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LETTERS TO THE EDITOR

ON THE DERIVATION OF THE EQUATIONS OF MOTION FOR A PARAMETRICALLY EXCITED CANTILEVER BEAM

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1. INTRODUCTION

One of the most readily assimilated mechanical structures, in which both forced and parametric vibration phenomena can occur, is the base excited cantilever beam. In the context of parametric vibrations this structure has been considered by many researchers either alone [1, 2], or as part of a larger structural system [3, 4], on the understanding that it is the *orientation* of the beam relative to the direction of excitation which determines whether the system is forced or parametric in nature. In the case of forced vibration the model can be linear or non-linear and is frequently characterized by modal differential equations of motion, noting that the equations of motion in this case always exhibit constant coefficients. This is not the case for parametric vibrations where the governing equations of motion contain certain terms in which the coefficients are time variant, irrespective of whether they are cast in the modal space or not, or whether they are linear or non-linear. The paper by Cartmell [5] was an attempt to unify the necessary kinematics and dynamics for a simple vertical beam with a lumped end mass which is undergoing a single frequency harmonic excitation in the stiff y direction, as shown in Figure 1. The development in reference [5] attempted to demonstrate that this system is parametric in nature by deriving the necessary kinematic relationships for combined bending and torsional motions of the beam and from these leading on to three non-linear modal equations of motion by recourse to a Lagrangian derivation. Much of the work in this paper was newly presented as a unification based around this particular physical problem and involved certain approximations, some justifiable in the engineering sense, others less so. It is the purpose of this letter to examine key stages of this useful analysis once again and to add justification to some of the principal features of the development in reference [5] whilst improving other, weaker aspects of that work by means of new and rigorous analysis. The motivation for this work is the enduring usefulness of simple structural systems for the investigation and interpretation of complex vibrational phenomena both for the researcher and the educator.

2. SUMMARY OF KINEMATICS

Figure 1 defines the beam length l as the portion of the beam emerging from the top of the base-clamp as far as the point where it enters the end mass, this is an improvement on the earlier schematic in reference [5] where the length l of the beam was defined as far as the

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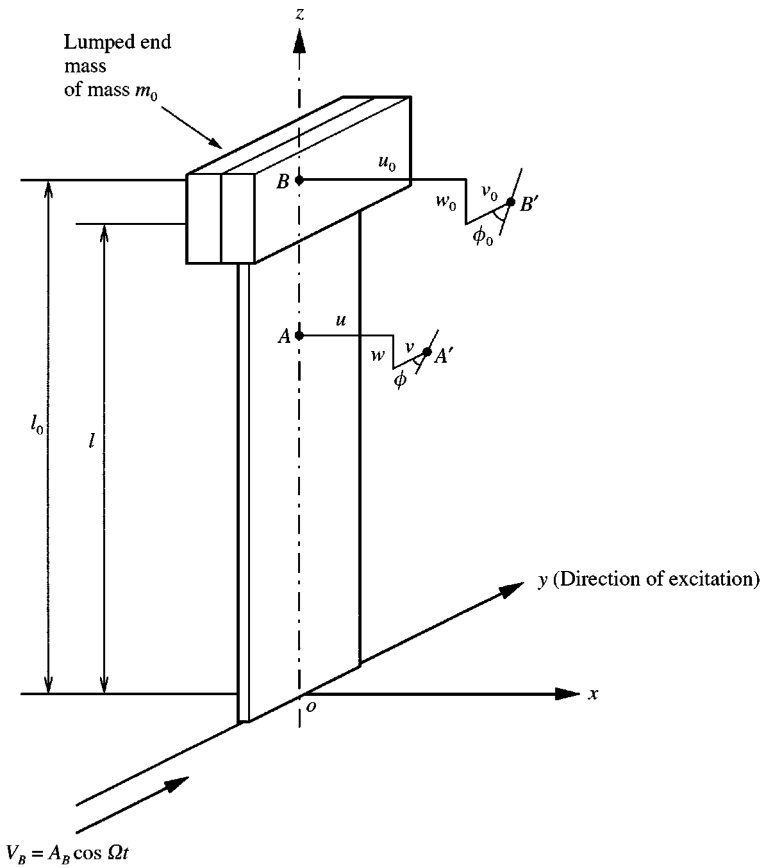

Figure 1. Physical representation of the system.

centre of the end mass. This was not strictly correct, but, as is shown later, does not turn out to be a serious error. Thus it is correct to say that $u_o(t) = u(l_o, t)$, $v_o(t) = v(l_o, t)$, $w_o(t) = w(l_o, t)$ and $\phi_o(t) = \phi(l_o, t)$, where u, v, w and ϕ are all functions of z and t. The o-subscripted displacements are definitionally considered to be at the centre of mass of the end mass, B, whereas the unsubscripted displacements are at some arbitrary location along the beam, shown here as A.

