{1,2}^{(i)}, \ldots, u_{1,t_1}^{(i)}$. Since the elements in $\mathbf{D}_1$ are the solutions of equation (13), $\mathbf{D}_1$ is the eigensubspace of λ_i.

From equation (8) we have $t_1 < m_i$, otherwise, $\mathbf{J}_{m_i}$ is a diagonal matrix. Next, we turn to the equation

$$(\mathbf{A} - \lambda_i \mathbf{I})^2 u_2^{(i)} = 0. \tag{14}$$

Assume that the dimension of $Ker(\mathbf{A} - \lambda_i \mathbf{I})^2$ is $t_2 (t_2 > 0)$. Similarly, there exists a set of orthogonal basis $u_{2,1}^{(i)}, u_{2,2}^{(i)}, \ldots, u_{2,t_2}^{(i)}$, which are known as the second order generalized modes. The basic vectors of $\mathbf{D}_1$ can be extended to

$$u_{1,1}^{(i)}, u_{1,2}^{(i)}, \ldots, u_{1,t_1}^{(i)}; \quad u_{2,1}^{(i)}, u_{2,2}^{(i)}, \ldots, u_{2,t_2}^{(i)}. \tag{15}$$

Therefore, the invariant subspace $\mathbf{D}_2$ spanned by the set of equation (15) contains all of the first and second order generalized modes. It can be shown that $\mathbf{D}_1 \subset \mathbf{D}_2$ and $t_2 \leqslant t_1$.

According to the above recursive procedure, we get the following condition for termination of the computation:

$$Ker(\mathbf{A} - \lambda \mathbf{I})^k = Ker(\mathbf{A} - \lambda \mathbf{I})^{k+1}. \tag{16}$$

In such a way we obtain a set of the orthogonal basis vectors

$$u_{1,1}^{(i)}, u_{1,2}^{(i)}, \ldots, u_{1,t_1}^{(i)}$$

$$u_{1,1}^{(i)}, u_{1,2}^{(i)}, \ldots, u_{1,t_1}^{(i)}; \quad u_{2,1}^{(i)}, u_{2,2}^{(i)}, \ldots, u_{2,t_2}^{(i)}$$

$$\cdots\cdots\cdots$$

$$u_{1,1}^{(i)}, u_{1,2}^{(i)}, \ldots, u_{1,t_1}^{(i)}; \ \cdots u_{k,1}^{(i)}, u_{k,2}^{(i)}, \ldots, u_{k,t_k}^{(i)}$$

and corresponding a set of the invariant-subspaces, $\mathbf{D}_1, \mathbf{D}_2, \ldots, \mathbf{D}_k$, they satisfy

$$(1)\ t_k \leqslant t_{k-1} \leqslant \cdots \leqslant t_2 \leqslant t_1;$$

$$(2)\ \mathbf{D}_1 \subset \mathbf{D}_2 \subset \cdots \subset \mathbf{D}_k.$$

Thus, $\mathbf{D}_k$ obtained by the recursive procedure is the generalized modal subspace we need.

The invariant subspace recursive procedure for computing the generalized modes of the defective systems is summarized as follows:

(1) Form $\mathbf{A}$ and compute $\lambda_i (i = 1, 2, \ldots, m_i)$ which are repeated defective eigenvalues;
(2) Let $\mathbf{M} = \mathbf{A} - \lambda_i \mathbf{I}$, $\Psi = \mathbf{I}$, $t = 0$;
(3) Identity if $m_i = t$ turn to 8, otherwise, turn to 4;
(4) Take the singular value decomposition of $\mathbf{M}$

$$\mathbf{M} = \Phi^{\mathrm{H}} \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_{t_1} & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} \Psi.$$

(5) Compute

$$\mathbf{M} = \tilde{\Psi}^{\mathrm{H}} \mathbf{M} \tilde{\Psi},$$

$$\tilde{\Psi} = \begin{bmatrix} \Psi & \\ & I \end{bmatrix}.$$

(6) Record $\Psi = \Psi \tilde{\Psi}$;
(7) $t = t + t1$, turn to 3;
(8) Output Ψ, stop.

Here, Ψ is the right generalized modes $\mathbf{U}$ that we need. The details of the invariant subspace recursive procedure can be found in reference [7].

4. MODAL OPTIMAL CONTROL ALGORITHM FOR SOLVING RICCATI EQUATIONS OF THE DEFECTIVE SYSTEMS

4.1. THE POTTER ALGORITHM [2]

Consider the control system indicated by the following state equation:

$$\begin{aligned} \dot{\mathbf{X}}(t) &= \mathbf{A}\mathbf{X} + \mathbf{B}Z(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{X}(t), \end{aligned} \tag{17}$$

where the state matrix $\mathbf{A}$ as given by equation (5). $\mathbf{X}(t) \in \mathbf{R}^{n \times 1}$ is the state vector, $Z(t)$ is the input, $\mathbf{y}(t) \in \mathbf{R}^{n \times 1}$ is the output vector, $\mathbf{B} \in \mathbf{R}^{n \times 1}$ and $\mathbf{C} \in \mathbf{R}^{q \times n}$ were called the actuator distribution matrix and sensor distribution matrix, respectively, indicating the locations of control forces and sensors.

The object is to determine an optimal control minimizing the quadratic performance measure [8].

$$J = \frac{1}{2}\mathbf{X}^{\mathrm{T}}(t_f)\mathbf{H}\mathbf{X}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} [\mathbf{X}^{\mathrm{T}}(t)\mathbf{Q}\mathbf{X}(t) + Z^{\mathrm{T}}(t)\mathbf{R}Z(t)]\,\mathrm{d}t, \tag{18}$$

where $\mathbf{H}$ and $\mathbf{Q}$ are real symmetric positive semidefinite matrices and $\mathbf{R}$ is a real symmetric positive-definite matrix. We assume that $\mathbf{X}(t_f)$ is free and t_f is fixed. The optimal control problem using the performance measure (18) can be interpreted as the problem of driving the initial state as close as possible to zero. The optimal feedback control gain matrix has the form

$$\mathbf{G} = \mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\tilde{\mathbf{K}}(t), \tag{19}$$

where $\tilde{\mathbf{K}}(t)$ is an $n \times n$ Riccati matrix and satisfies the Riccati equation

$$\dot{\tilde{\mathbf{K}}}(t) = -\mathbf{Q} - \mathbf{A}^{\mathrm{T}}(t)\tilde{\mathbf{K}}(t) - \tilde{\mathbf{K}}(t)\mathbf{A}(t) + \tilde{\mathbf{K}}(t)\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\tilde{\mathbf{K}}(t) \tag{20}$$

and subject to the boundary condition

$$\tilde{\mathbf{K}}(t_f) = \mathbf{H}(t_f) = \mathbf{H}. \tag{21}$$

If $\mathbf{A}$, $\mathbf{B}$, $\mathbf{Q}$, and $\mathbf{R}$ are constant, the Riccati matrix approaches a constant value as the final time increases without bounds, $\tilde{\mathbf{K}}(t) \to \tilde{\mathbf{K}} =$ constant as $t_f \to \infty$. In this case, the matrix Riccati equation, equation (20), reduces to

$$-\mathbf{Q} - \mathbf{A}^{\mathrm{T}}\tilde{\mathbf{K}} - \tilde{\mathbf{K}}\mathbf{A} + \tilde{\mathbf{K}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\tilde{\mathbf{K}} = 0, \tag{22}$$

which constitutes a set of algebraic equations called the steady state matrix Riccati equation.