Figure 2 shows an element of the beam, for convenience taken at point A in Figure 1, in both the undeformed and the deformed states. It can be clearly seen that $u(z, t)$ represents motion purely in the Oxz-plane (known informally as the flexible plane) whereas $v(z, t)$ defines motion purely in the stiff plane, Oyz. Rotation about the deformed z-axis, *i.e.* the Z-axis, is denoted by $\phi(z, t)$. In reference [5] it was shown that application of the Euler–Kirchhoff–Love theory for rods [6] could lead to relatively simple kinematic expressions for curvatures about the deformed X and Y axes for the element under the assumption of small bending, u, in the x direction and small bending, v, in the y direction. Denoting elemental curvatures about X and Y, as κ_1 and κ_2, respectively, it was shown in reference [5] that these are given by

$$\kappa_1 = u''\phi - v'' \quad \text{and} \quad \kappa_2 = v''\phi + u'', \tag{1}$$

where, in this letter, $'$ denotes differentiation with respect to z and $\cdot$ denotes differentiation with respect to t.

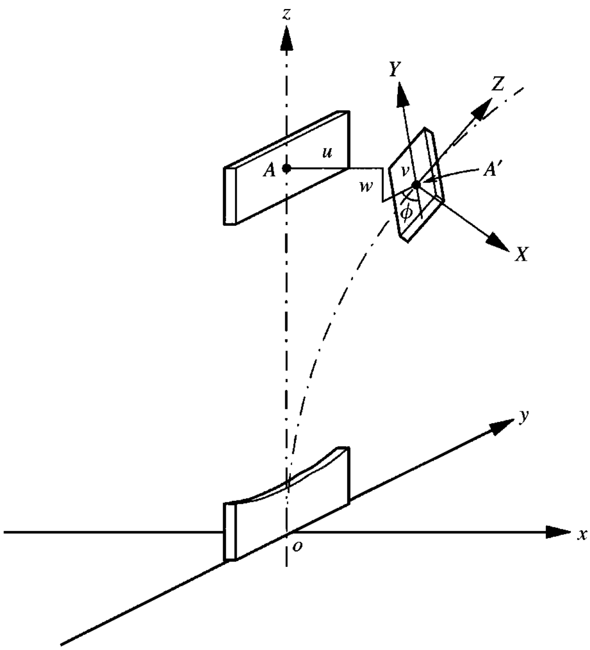

Figure 2. An element of the beam in both its undeformed and deformed states.

The earlier assumption, which continues to be justifiable, was that the beam is extremely stiff in the Oyz-plane, and therefore in the deformed Y direction which led to the further assumption that κ_1 is virtually zero, from which it is possible to write

$$u''\phi = v'' \tag{2}$$

and

$$\kappa_2 = u''(1 + \phi^2). \tag{3}$$

A useful physical perspective on this is to consider that at a fixed time, given small $u(z)$ and $v(z)$ then $\phi(z)$ must satisfy equation (2), provided $\phi(z)$ turns out to be small as well, in order to ensure that κ_1 equals zero, Thus, on this basis, the analysis in reference [5] remains formally unchanged up to this point (equation (38) in that paper) but with the slight notational and definitional improvements given above taken on board. It can also be seen in reference [7] that equations (1) and (2) have been restated there, but with no explicit development.

At this point it is expedient to consider a fixed time so that under the assumptions of small displacement $u(z)$ and $\kappa_1 = 0$, the smaller displacement $v(z)$, in the direction of the excitation, can be expressed as a function of $u(z)$ and $\phi(z)$ by further consideration of the system geometry. This is shown in Figure 3, wherein it should be noted that two reference points are taken, P and Q, at distances z and $z + \delta z$, respectively, along the deformed Z-axis. It is helpful to strengthen this argument by emphasizing that because the displacement u in the x direction is small and the displacement v in the y direction is even smaller, then the distance z up the z-axis and arc length s up the deformed Z-axis are approximately equal and can therefore be used interchangeably. Tangents to points P and Q can then be projected onto an end-plane located at $z = l$. Points P and Q are analogous to the points