Using the Potter's algorithm, the solution of equation (22) can be reduced to an algebraic eigenvalue problem as follows [2]:

$$\mathbf{M}_1 \begin{bmatrix} \mathbf{E} \\ \cdots \\ \mathbf{F} \end{bmatrix} = \begin{bmatrix} \mathbf{E} \\ \cdots \\ \mathbf{F} \end{bmatrix} \mathbf{J}_1, \tag{23}$$

where

$$\mathbf{M}_1 = \begin{bmatrix} \mathbf{A}^{\mathrm{T}} & \vdots & \mathbf{Q} \\ \cdots\cdots & \vdots & \cdots \\ \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}} & \vdots & -\mathbf{A} \end{bmatrix}. \tag{24}$$

Solving eigenproblem (23), we can obtain the eigenvalues with positive real parts and the corresponding eigenvector matrix $\begin{bmatrix} \mathbf{E} \\ \cdots \\ \mathbf{F} \end{bmatrix}$. The steady state solution of the matrix Riccati equation (21) can be obtained as

$$\tilde{\mathbf{K}} = \mathbf{E}\mathbf{F}^{-1}. \tag{25}$$

Equation (24) indicates that the Potter's algorithm requires to compute all the eigenvalues and corresponding eigenvectors of $\mathbf{M}_1$ of order $2n$. If the degrees of freedom of the large complex system, n, is very large, the use of the Potter's algorithm is difficult.

4.2. MODAL OPTIMAL CONTROL ALGORITHM FOR SOLVING THE RICCATI EQUATION OF THE DEFECTIVE SYSTEMS

Assume that $\lambda_1 = \lambda_2 = \cdots = \lambda_m = \lambda$ are defective eigenvalues of $\mathbf{A}$, and rest of the eigenvalues $\lambda_{m+1}, \ldots, \lambda_n$ are distinct, the corresponding right and left generalized modal are $\mathbf{U}$ and $\mathbf{V}$, which can be obtained by the invariant subspace recursive method given by section 3.

Transforming equation (17) into the generalized modal co-ordinates through the following modal transformation

$$x(t) = \mathbf{U}\xi(t) \tag{26}$$

yields

$$\begin{bmatrix} \dot{\xi}_m \\ \cdots \\ \dot{\xi}_d \end{bmatrix} = \begin{bmatrix} \mathbf{J}_m & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & \mathbf{\Lambda}_d \end{bmatrix} \begin{bmatrix} \xi_m \\ \cdots \\ \xi_d \end{bmatrix} + \begin{bmatrix} \mathbf{V}_m^{\mathrm{H}} \\ \cdots \\ \mathbf{V}_d^{\mathrm{H}} \end{bmatrix} \mathbf{B}Z(t), \tag{27}$$

where $\mathbf{J}_m$ is the Jordan form matrix with m defective repeated eigenvalue λ, $\mathbf{\Lambda}_d$ the diagonal matrix with $(n-m)$ distinct eigenvalues

$$J_m = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}_{m \times m}, \tag{28}$$

$$\mathbf{\Lambda}_d = \begin{bmatrix} \lambda_{m+1} & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}_{(n-m)\times(n-m)}, \tag{29}$$

In equation (27), the right and left generalized modal matrices can be expressed as the partitional form

$$\mathbf{U} = [\mathbf{U}_m, \mathbf{U}_{n-m}], \qquad \mathbf{V} = [\mathbf{V}_m, \mathbf{V}_{n-m}]. \tag{30, 31}$$

ξ_m and ξ_d donate the generalized modal co-ordinates corresponding to the defective repeated and distinct eigenvalues, respectively.

From equation (27) we obtain the modal control equations corresponding to the defective repeated eigenvalues and distinct eigenvalues

$$\dot{\xi}_m = \mathbf{J}_m \xi_m + \mathbf{V}_m^{\mathrm{H}} \mathbf{B} z_m(t), \tag{32}$$

$$\dot{\xi}_d = \mathbf{\Lambda}_d \xi_d + \mathbf{V}_{n-m}^{\mathrm{H}} \mathbf{B} z_d(t). \tag{33}$$

If the following notations are introduced:

$$\mathbf{P}_m = \mathbf{V}_m^{\mathrm{H}} \mathbf{B}, \quad \mathbf{P}_d = \mathbf{V}_{n-m}^{\mathrm{H}} \mathbf{B}. \tag{34}$$

Equation (32) and (33) become

$$\dot{\xi}_m = \mathbf{J}_m\xi_m + \mathbf{P}_m z_m(t), \tag{35}$$

$$\dot{\xi}_d = \mathbf{\Lambda}_d\xi_d + \mathbf{P}_d z_d(t). \tag{36}$$

Equations (35) and (36) indicate that the state control equation (27) has been transferred into two set of modal control equations in terms of the modal transformation, the first is in the modal subspace corresponding to the defective repeated eigenvalues, the second is in the modal subspace corresponding to the rest of the distinct eigenvalues. Since $\mathbf{J}_m$ is the Jordan form matrix, equation (35) is still coupled, $\mathbf{J}_m$ is an order of $m \times m$, $m \ll n$. Therefore, the solution based on equation (35) for optimal control is simpler than that of the original state control equation (27), and the conventional algorithm for optimal control, such as the Potter's algorithm, can be conveniently applied. In the following section, we extend the Potter's algorithm to deal with the optimal control based on the modal control equation (35) in the modal subspace corresponding to the defective repeated eigenvalues.