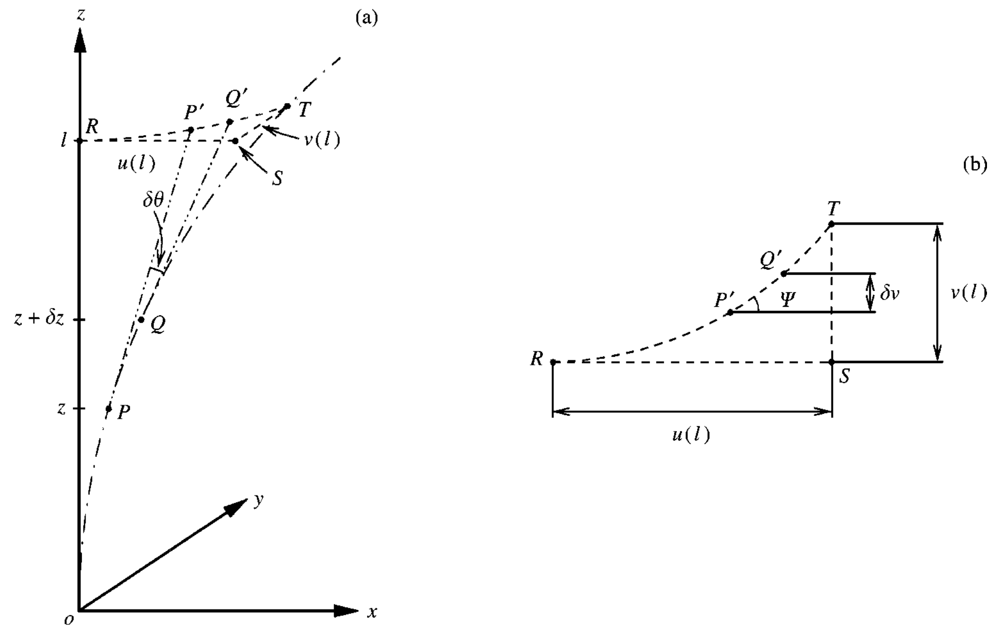

Figure 3. (a) The tangents at P and Q projected onto the "end-plane" RST ($z = l$); (b) plan view of the "end-plane" RST.

O and P in Figure 5 in reference [5], but it should be noted that strictly speaking the tangent to point O in Figure 5 of reference [5] is in fact the undeformed z-axis, therefore points P and Q, as defined above, now provide the necessarily correct points of emanation for the tangents projecting onto the end-plane. In keeping with the development of reference [5] the length of both tangents PP' and QQ' is approximately $l - z$ and so the length of the chord $P'Q'$ is approximately

$$(l - z)\,\delta\theta. \tag{4}$$

In Figure 3(b) it is shown that the angle between the positive x direction and the chord $P'Q'$ is defined as ψ and it is newly shown in Appendix A that

$$\psi(z) = \phi(z). \tag{5}$$

This underpins the intuitive statements in Figure 5(b) in reference [5] that this angle is ϕ. Consequently, since ϕ is small ($\sin\phi \approx \phi$) it is acceptable to state that

$$\delta v \approx \phi(l - z)\delta\theta. \tag{6}$$

Following the analysis of reference [5] a little further indicates that the curvature about the y-axis is approximately equal to

$$\kappa_2 = \mathrm{d}\theta/\mathrm{d}z. \tag{7}$$

So, by taking the limit as $\delta z \to 0$, meaning physically that the point Q tends to coalesce with point P in Figure 3(a), and then substituting equation (7) into equation (6) the following equation emerges:

$$\mathrm{d}v/\mathrm{d}z = (l - z)\phi\kappa_2. \tag{8}$$

Substituting the curvature equation (3) into equation (8) and then integrating from $z = 0$ to $z = l$ gives

$$v(l) = \int_0^l (l - z) u''(\phi + \phi^3) \, \mathrm{d}z. \tag{9}$$

Finally, as in reference [5], the $u''\phi^3$ term is neglected because ϕ has justifiably been assumed to be small throughout the foregoing analysis. Thus it is found that

$$v(l) = \int_0^l (l - z) u'' \phi \, \mathrm{d}z. \tag{10}$$

This is directly comparable with equation (47) in reference [5], with the difference that the upper limit is correctly stated here as l. Equation (10) appeared with the correct upper limit in an earlier, pre-publication, version of reference [5] but was later erroneously modified after considerable discussion with others. Additionally, the correct form of equation (10) can be found in the work of Bux [8] and Ibrahim and Hijawi [7], but, unlike in reference [5], it was just stated and no attempt was made to fully derive it. It can also be found, summarized in reference [9].

The above analysis, which is based around Figure 3, provides a geometrical proof of equation (10). However, it is also possible to obtain equation (10) simply and directly from equation (2) and this derivation is newly presented here as follows.

Integrating equation (2) once from $z = 0$ to $z = \zeta$ gives

$$v'(\zeta) - v'(0) = \int_0^\zeta u''(z) \phi(z) \, \mathrm{d}z, \tag{11}$$

but $v'(0) = 0$ because the beam is clamped at $z = 0$. Therefore

$$v'(\zeta) = \int_0^\zeta u''(z) \phi(z) \, \mathrm{d}z. \tag{12}$$

Integrating again and using the fact that $v(0) = 0$, because the beam is clamped at $z = 0$, yields

$$v(l) = \int_0^l \left(\int_0^\zeta u''(z) \phi(z) \, \mathrm{d}z \right) \mathrm{d}\zeta. \tag{13}$$

Finally, by changing the order of integration in equation (13), the following equation is obtained:

$$\begin{aligned} v(l) &= \int_0^l \left(\int_z^l u''(z) \phi(z) \, \mathrm{d}\zeta \right) \mathrm{d}z \\ &= \int_0^l \left(\int_z^l \mathrm{d}\zeta \right) u''(z) \phi(z) \, \mathrm{d}z \\ &= \int_0^l (l - z) \, u''(z) \phi(z) \, \mathrm{d}z. \end{aligned} \tag{14}$$

This is equation (10).

After having derived equation (10), the next step is to reintroduce t and to propose that $u(z, t)$ and $\phi(z, t)$ are separable in time and space, such that

$$u(z, t) = f_1(z)u_1(t) + f_2(z)u_2(t) \tag{15}$$

and

$$\phi(z, t) = g_1(z)\phi_1(t), \tag{16}$$

in which the spatial dependency is represented by the mode shapes $f_1(z)$, $f_2(z)$ and $g_1(z)$ for the fundamental and second bending and fundamental torsion modes, respectively, and the temporal dependency is defined by the corresponding modal co-ordinates $u_1(t)$, $u_2(t)$ and $\phi_1(t)$. It is important to reinforce the point made in reference [5] that this three-mode approximation is sufficient for many problems but not for all and that the analyst could (and should) extend the modal contribution at this point if more complicated inter-modal phenomena are likely to be encountered. Equations (15) and (16) can then be substituted into equation (10) to give the final form of the kinematic equation for $v(l, t)$ as

$$v(l, t) = \int_0^l (l - z) g_1 \phi_1 (f_1'' u_1 + f_2'' u_2)\, \mathrm{d}z, \tag{17}$$

which can be rewritten in the form

$$v(l, t) = B_1 \phi_1 u_1 + B_2 \phi_1 u_2, \tag{18}$$

where

$$B_1 = \int_0^l (l - z)\, g_1 f_1''\, \mathrm{d}z \quad \text{and} \quad B_2 = \int_0^l (l - z)\, g_1 f_2''\, \mathrm{d}z. \tag{19}$$

Equations (17), (18), (19a) and (19b) are directly equivalent to equations (48), (49), (50) and (51) in reference [5], respectively, but with improved notation and the correct upper integration limit shown here.

3. ORTHOGONALITY OF THE BENDING MODES

To obtain the mode shapes for the bending u in the x direction (that is, to obtain $f_1(z)$, $f_2(z)$, $f_3(z)$, ...) the following procedure is performed.

Consider the case of pure bending in the x direction, as shown in Figure 4. The beam is assumed to have constant mass per unit length, m, and constant flexural rigidity about the y-axis, EI_y. From Figure 4 it can be seen that

$$q = u(l, t) \quad \text{and} \quad \left.\frac{\partial u(z, t)}{\partial z}\right|_{z=l} = \tan\alpha \approx \alpha, \tag{20}$$

where $\tan\alpha \approx \alpha$ because α is small. Consequently, the x-displacement of the mass centre is $q + (l_o - l)\sin\alpha \approx q + (l_o - l)\alpha$, since α is small. The shear force at the top of the beam, $z = l$, is V. Therefore