The object is to determine an optimal control minimizing the modal quadratic performance measure based on equation (35)

$$J_m = \frac{1}{2}\xi_m^{\mathrm{T}}(t_f)\mathbf{H}_m\xi_m(t) + \frac{1}{2}\int_{t_0}^{t_f}[\xi_m^{\mathrm{T}}(t)\mathbf{Q}_m\xi_m + z_m^{\mathrm{T}}(t)\mathbf{R}_m z_m(t)]\,\mathrm{d}t. \tag{37}$$

Hence, the optimal modal control is given by

$$\begin{aligned} z_m(t) &= -\mathbf{R}_m^{-1}\mathbf{P}_m^{\mathrm{T}}\mathbf{K}_m(t)\xi_m(t) \\ &= \mathbf{G}_m\xi_m(t), \end{aligned} \tag{38}$$

where the modal gain matrix $\mathbf{G}_m$ is given by

$$\mathbf{G}_m = -\mathbf{R}_m^{-1}\mathbf{P}_m^{\mathrm{T}}\mathbf{K}_m(t). \tag{39}$$

The modal Riccati matrix, $(\mathbf{K}_m)_{m\times m}$, satisfies the Riccati equation

$$\dot{\mathbf{K}}_m(t) = -\mathbf{Q}_m - \mathbf{J}_m^{\mathrm{T}}\mathbf{K}_m(t) - \mathbf{K}_m(t)\mathbf{J}_m + \mathbf{K}_m(t)\mathbf{P}_m\mathbf{R}_m^{-1}\mathbf{P}_m^{\mathrm{T}}\mathbf{K}_m(t). \tag{40}$$

If $\mathbf{Q}_m$, $\mathbf{R}_m$, $\mathbf{J}_m$ and $\mathbf{P}_m$ are constant, the modal Riccati matrix $\mathbf{K}_m(t) \to \mathbf{K}_m = \text{constant}$, as the $t_f \to \infty$. In this case, equation (40) becomes

$$-\mathbf{Q}_m - \mathbf{J}_m^{\mathrm{T}}\mathbf{K}_m - \mathbf{K}_m\mathbf{J}_m + \mathbf{K}_m\mathbf{P}_m\mathbf{R}_m^{-1}\mathbf{P}_m^{\mathrm{T}}\mathbf{K}_m = 0. \tag{41}$$

If the Potter's algorithm is used to solve equation (41) for $\mathbf{K}_m$, we have to solve the following eigenproblem and retain the eigenvalues with positive real parts:

$$\begin{bmatrix} \mathbf{J}_m^{\mathrm{T}} & \vdots & \mathbf{Q}_m \\ \cdots\cdots & \vdots & \cdots \\ \mathbf{P}_m\mathbf{R}_m^{-1}\mathbf{P}_m^{\mathrm{T}} & \vdots & -\mathbf{J}_m \end{bmatrix} \begin{bmatrix} \mathbf{E}_m \\ \cdots \\ \mathbf{F}_m \end{bmatrix} = \begin{bmatrix} \mathbf{E}_m \\ \cdots \\ \mathbf{F}_m \end{bmatrix} \mathbf{J}_1. \tag{42}$$

The steady state solution of the modal Riccati matrix equation (41) is as follows:

$$\mathbf{K}_m = \mathbf{E}_m\mathbf{F}_m^{-1}. \tag{43}$$

Equation (42) shows that if the Potter's algorithm is extended to that of the modal subspace corresponding to m repeated eigenvalues of the defective system, only $2m$ eigenvalues and eigenvectors are required for computation in equation (42).

Using equation (39) we obtain the modal gain matrix

$$\mathbf{G}_m = -\mathbf{R}_m^{-1}\mathbf{P}_m^{\mathrm{T}}\mathbf{E}_m\mathbf{F}_m^{-1}. \tag{44}$$

Using equation (38), the optimal modal feedback control is given by

$$\begin{aligned} z_m(t) &= -\mathbf{R}_m^{-1}\mathbf{P}_m^{\mathrm{T}}\mathbf{E}_m\mathbf{F}_m^{-1}\xi_m(t) \\ &= \mathbf{G}_m\xi_m(t). \end{aligned} \tag{45}$$

Substituting equation (45) into equation (32), we obtain the closed-loop modal equation in the modal subspace corresponding to the m repeated eigenvalues

$$\begin{aligned} \dot{\xi}_m &= \mathbf{J}_m\xi_m + \mathbf{V}_m^{\mathrm{H}}\mathbf{B}z_m(t) \\ &= (\mathbf{J}_m - \mathbf{V}_m^{\mathrm{H}}\mathbf{B}\mathbf{R}_m^{-1}\mathbf{P}_m^{\mathrm{T}}\mathbf{E}_m\mathbf{F}_m^{-1})\xi_m. \end{aligned} \tag{46}$$

The characteristic equation for the closed-loop system in modal subspace corresponding to the m repeated eigenvalues is

$$\det[\mathbf{J}_m - \mathbf{V}_m^{\mathrm{H}}\mathbf{B}\mathbf{R}_m^{-1}\mathbf{P}_m^{\mathrm{T}}\mathbf{E}_m\mathbf{F}_m^{-1} - \rho\mathbf{I}] = 0. \tag{47}$$

From equation (47), m closed-loop eigenvalues can be obtained.

Using the generalized modal orthogonal condition (12), and equation (45) the actual feedback control corresponding to the m repeated eigenvalues of the defective system can be obtained

$$Z_m(t) = -\mathbf{R}_m^{-1}\mathbf{P}_m^{T}\mathbf{E}_m\mathbf{F}_m^{-1}\mathbf{V}_m^{\mathrm{H}}\mathbf{x}(t). \tag{48}$$

If the Potter's algorithm is used to deal with the optimal control based on the modal equation (36), we have to solve the following eigenproblem and retain the eigenvalues with positive real parts:

$$\begin{bmatrix} \mathbf{\Lambda}_d & \vdots & \mathbf{Q}_d \\ \cdots & \vdots & \cdots \\ \mathbf{P}_d\mathbf{R}_d^{-1}\mathbf{P}_d & \vdots & -\mathbf{\Lambda}_d \end{bmatrix}\begin{bmatrix} \mathbf{E}_d \\ \cdots \\ \mathbf{F}_d \end{bmatrix} = \begin{bmatrix} \mathbf{E}_d \\ \cdots \\ \mathbf{F}_d \end{bmatrix}\mathbf{J}_2, \tag{49}$$

where $\mathbf{\Lambda}_d$, $\mathbf{Q}_d$, $\mathbf{R}_d$, $\mathbf{E}_d$ and $\mathbf{F}_d$ are the corresponding notations associated with the distinct eigenvalues, and the corresponding solution of the modal Riccati equation is

$$\mathbf{K}_d = \mathbf{E}_d\mathbf{F}_d^{-1}. \tag{50}$$

The modal gain matrix corresponding to the distinct eigenvalues is

$$\mathbf{G}_d = -\mathbf{R}_d^{-1}\mathbf{P}_d^{\mathrm{T}}\mathbf{E}_d\mathbf{F}_d^{-1} \tag{51}$$

and the optimal modal feedback control is

$$Z_d(t) = \mathbf{G}_d\xi_d. \tag{52}$$

From equation (36), we obtain that the closed-loop modal equation is the modal subspace corresponding $(n-m)$ distinct eigenvalues

$$\dot{\xi}_d = \mathbf{\Lambda}_d \xi_d + \mathbf{P}_d \mathbf{G}_d \xi_d = (\mathbf{\Lambda}_d + \mathbf{P}_d \mathbf{G}_d)\xi_d. \tag{53}$$