$$V = m_o(\ddot{q} + (l_o - l)\ddot{\alpha}). \tag{21}$$

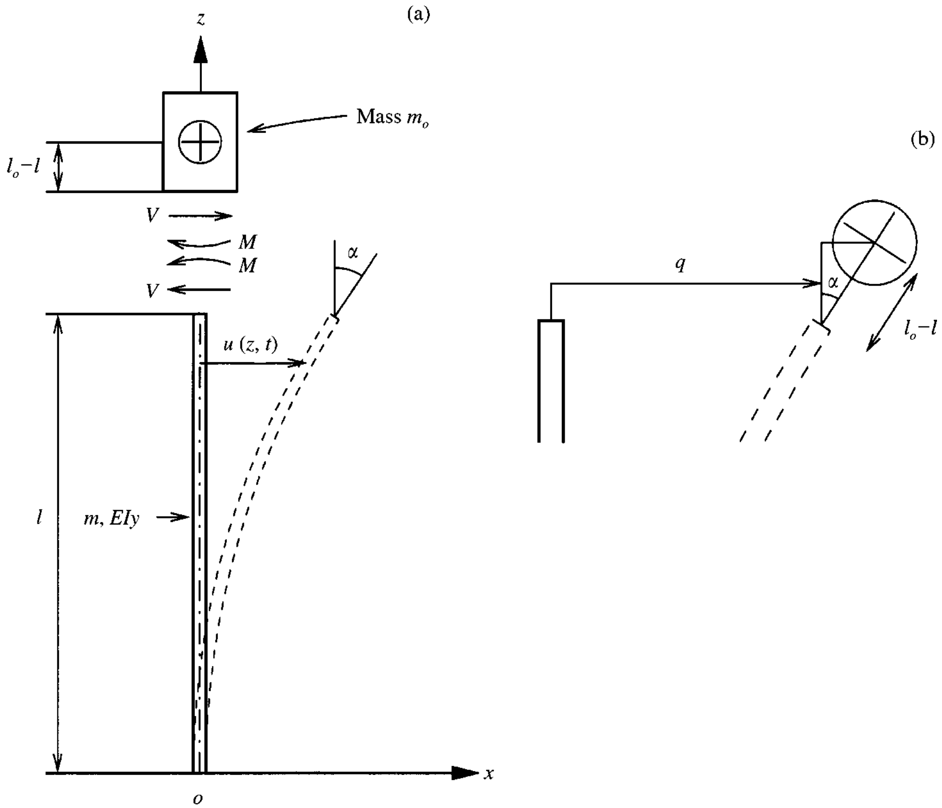

Figure 4. (a) The beam bending purely in the x-direction; (b) a close-up.

Also, the bending moment in the beam at $z = l$ is M, so

$$I_G\ddot{\alpha} = M - V(l_o - l), \tag{22}$$

where I_G is the moment of inertia of the end mass about an axis through its centre, parallel to the y-axis.

As alternatives to equations (21) and (22), the shear force V can be expressed in terms of the third spatial derivative of u at $z = l$, as follows:

$$V = EI_y \left.\frac{\partial^3 u(z,t)}{\partial z^3}\right|_{z=l} \tag{23}$$

and the bending moment M can be expressed in terms of the second spatial derivative of u at $z = l$, as follows:

$$M = -EI_y \left.\frac{\partial^2 u(z,t)}{\partial z^2}\right|_{z=l}. \tag{24}$$

Substituting equation (23) into equation (21), and using equations (20a) and (20b), yields

$$EI_y u''' = m_o(\ddot{u} + (l_o - l)\ddot{u}') \quad \text{at} \quad z = l. \tag{25}$$

Also, substituting equations (23) and (24) into equation (22), and using equation (20b), gives

$$I_G\ddot{u}' = -EI_y(u'' + (l_o - l)u''') \quad \text{at} \quad z = l. \tag{26}$$

Now suppose the system is oscillating in just one mode. In that case the bending u is given by

$$u(z, t) = f(z)\cos\omega t, \tag{27}$$

where ω is the natural frequency of that mode and $f(z)$ is the standard Euler–Bernoulli mode shape for a beam clamped at $z = 0$ ($u(0, t) = u'(0, t) = 0$). That is

$$f(z) = C_1(\sin\lambda z - \sinh\lambda z) + C_2(\cos\lambda z - \cosh\lambda z), \tag{28}$$

where $\lambda^4 = \omega^2 m/EI_y$. Upon using equation (27), equations (25) and (26) become, respectively,

$$EI_y f'''(l) = -\omega^2 m_o(f(l) + (l_o - l)f'(l)) \tag{29}$$

and

$$-\omega^2 I_G f'(l) = -EI_y(f''(l) + (l_o - l)f'''(l)). \tag{30}$$

By substituting equation (28) into equations (29) and (30) two new equations are obtained which can be used to find the ratio of C_2 to C_1 and to find the *frequency equation*. From the frequency equation the eigenvalues, λ_i, and therefore the corresponding natural frequencies, ω_i, can be found.

The mode shapes which result from the above procedure are orthogonal in the following sense:

$$\int_0^l m f_i(z)f_j(z)\,\mathrm{d}z + m_o(f_i(l) + (l_o - l)(f_i'(l))(f_j(l) + (l_o - l)f_j'(l)) + I_G f_i'(l)f_j'(l) = 0 \quad \text{for } i \neq j. \tag{31}$$

See Appendix B for a proof of equation (31). Furthermore, these mode shapes can be normalized so that

$$\int_0^l m(f_i(z))^2\,\mathrm{d}z + m_o(f_i(l) + (l_o - l)f_i'(l))^2 + I_G(f_i'(l))^2 = m_o \quad \text{for} \quad i = 1, 2, \ldots\ . \tag{32}$$

With this normalization the mode shapes are non-dimensional.

Now consider the kinetic energy, T, in the pure bending case. This is given by the equation

$$T = \frac{1}{2}\int_0^l m(\dot{u}(z, t))^2\,\mathrm{d}z + \frac{1}{2}m_o(\dot{q} + (l_o - l)\dot{\alpha})^2 + \frac{1}{2}I_G\dot{\alpha}^2. \tag{33}$$

At this point, suppose only the first two bending modes are taken into account, as in equation (15). Then, by first substituting equations (20a) and (20b) into equation (33), and then substituting in equation (15), and finally using equations (31) and (32), equation (33) becomes

$$T = \tfrac{1}{2}m_o(\dot{u}_1^2 + \dot{u}_2^2). \tag{34}$$

Returning to the main problem of combined bending and torsion, the expression for the kinetic energy becomes

$$T = \tfrac{1}{2} m_o (\dot{u}_1^2 + \dot{u}_2^2 + \dot{w}_o^2 + [\dot{v}_o + \dot{V}_B]^2) + \tfrac{1}{2} I_o \dot{\phi}_o^2, \tag{35}$$

where w_o is the vertical drop of the end mass centre, v_o is its small displacement in the y direction, V_B is the support excitational displacement, I_o is the moment of inertia of the end mass about the deformed Z-axis and ϕ_o is the twist angle, measured at the mass centre. Using equation (16) shows that the twist angle, $\phi_o(t)$, satisfies

$$\phi_o(t) = \phi(l_o, t) = \phi(l, t) = g_1(l)\phi_1(t). \tag{36}$$

Consequently, if $g_1(z)$ is normalized so that $g_1(l) = 1$, then the following equation is obtained:

$$\phi_o(t) = \phi_1(t). \tag{37}$$

The previous analysis shows that equation (35), which is equation (52) in reference [5], is correct, in that the modes for u (u_1 and u_2) do decouple. That is, there is no cross term involving $\dot{u}_1 \dot{u}_2$ in equation (35). Furthermore, by virtue of equations (33) and (34), equation (35) also takes into account the kinetic energy of the *beam* due to its bending in the x direction and the rotational kinetic energy of the end mass about an axis through its centre and parallel to the y-axis.

The total system potential energy is given by

$$U = \int_0^l \frac{1}{2} EI_y (u'')^2 \, \mathrm{d}z + \int_0^l \frac{1}{2} cGJ(\phi')^2 \, \mathrm{d}z - m_o g w_o, \tag{38}$$

where the first and second terms are the strain energies due to bending and torsion, respectively, (see reference [10]) and the third term is the gravitational potential energy of the end mass. The beam is assumed to have constant torsional rigidity about the z-axis, GJ, and the constant c is to account for its non-circular cross-section. Equation (38) is directly comparable with equation (53) in reference [5], with the difference that the sign in front of the gravitational potential energy term is correctly stated here as a minus sign. This is because w_o is the vertical *drop* of the end mass centre.

For equations (35) and (38) it is assumed that $l_o - l \ll l$. Under this assumption

$$v_o(t) = v(l_o, t) \approx v(l, t) = B_1 \phi_1 u_1 + B_2 \phi_1 u_2. \tag{39}$$

In obtaining the above equation, equation (18) was used. Also, under the assumption $l_o - l \ll l$

$$w_o(t) = w(l_o, t) \approx w(l, t) = \frac{1}{2} \int_0^l (u'(z, t))^2 \, \mathrm{d}z. \tag{40}$$