The corresponding characteristic equation is

$$\det[\mathbf{\Lambda}_d + \mathbf{P}_d \mathbf{G}_d - \rho \mathbf{I}] = 0 \tag{54}$$

and the actual feedback control can be expressed as

$$Z_d(t) = \mathbf{G}_d \mathbf{V}_{n-m}^{\mathrm{H}} \mathbf{X}(t). \tag{55}$$

The input $Z(t)$ in equation (17) is

$$\begin{aligned} Z(t) &= Z_m(t) + Z_d(t) \\ &= \mathbf{G}_m \mathbf{V}_m^{\mathrm{H}} \mathbf{X}(t) + \mathbf{G}_d \mathbf{V}_{n-m}^{\mathrm{H}} \mathbf{X}(t) \\ &= (\mathbf{G}_m \mathbf{V}_m^{\mathrm{H}} + \mathbf{G}_d \mathbf{V}_{n-m}^{\mathrm{H}})\mathbf{X}(t). \end{aligned} \tag{56}$$

5. MODAL OPTIMAL CONTROL ALGORITHM FOR NEAR DEFECTIVE SYSTEMS WITH CLOSE EIGENVALUES

The numerical analysis results show that if some changes of parameters in the defective systems are made, the system with defective repeated eigenvalues can be perturbed into one with close eigenvalues and the corresponding eigenvectors to near parallel, which is called a near defective system [7]. For such a special case from the view point of mathematics, although the close eigenvalues are distinct, the dynamic characteristic of the system is still defective. Thus, the formula for obtaining the gain matrix in equation (51) of systems with the distinct eigenvalues cannot be used for the case of the near defective system. In addition, the formula for obtaining the gain matrix in equation (44) of systems with repeated eigenvalues of defective system as discussed in the above cannot be also directly used to deal with a near defective system with close eigenvalues. Therefore, to discuss the optimal control problem a near defective system with close eigenvalues is necessary.

Assume that the first m eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ of $\mathbf{A}$ are close, and the rest of the eigenvalues, $\lambda_{m+1}, \ldots, \lambda_n$, are distinct, and the modal matrices $\mathbf{U}$ and $\mathbf{V}$ satisfy the following equations:

$$\mathbf{AU} = \mathbf{UJ}, \qquad \mathbf{A}^{\mathrm{H}}\mathbf{V} = \mathbf{VJ}^{\mathrm{H}} \tag{57, 58}$$

and the orthogonal condition

$$\mathbf{U}^{\mathrm{H}}\mathbf{V} = \mathbf{V}^{\mathrm{H}}\mathbf{U} = \mathbf{I}. \tag{59}$$

The modal matrices $\mathbf{U}$ and $\mathbf{V}$ can be partitioned as

$$\mathbf{U} = [\mathbf{U}_m, \mathbf{U}_{n-m}], \qquad \mathbf{V} = [\mathbf{V}_m, \mathbf{V}_{n-m}], \tag{60, 61}$$

where $\mathbf{U}_m = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m]$ and $\mathbf{V}_m = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m]$ are the right and left modal matrices corresponding to eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_m$, $\mathbf{U}_{n-m}$ and $\mathbf{V}_{n-m}$ are the right and left modal matrices corresponding to eigenvalues, $\lambda_{m+1}, \ldots, \lambda_n$. The matrix $\mathbf{A}$ can be expressed as

$$\begin{aligned}\mathbf{A} &= \mathbf{UJV}^{\mathrm{H}} \\ &= \mathbf{U}_m\mathbf{J}_m\mathbf{V}_m^{\mathrm{H}} + \mathbf{U}_{n-m}\mathbf{J}_{n-m}\mathbf{V}_{n-m}^{\mathrm{H}},\end{aligned} \tag{62}$$

where

$$\mathbf{J}_m = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix}, \quad \mathbf{J}_{n-m} \begin{bmatrix} \lambda_{m+1} & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}. \tag{63}$$

$\mathbf{J}_m$ can be expressed as

$$\begin{aligned}\mathbf{J}_m &= \begin{bmatrix} \lambda_0 & 1 & & \\ & \lambda_0 & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_0 \end{bmatrix} + \begin{bmatrix} \lambda_1 - \lambda_0 & -1 & & \\ & \lambda_2 - \lambda_0 & \ddots & \\ & & \ddots & -1 \\ & & & \lambda_m - \lambda_0 \end{bmatrix}, \\ &= \mathbf{J}_{m0} + \delta\mathbf{J}_0\end{aligned} \tag{64}$$

where

$$\mathbf{J}_{m0} = \begin{bmatrix} \lambda_0 & 1 & & \\ & \lambda_0 & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_0 \end{bmatrix}, \tag{65}$$

$$\delta\mathbf{J}_m = \begin{bmatrix} \lambda_1 - \lambda_0 & -1 & & \\ & \lambda_2 - \lambda_0 & \ddots & \\ & & \ddots & -1 \\ & & & \lambda_m - \lambda_0 \end{bmatrix}, \tag{66}$$

$$\lambda_0 = \frac{1}{m}\sum_{j=1}^{m} \lambda_j. \tag{67}$$

Substituting equation (64) into equation (62) yields

$$\begin{aligned}\mathbf{A} &= \mathbf{U}_m\mathbf{J}_{m0}\mathbf{V}_m^{\mathrm{H}} + \mathbf{U}_m\delta\mathbf{J}_m\mathbf{V}_m^{\mathrm{H}} + \mathbf{U}_{n-m}\mathbf{J}_{n-m}\mathbf{V}_{n-m}^{\mathrm{H}} \\ &= \mathbf{A}_r + \delta\mathbf{A},\end{aligned} \tag{68}$$

where

$$\mathbf{A}_r = \mathbf{U}_m\mathbf{J}_{m0}\mathbf{V}_m^{\mathrm{H}} + \mathbf{U}_{n-m}\mathbf{J}_{n-m}\mathbf{V}_{n-m}^{\mathrm{H}}, \tag{69}$$

$$\delta\mathbf{A} = \mathbf{U}_m\delta\mathbf{J}_m\mathbf{V}_m^{\mathrm{H}}. \tag{70}$$

If $\lambda_i (i = 1, 2, \ldots, m)$ are the close eigenvalues, and $\delta = \max_i |\lambda_i - \lambda_0|$, it can be shown that the error matrix $\delta\mathbf{A} = \mathbf{U}_m \delta\mathbf{J}_m \mathbf{V}_m^{\mathrm{H}}$ is a small perturbation one and its norm satisfies

$$\begin{aligned} \|\delta\mathbf{A}\|_2 &\leqslant \|\mathbf{U}_m\|_2 \, \|\delta\mathbf{J}_m\|_2 \, \|\mathbf{V}_m^{\mathrm{H}}\|_2 \\ &\leqslant \|\delta\mathbf{J}_m\|_2 \leqslant \delta^m. \end{aligned} \tag{71}$$

Since the eigenvalues cannot be changed by the orthogonal transform, the eigenvalues of $\mathbf{A}_r$ are equal to m repeated defective eigenvalues, λ_0, and $n - m$ distinct eigenvalues. Equation (68) indicates that matrix $\mathbf{A}$ is equal to the sum of the defective matrix $\mathbf{A}_r$ with m repeated eigenvalues and the perturbation matrix $\delta\mathbf{A}$, and that the dynamic characteristic of $\mathbf{A}$ is also defective.