Ignoring the approximately equals sign in equation (40) and using equation (15) gives

$$w_o = \frac{1}{2} \left(\int_0^l (f_1')^2 \mathrm{d}z \right) u_1^2 + \left(\int_0^l f_1' f_2' \, \mathrm{d}z \right) u_1 u_2 + \frac{1}{2} \left(\int_0^l (f_2')^2 \, \mathrm{d}z \right) u_2^2. \tag{41}$$

Finally, substituting equations (15), (16) and (41) into equation (38) yields

$$
\begin{aligned}
U = \frac{1}{2}\left[EI_y \int_0^l (f_1'')^2\,\mathrm{d}z - m_o g \int_0^l (f_1')^2\,\mathrm{d}z\right]u_1^2 \\
+ \left[EI_y \int_0^l f_1''\, f_2''\mathrm{d}z - m_o g \int_0^l f_1' f_2'\,\mathrm{d}z\right]u_1\,u_2 \\
+ \frac{1}{2}\left[EI_y \int_0^l (f_2'')^2\,\mathrm{d}z - m_o g \int_0^l (f_2')^2\,\mathrm{d}z\right]u_2^2 \\
+ \frac{1}{2}cGJ\left[\int_0^l (g_1')^2\,\mathrm{d}z\right]\phi_1^2.
\end{aligned} \tag{42}
$$

Equations (41) and (42) are the equivalents of equations (55) and (56) in reference [5] respectively. However, the cross terms (those involving $u_1 u_2$) are correctly shown here and they do not disappear, as shown in reference [5].

4. CONCLUSIONS

The purpose of this letter has been to re-examine the study by Cartmell [5]. In that paper Cartmell investigated the kinematics and dynamics of a slender vertical beam with an end mass, which was excited at its base by a harmonic excitation in the stiff direction.

This letter has strengthened the geometrical proof of the important equation for the small displacement, $v(l)$, of the top of the beam in the y direction. That is, it has strengthened the proof of equation (47) in reference [5], which corresponds to equation (10) here. Furthermore, it has been shown that the correct upper limit of integration of the integral in the equation for $v(l)$ is l and not, as shown in reference [5], $l/2$. This is in agreement with a pre-publication version of reference [5] and work by Cartmell [9], Bux [8] and Ibrahim and Hijawi [7], and as a result a conflict in the literature has been resolved. This letter has gone on to give a new, simple and direct proof of equation (10). The equation for $v(l)$ is very important because it describes the coupling between the torsion of the beam, the bending of the beam in the flexible direction and the bending of the beam in the stiff direction, which is the direction of excitation.

Next, a rigorous analysis has been presented in this letter to show that the modal co-ordinates for the bending u in the x direction *do* decouple in the expression for the kinetic energy, as shown in reference [5]. More specifically, this means that equation (52) in reference [5], which is equation (35) here, is correct, in that it does not contain a cross term (a term involving $\dot{u}_1\dot{u}_2$). In addition, the above analysis revealed that the expression for the kinetic energy not only takes into account the kinetic energy of the end mass due to its displacement in the x direction but also takes into account the kinetic energy of the *beam* due to its bending in the x direction and the rotational kinetic energy of the end mass about an axis through its centre and parallel to the y-axis.

Finally, it has been shown in this letter that, unfortunately, the modal co-ordinates for the bending u in the x direction *do not* decouple in the expression for the potential energy. That is, the cross term (the term involving $u_1 u_2$) does not disappear, as shown in equation (56) in reference [5], but appears as shown here in equation (42).

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APPENDIX A

To show that $\psi(z) = \phi(z)$, first consider the deformed Z-axis shown in Figure 3(a). The equation of this curve is given by

$$\boldsymbol{r}(z) = (u(z),\ v(z),\ z). \tag{A.1}$$

Therefore, the tangent vector to the curve, at height z, is

$$\boldsymbol{r}' = (u'(z),\ v'(z),\ 1). \tag{A.2}$$

Now, the general equation of a line in space is

$$\boldsymbol{l} = \boldsymbol{a} + t\boldsymbol{d}, \tag{A.3}$$

where $\boldsymbol{a}$ is a point on the line, $\boldsymbol{d}$ is a vector parallel to the line and t is a parameter describing the line. Consequently, by using equations (A.1) and (A.2) in equation (A.3), the equation of the tangent to the deformed Z-axis, at height z, is

$$\begin{aligned}\boldsymbol{l} &= (u(z),\ v(z),\ z) + t\,(u'(z),\ v'(z),\ 1)\\ &= (u(z) + tu'(z),\ v(z) + tv'(z),\ z + t).\end{aligned} \tag{A.4}$$