Transforming the state control equation (17) into the generalized modal co-ordinates through the following modal transformation:

$$\mathbf{x}(t) = [\mathbf{U}_m, \, \mathbf{U}_{n-m}] \begin{bmatrix} \xi_m \\ \cdots \\ \xi_d \end{bmatrix} \tag{72}$$

yields

$$\begin{bmatrix} \dot{\xi}_m \\ \cdots \\ \dot{\xi}_d \end{bmatrix} = \begin{bmatrix} \mathbf{J}_m & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & \mathbf{\Lambda}_d \end{bmatrix} \begin{bmatrix} \xi_m \\ \cdots \\ \xi_d \end{bmatrix} + \begin{bmatrix} \mathbf{V}_m^{\mathrm{H}} \\ \cdots \\ \mathbf{V}_{n-m}^{\mathrm{H}} \end{bmatrix} \mathbf{B} Z(t). \tag{73}$$

From equation (73), we have

$$\dot{\xi}_m = \mathbf{J}_m \xi_m + \mathbf{V}_m^{\mathrm{H}} \mathbf{B} Z_m(t), \tag{74}$$

$$\dot{\xi}_d = \mathbf{\Lambda}_d \xi_d + \mathbf{V}_{n-m}^{\mathrm{H}} \mathbf{B} Z_d(t). \tag{75}$$

Using equation (64), equation (74) becomes

$$\dot{\xi}_m = (\mathbf{J}_{m0} + \delta\mathbf{J}_m) \xi_m + \mathbf{V}_m^{\mathrm{H}} \mathbf{B} Z_m(t), \tag{76}$$

where $\mathbf{J}_{m0}$ and $\delta\mathbf{J}_m$ are given by equation (65) and (66).

Equation (76) can be approximated by the following equation:

$$\dot{\xi}_m \approx \mathbf{J}_{m0} \xi_m + \mathbf{V}_m^{\mathrm{H}} \mathbf{B} Z_m(t). \tag{77}$$

Equation (77) is the same as the modal control equation (35) corresponding to the defective system with m repeated eigenvalues λ_0 which is equal to the average of the m close eigenvalues λ_i, i.e., $\lambda_0 = 1/m \sum_{i=1}^{m} \lambda_i$. Therefore, the modal optimal control algorithm for solving the Riccati equations of the defective systems discussed in the above can be used to deal with those of the approximate system of the near defective system with close eigenvalues.

Using equation (45), optimal modal feedback control of the defective system is as follows:

$$z_{m0}(t) = -\mathbf{R}_m^{-1}\mathbf{P}_m^{\mathrm{T}}\mathbf{E}_{m_0}\mathbf{F}_{m_0}^{-1}\xi_m(t). \tag{78}$$

If this modal feedback control law is used, we obtain the closed-loop modal equation of a near defective system

$$\begin{aligned}\dot{\xi}_m &= \mathbf{J}_{m0}\xi_m + \mathbf{V}_m^{\mathrm{H}}\mathbf{B}z_m(t)\\ &= (\mathbf{J}_{m0} - \mathbf{V}_m^{\mathrm{H}}\mathbf{B}\mathbf{R}_m^{-1}\mathbf{P}_m^{\mathrm{T}}\mathbf{E}_{m_0}\mathbf{F}_{m_0}^{-1})\xi_m\end{aligned} \tag{79}$$

and the corresponding characteristic equation is

$$\det[\mathbf{J}_{m0} - \mathbf{V}_m^{\mathrm{H}}\mathbf{B}\mathbf{R}_m^{-1}\mathbf{P}_m^{\mathrm{T}}\mathbf{E}_{m0}\mathbf{F}_{m0}^{-1} - \rho\mathbf{I}] = 0, \tag{80}$$

which can be used to compute m closed-loop eigenvalues.

It should be noted that if the control equation of a near defective system with close eigenvalues is expressed as one of the non-defective system with distinct eigenvalues

$$\dot{\xi}(t) = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)\xi(t) + \mathbf{V}^{\mathrm{H}}\mathbf{B}Z(t) \tag{81}$$

thus, we have that since equation (81) represents a set of independent equations, the analysis results based on this equation will be misleading. In addition, because the system is near defective, the right and left modal matrix $\mathbf{U}$ and $\mathbf{V}$ in equation (57) and (58) cannot be obtained using the following equations:

$$\mathbf{A}\mathbf{U} = \mathbf{U}\mathbf{\Lambda}, \qquad \mathbf{A}^{\mathrm{H}}\mathbf{V} = \mathbf{V}\tilde{\mathbf{\Lambda}}, \tag{82, 83}$$

where $\mathbf{\Lambda} = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\tilde{\mathbf{\Lambda}}$ is the conjugate of $\mathbf{\Lambda}$.

From above discussions it is known that for a near defective system with close eigenvalues, we should use the invariant subspace recursive method to compute the generalized modal matrices $\mathbf{U}$ and $\mathbf{V}$ based on equation (9) and (11), and then the method presented to obtain modal optimal control law of the defective system with repeated eigenvalue can be used to deal with the problem of a defective system with close eigenvalues.

6. PERTURBATION ANALYSIS OF GAIN MATRIX OF NEAR DEFECTIVE SYSTEMS

From the above discussion, it can be seen that since the feedback gain matrix is obtained by neglecting by perturbation of close m eigenvalues from their average value, the stability analysis for a near defective system is required. To this end, the perturbation analysis of the gain matrix is given as follows.

Recalling the modal control equation (76) for near defective systems, equation (42) becomes

$$\begin{bmatrix}(\mathbf{J}_{m_0}^{\mathrm{T}} + \delta\mathbf{J}_m)^{\mathrm{T}} & \vdots & \mathbf{Q}_m\\ \cdots\cdots & \vdots & \cdots\cdots\\ \mathbf{P}_m\mathbf{R}_m^{-1}\mathbf{P}_m^{\mathrm{T}} & \vdots & -(\mathbf{J}_{m_0} + \delta\mathbf{J}_m)\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{E}}_m\\ \cdots\\ \tilde{\mathbf{F}}_m\end{bmatrix} = \begin{bmatrix}\tilde{\mathbf{E}}_m\\ \cdots\\ \tilde{\mathbf{F}}_m\end{bmatrix}\tilde{\mathbf{J}}_1 \tag{84}$$

or

$$\left(\begin{bmatrix} \mathbf{J}_{m_0}^{\mathrm{T}} & \vdots & \mathbf{Q}_m \\ \cdots\cdots & \vdots & \cdots\cdots \\ \mathbf{P}_m\mathbf{R}_m^{-1}\mathbf{P}_m^{\mathrm{T}} & \vdots & -\mathbf{J}_{m_0} \end{bmatrix} + \begin{bmatrix} \delta\mathbf{J}_m^{\mathrm{T}} & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & -\delta\mathbf{J}_m \end{bmatrix}\right) \begin{bmatrix} \mathbf{E}_{m_0} + \Delta\mathbf{E}_{m_0} \\ \cdots\cdots \\ \mathbf{F}_{m_0} + \Delta\mathbf{F}_{m_0} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{E}_{m_0} + \Delta\mathbf{E}_{m_0} \\ \cdots\cdots \\ \mathbf{F}_{m_0} + \Delta\mathbf{F}_{m_0} \end{bmatrix} (\mathbf{J}_{10} + \Delta\mathbf{J}_{10}). \tag{85}$$