The above equation is the equation of the line PP' in Figure 3(a). This line intersects the plane $z = l$ (the "end-plane") when $t = l - z$. That is, this line intersects the plane $z = l$ at the point

$$(u(z) + (l - z)u'(z),\ v(z) + (l - z)v'(z),\ l). \tag{A.5}$$

From Figure 3(b) it can be seen that the point P' traces out the curve RT as z moves from 0 to l (see Figure 3(a)). Let P' have co-ordinates $(x(z), y(z))$. Then from equation (A.5)

$$x(z) = u(z) + (l - z)u'(z), \tag{A.6}$$

$$y(z) = v(z) + (l - z)v'(z). \tag{A.7}$$

Also, from Figure 3(b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\psi \tag{A.8}$$

but

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}z}{\mathrm{d}x/\mathrm{d}z} = \frac{v'(z) - v'(z) + (l - z)v''(z)}{u'(z) - u'(z) + (l - z)u''(z)} = \frac{v''(z)}{u''(z)}. \tag{A.9}$$

Upon using equation (2) (that is, using $u''\phi = v''$) the equation above becomes

$$\mathrm{d}y/\mathrm{d}x = \phi. \tag{A.10}$$

Finally, combining equation (A.8) with equation (A.10) and using the fact that ψ is small, so that $\tan\psi \approx \psi$, gives

$$\psi(z) = \phi(z). \tag{A.11}$$

APPENDIX B

To show that the mode shapes $(f_1(z), f_2(z), f_3(z), \ldots)$ for the bending u in the x direction are orthogonal in the following sense:

$$\int_0^l m f_i(z) f_j(z)\,\mathrm{d}z + m_o\,(f_i(l) + (l_o - l) f_i'(l))\,(f_j(l) + (l_o - l) f_j'(l)) + I_G f_i'(l) f_j'(l) = 0 \quad \text{for} \quad i \neq j. \tag{B.1}$$

Consider two mode shapes: $f_i(z)$ and $f_j(z)$, with $i \neq j$. The mode shape $f_i(z)$ satisfies the differential equation

$$EI_y \frac{\mathrm{d}^4 f_i(z)}{\mathrm{d}z^4} = \omega_i^2 m f_i(z) \quad \text{for} \quad 0 < z < l \tag{B.2}$$

and the mode shape $f_j(z)$ satisfies the differential equation

$$EI_y \frac{\mathrm{d}^4 f_j(z)}{\mathrm{d}z^4} = \omega_j^2 m f_j(z) \quad \text{for} \quad 0 < z < l. \tag{B.3}$$

Multiplying equation (B.3) by $f_i(z)$ and equation (B2) by $f_j(z)$, integrating them both by parts twice over the domain $0 < z < l$, and finally subtracting them gives

$$(\omega_j^2 - \omega_i^2) \int_0^l m f_i(z) f_j(z)\,\mathrm{d}z = f_i(l) EI_y f_j'''(l) - f_i'(l) EI_y f_j''(l) - f_j(l) EI_y f_i'''(l) + f_j'(l) EI_y f_i''(l). \tag{B.4}$$

In obtaining equation (B.4) the fact that $f_i(0) = f_i'(0) = f_j(0) = f_j'(0) = 0$ (the beam is clamped at $z = 0$) has been used.

Now, from equations (29) and (30) it is possible to solve for $f''(l)$ in terms of $f(l)$ and $f'(l)$, as follows:

$$EI_y f''(l) = \omega^2 (m_o(l_o - l)(f(l) + (l_o - l) f'(l)) + I_G f'(l)). \tag{B.5}$$

Then, substituting equations (29) and (B.5) into equation (B.4) and performing some algebra gives

$$\begin{aligned}(\omega_j^2 - \omega_i^2) \int_0^l m f_i(z) f_j(z)\,\mathrm{d}z \\ = -(\omega_j^2 - \omega_i^2)[m_o(f_i(l) + (l_o - l) f_i'(l))(f_j(l) + (l_o - l) f_j'(l)) \\ + I_G f_i'(l) f_j'(l)].\end{aligned} \tag{B.6}$$

Finally, taking everything in equation (B.6) onto one side and using the fact that $\omega_i \neq \omega_j$, yields

$$\begin{aligned}\int_0^l m f_i(z) f_j(z)\,\mathrm{d}z + m_o(f_i(l) + (l_o - l) f_i'(l))(f_j(l) + (l_o - l) f_j'(l)) \\ + I_G f_i'(l) f_j'(l) = 0.\end{aligned} \tag{B.7}$$