Equation (85) can be expressed in the following form

$$(\mathbf{M}_{10} + \Delta\mathbf{M}_{10})\left(\begin{bmatrix} \mathbf{E}_{m_0} \\ \cdots \\ \mathbf{F}_{m_0} \end{bmatrix} + \begin{bmatrix} \Delta\mathbf{E}_{m_0} \\ \cdots \\ \Delta\mathbf{F}_{m_0} \end{bmatrix}\right) = \left(\begin{bmatrix} \mathbf{E}_{m_0} \\ \cdots \\ \mathbf{F}_{m_0} \end{bmatrix} + \begin{bmatrix} \Delta\mathbf{E}_{m_0} \\ \cdots \\ \Delta\mathbf{F}_{m_0} \end{bmatrix}\right)(\mathbf{J}_{10} + \Delta\mathbf{J}_{10}). \tag{86}$$

In the above equations, $\begin{bmatrix} \Delta\mathbf{E}_{m_0} \\ \cdots \\ \Delta\mathbf{F}_{m_0} \end{bmatrix}$ is the change of the eigenvector matrix $\begin{bmatrix} \mathbf{E}_{m_0} \\ \cdots \\ \mathbf{F}_{m_0} \end{bmatrix}$ and $\Delta\mathbf{M}_{10}$ is the change reduced by the perturbation of close m eigenvalues. The adjoint system of equation (86) is

$$(\mathbf{M}_{10} + \Delta\mathbf{M}_{10})^{\mathrm{T}}\left(\begin{bmatrix} \mathbf{H}_{m_0} \\ \cdots \\ \mathbf{P}_{m_0} \end{bmatrix} + \begin{bmatrix} \Delta\mathbf{H}_{m_0} \\ \cdots \\ \Delta\mathbf{P}_{m_0} \end{bmatrix}\right) = \left(\begin{bmatrix} \mathbf{H}_{m_0} \\ \cdots \\ \mathbf{P}_{m_0} \end{bmatrix} + \begin{bmatrix} \Delta\mathbf{H}_{m_0} \\ \cdots \\ \Delta\mathbf{P}_{m_0} \end{bmatrix}\right)(\mathbf{J}_{10} + \Delta\mathbf{J}_{10}). \tag{87}$$

Using the perturbation theory, $\begin{bmatrix} \Delta\mathbf{E}_{m_0} \\ \cdots \\ \Delta\mathbf{F}_{m_0} \end{bmatrix}$ can be obtained [7]

$$\begin{bmatrix} \Delta\mathbf{E}_{m_0} \\ \cdots \\ \Delta\mathbf{F}_{m_0} \end{bmatrix} = \begin{bmatrix} \mathbf{E}_{m_0} \\ \cdots \\ \mathbf{F}_{m_0} \end{bmatrix} \mathbf{C}^1, \tag{88}$$

where

$$\mathbf{C}_{ij}^1 = \frac{-R_{ij}^1}{s_{0j} - s_{0i}} \quad (j \neq i,\ i, j = 1, 2, \dots), \tag{89}$$

$$\mathbf{R}^1 = \begin{bmatrix} \mathbf{H}_{m_0} \\ \cdots \\ \mathbf{P}_{m_0} \end{bmatrix}^{\mathrm{T}} \Delta\mathbf{M}_{10} \begin{bmatrix} \mathbf{E}_{m_0} \\ \cdots \\ \mathbf{F}_{m_0} \end{bmatrix}. \tag{90}$$

Therefore, the feedback gain matrix of actual near defective system is

$$\mathbf{G}_m = -\mathbf{R}_m^{-1}\mathbf{P}_m^{\mathrm{T}}(\mathbf{E}_{m0} + \Delta\mathbf{E}_{m0})(\mathbf{F}_{m0} + \Delta\mathbf{F}_{m0})^{-1} \tag{91}$$

and corresponding optimal modal feedback control is

$$Z_m(t) = -\mathbf{R}_m^{-1}\mathbf{P}_m^{\mathrm{T}}(\mathbf{E}_{m0} + \Delta\mathbf{E}_{m0})(\mathbf{F}_{m0} + \Delta\mathbf{F}_{m0})^{-1}\xi_m. \tag{92}$$

The closed-loop modal equation of an actual near defective system is

$$\dot{\xi}_m = (\mathbf{J}_{m0} - \mathbf{V}_m^{\mathrm{H}}\mathbf{B}\mathbf{R}_m^{-1}\mathbf{P}_m^{\mathrm{T}}(\mathbf{E}_{m0} + \Delta\mathbf{E}_{m0})(\mathbf{F}_{m0} + \Delta\mathbf{F}_{m0})^{-1})\xi_m \qquad (93)$$

and the corresponding characteristic equation is

$$\det[\mathbf{J}_{m0} - \mathbf{V}_m^{\mathrm{H}}\mathbf{B}\mathbf{R}_m^{-1}\mathbf{P}_m^{\mathrm{T}}(\mathbf{E}_{m0} + \Delta\mathbf{E}_{m0})(\mathbf{F}_{m0} + \Delta\mathbf{F}_{m0})^{-1} - \rho\mathbf{I}] = 0, \qquad (94)$$

which can be used to compute m closed-loop eigenvalues of an actual near defective system.

The modal optimal control algorithm for near defective systems with close eigenvalues is summarized as follows:

(1) From state matrix $\mathbf{A}$, and compute close m eigenvalues, $\lambda_1, \ldots, \lambda_m$, of near defective system.
(2) Compute

$$\lambda_0 = \frac{1}{m}\sum_{j=1}^{m}\lambda_j.$$

(3) Compute generalized modal matrices $\mathbf{U}$ and $\mathbf{V}$ using the invariant subspace recursive procedure presented in Section 3.
(4) Form approximate defective system using equation (77).
(5) Compute eigenvectors $\begin{bmatrix}\mathbf{E}_{m_0}\\ \cdots\\ \mathbf{F}_{m_0}\end{bmatrix}$ using equation (42).
(6) Compute $\mathbf{G}_{m0}$ from equation (44) for an approximate system with a defective repeated eigenvalue λ_0

$$\mathbf{G}_{m0} = -\mathbf{R}_m^{-1}\mathbf{P}_m^{\mathrm{T}}\mathbf{E}_{m0}\mathbf{F}_{m0}^{-1}.$$

(7) Using equation (88), compute

$$\begin{bmatrix}\Delta\mathbf{E}_{m_0}\\ \cdots\\ \Delta\mathbf{F}_{m_0}\end{bmatrix}.$$

(8) Compute $\mathbf{G}_m$ from equation (91) for an actual near defective system.
(9) Compute m closed-loop eigenvalues from equation (94) for an actual near defective system.

7. NUMERICAL EXAMPLE

In order to illustrate the application of the present modal control theory, a numerical example of a near defective system is given as follows.

Assume that the state matrix is given by

$$\mathbf{A} = \begin{bmatrix} 17 & 0 & -24{\cdot}99999\\ 0 & 3 & 0\\ 9 & 0 & -13\end{bmatrix}$$

and the control matrix $\mathbf{B}$ for single input control force, $Z(t)$, is $\mathbf{B} = \begin{bmatrix}\mathbf{1}\\ \mathbf{1}\\ \mathbf{1}\end{bmatrix}$. The system has two close eigenvalues, i.e., $\mathbf{J} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$

$$\lambda_1 = 2{\cdot}009487, \quad \lambda_2 = 1{\cdot}990513, \quad \lambda_3 = 3{\cdot}0.$$

The algebra average of λ_1, λ_2, is

$$\lambda_0 = \frac{1}{2}\sum_{i=1}^{2} \lambda_i = 2{\cdot}0.$$

The Jordan form matrix is

$$\mathbf{J}_m = \mathbf{J}_{m_0} + \delta\mathbf{J}_0 = \begin{bmatrix} \lambda_0 & 1 \\ & \lambda_0 \end{bmatrix} + \begin{bmatrix} \lambda_1 - \lambda_0 & -1 \\ & \lambda_2 - \lambda_0 \end{bmatrix}.$$

Therefore, a near defective system with close eigenvalues can be transformed into one of the defective system. With a recursive procedure the right and left generalized modes, $\mathbf{U} = [U_m, U_d]$ and $\mathbf{V} = [V_m, V_d]$, corresponding to λ_0 and λ_3 can be obtained as follows:

$$\mathbf{U} = \begin{bmatrix} 0{\cdot}857493 & -0{\cdot}514496 & 0{\cdot}0000000 \\ 0{\cdot}000000 & 0{\cdot}000000 & 1{\cdot}0000000 \\ 0{\cdot}514496 & 0{\cdot}857493 & 0{\cdot}0000000 \end{bmatrix},$$

$$\mathbf{V} = \begin{bmatrix} 0{\cdot}857493 & -0{\cdot}514496 & 0{\cdot}0000000 \\ 0{\cdot}000000 & 0{\cdot}000000 & 1{\cdot}0000000 \\ 0{\cdot}514496 & 0{\cdot}857493 & 0{\cdot}0000000 \end{bmatrix}.$$

If $\mathbf{H}_m$, $\mathbf{Q}_m$, and $\mathbf{R}_m$ in equation (37) are given by

$$\mathbf{H}_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{Q}_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{R}_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\mathbf{P}_m = \mathbf{V}_m^{\mathrm{H}}\mathbf{B} = \begin{bmatrix} 1{\cdot}371989 \\ 0{\cdot}342997 \end{bmatrix}.$$

We obtain the following eigenproblem:

$$\begin{bmatrix} 2{\cdot}000000 & 0{\cdot}000000 & 1{\cdot}000000 & 0{\cdot}000000 \\ 1{\cdot}000000 & 2{\cdot}000000 & 0{\cdot}000000 & 1{\cdot}000000 \\ 1{\cdot}882354 & 0{\cdot}470588 & -2{\cdot}000000 & -1{\cdot}000000 \\ 0{\cdot}470588 & 0{\cdot}117647 & 0{\cdot}000000 & -2{\cdot}000000 \end{bmatrix} \begin{bmatrix} \mathbf{E}_{m_0} \\ \cdots \\ \mathbf{F}_{m_0} \end{bmatrix} = \begin{bmatrix} \mathbf{E}_{m_0} \\ \cdots \\ \mathbf{F}_{m_0} \end{bmatrix} \mathbf{J}_1 .$$

Solving this eigenproblem, we obtain the eigenvalues with positive real part

$$s_1 = 2{\cdot}581228, \quad s_2 = 1{\cdot}826817,$$

$$\begin{bmatrix} \mathbf{E}_{m_0} \\ \cdots \\ \mathbf{F}_{m_0} \end{bmatrix} = \begin{bmatrix} -0{\cdot}434316 & -0{\cdot}178581 \\ -0{\cdot}862084 & 0{\cdot}983404 \\ -0{\cdot}252437 & 0{\cdot}030927 \\ -0{\cdot}066752 & 0{\cdot}008272 \end{bmatrix}.$$

The steady state solution of the modal Riccati matrix is

$$\mathbf{K}_{m_0} = \mathbf{E}_{m_0}\mathbf{F}_{m_0}^{-1} = 10^3 \times \begin{bmatrix} 0{\cdot}652663 & -2{\cdot}461682 \\ -2{\cdot}461682 & 9{\cdot}322307 \end{bmatrix}$$

and the corresponding modal gain is

$$\mathbf{G}_{m_0} = -\mathbf{R}_m^{-1}\mathbf{P}_m^{\mathrm{T}}\mathbf{E}_{m_0}\mathbf{F}_{m_0}^{-1} = 10^2 \times [-0{\cdot}510966 \quad 1{\cdot}798767].$$

The optimal modal control is given by

$$\begin{aligned} Z_{m_0}(t) &= -\mathbf{R}_m^{-1}\mathbf{P}_m^{\mathrm{T}}\mathbf{E}_{m_0}\mathbf{F}_{m_0}^{-1}\xi_m(t) \\ &= 10^2 \times [-0{\cdot}510966 \;\; 1{\cdot}798767]\xi_m(t) \end{aligned}$$

and the corresponding actual control law in the state space is

$$\begin{aligned} Z_{m0}(t) &= -\mathbf{R}_m^{-1}\mathbf{P}_m^{\mathrm{T}}\mathbf{E}_{m_0}\mathbf{F}_{m_0}^{-1}\mathbf{V}_m^{\mathrm{H}}\mathbf{x}(t) \\ &= 10^2 \times [-1{\cdot}363608 \;\; 0 \;\; 1{\cdot}279539]\,\mathbf{x}(t). \end{aligned}$$

From the following characteristic equation:

$$\det[\mathbf{J}_{m_0} + \mathbf{P}_m\mathbf{G}_{m_0} - \rho\mathbf{I}] = 0,$$

we obtain the eigenvalues of the modal closed-loop system as

$$\rho_1 = -2{\cdot}581236, \quad \rho_2 = -1{\cdot}825602.$$

For the distinct eigenvalues, $\mathbf{\Lambda}_d = \lambda_3 = 3{\cdot}0$, if $\mathbf{H}_d = 1$, $\mathbf{Q}_d = 1$, $\mathbf{R}_d = 1$, $\mathbf{P}_d = \mathbf{V}_3^{\mathrm{H}}\mathbf{B} = 1$, we have the following eigenproblem:

$$\begin{bmatrix} 3{\cdot}0 & 1 \\ 1{\cdot}0 & -3{\cdot}0 \end{bmatrix} \begin{bmatrix} \mathbf{E}_d \\ \cdots \\ \mathbf{F}_d \end{bmatrix} = \begin{bmatrix} \mathbf{E}_d \\ \cdots \\ \mathbf{F}_d \end{bmatrix} \mathbf{J}_2 .$$

Solving this eigenproblem, we obtain the eigenvalue with positive real part

$$s_3 = 3{\cdot}162278$$

and corresponding eigenvector

$$\begin{bmatrix} \mathbf{E}_d \\ \cdots \\ \mathbf{F}_d \end{bmatrix} = \begin{bmatrix} -0{\cdot}987087 \\ \cdots\cdots \\ -0{\cdot}160182 \end{bmatrix}.$$

The solution of the modal Riccati equation is

$$\mathbf{K}_d = \mathbf{E}_d\mathbf{F}_d^{-1} = 6{\cdot}162278$$

and the corresponding modal gain is

$$\mathbf{G}_d = -\mathbf{R}_d^{-1}\mathbf{P}_d^{\mathrm{T}}\mathbf{K}_d = -6{\cdot}162278.$$

The optimal modal control is

$$\mathbf{Z}_d(t) = \mathbf{G}_d\xi_d = -\,6{\cdot}162278\,\xi_d$$

and the corresponding characteristic equation of the modal closed-loop system is

$$\det[\mathbf{\Lambda}_d + \mathbf{P}_d\mathbf{G}_d - \rho\mathbf{I}] = 0.$$

So, we get the eigenvalue of the modal closed-loop system as

$$\rho_3 = -\,3{\cdot}162278.$$

Using equation (66), we obtain the input $Z(t)$ in equation (13) as

$$\begin{aligned} Z(t) &= Z_{m0}(t) + Z_d(t) \\ &= (\mathbf{G}_{m_0}\mathbf{V}_m^{\mathrm{H}} + \mathbf{G}_d\mathbf{V}_3^{\mathrm{H}})\mathbf{X}(t) \\ &= (10^2[-\,1{\cdot}363608 \quad 0 \quad 1{\cdot}279540] + [0 \quad -\,6{\cdot}162278 \quad 0])\mathbf{X}(t) \\ &= 10^2[-\,1{\cdot}363608 \quad -\,0{\cdot}061623 \quad 1{\cdot}279540]\mathbf{X}(t). \end{aligned}$$

In the following, we give the perturbation analysis of gain matrix $\mathbf{G}_{m0}$. To obtain $\begin{bmatrix} \Delta\mathbf{E}_{m0} \\ \cdots \\ \Delta\mathbf{F}_{m0} \end{bmatrix}$, we need to solve the following eigenproblem:

$$\begin{bmatrix} 2{\cdot}0 & 0{\cdot}0 & 1{\cdot}0 & 0{\cdot}0 \\ 1{\cdot}0 & 2{\cdot}0 & 0{\cdot}0 & 1{\cdot}0 \\ 1{\cdot}8822354 & 0{\cdot}470588 & -\,2{\cdot}0 & -\,1{\cdot}0 \\ 0{\cdot}470588 & 0{\cdot}117647 & 0{\cdot}0 & -\,2{\cdot}0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{H}_{m0} \\ \cdots \\ \mathbf{P}_{m0} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{m0} \\ \cdots \\ \mathbf{P}_{m0} \end{bmatrix} \mathbf{J}_{10}.$$

For the eigenvalues, $s_{01} = 2{\cdot}581228$, $s_{02} = 1{\cdot}826817$, we have

$$\begin{bmatrix} \mathbf{H}_{m0} \\ \cdots \\ \mathbf{P}_{m0} \end{bmatrix} = \begin{bmatrix} -\,0{\cdot}963011 & -\,0{\cdot}835533 \\ -\,0{\cdot}168343 & 0{\cdot}470927 \\ -\,0{\cdot}210208 & -\,0{\cdot}218336 \\ 0{\cdot}009138 & 0{\cdot}180114 \end{bmatrix}.$$

Using equation (90) yields

$$\begin{aligned} \mathbf{R}_1 &= \begin{bmatrix} \mathbf{H}_{m0} \\ \cdots \\ \mathbf{P}_{m0} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \delta\mathbf{J}_m^{\mathrm{T}} & \vdots & 0 \\ \cdots & \vdots & \cdots \\ & \vdots & -\,\delta\mathbf{J}_m \end{bmatrix} \begin{bmatrix} \mathbf{E}_{m0} \\ \cdots \\ \mathbf{F}_{m0} \end{bmatrix} \\ &= \begin{bmatrix} -\,0{\cdot}057001 & -\,0{\cdot}028537 \\ 0{\cdot}225763 & 0{\cdot}079392 \end{bmatrix}. \end{aligned}$$

From equation (88), we have

$$\begin{bmatrix} \Delta \mathbf{E}_{m0} \\ \cdots \\ \Delta \mathbf{F}_{m0} \end{bmatrix} = \begin{bmatrix} \mathbf{E}_{m0} \\ \cdots \\ \mathbf{F}_{m0} \end{bmatrix} \mathbf{C}^1 = \begin{bmatrix} -0{\cdot}053442 & -0{\cdot}016429 \\ 0{\cdot}294291 & -0{\cdot}032610 \\ 0{\cdot}009255 & -0{\cdot}009549 \\ 0{\cdot}002475 & -0{\cdot}002525 \end{bmatrix},$$

$$\mathbf{C}^1_{ij} = \frac{-\mathbf{R}^1_{ij}}{s_{0j} - s_{0i}}, \quad i = 1,2; \quad j = 1,2; \quad i \neq j.$$

So, from equation (91) and (94), we obtain

$$\mathbf{G}_m = 10^2 \times [-0{\cdot}511930 \;\; 1{\cdot}802409],$$

$$\rho_1 = -2{\cdot}581860, \quad \rho_2 = -1{\cdot}832298.$$

These results show that the perturbations of the gain matrix and eigenvalues of the closed-loop system of a near defective system are small, and the stability can be guaranteed.

8. CONCLUSIONS

The vibration control of the systems with repeated or close eigenvalues is an important problem in engineering. This paper focuses on the case of a defective or near defective systems with repeated or close eigenvalues and presents a modal optimal control algorithm based on the generalized modal co-ordinates. From the view point of mathematics, although the close eigenvalues of near defective system are distinct, the dynamic characteristic of the system is still defective. For such a case, we have to use an invariant subspace recursive method to obtain the reliable generalized modes, and then the standard modal optimal control method, such as the Potter's method, can be extended to deal with the corresponding modal optimal control problem. A near defective system can be approximated by the defective one with repeated eigenvalues equal to the average value of the close eigenvalues. Since the norm of the perturbation matrix $\delta\mathbf{A}$ of the approximate defective systems is small, the changes in closed-loop eigenvalues of defective system are small, and the stability can be checked by perturbation analysis. The conclusions are supported by the given numerical example.

ACKNOWLEDGMENT

This work is supported by the National Science Foundation (19872028) and the Mechanical Technology Development Foundation of China.

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