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PERIODICITY OF AVERAGED HISTORIES OF CHAOTIC OSCILLATORS

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This study is dedicated to demonstrate the periodicities embedded in the averaged responses of chaotic systems with periodic excitations. Recent studies in the field of non-linear oscillations often found random-like responses for some deterministic non-linear systems with periodic excitations, which were then named "chaotic systems". However, in this study, by discretizing the initial conditions on a chosen domain and averaging the corresponding responses, the averaged response can be calculated for the chaotic motions of Duffing, van der Pol and piecewise linear systems. These averaged responses exhibit near-periodicities with primary frequency components at excitation frequency, odd multiples or half multiples of excitation frequency. It is also found that this periodicity becomes more evident as the number of discretized initial conditions over a fixed domain. These results were obtained and validated by simulations.

1. INTRODUCTION

A number of research studies [1–10] have been dedicated to investigate the dynamics of chaotic non-linear systems that are the systems subjected to periodic excitation but exhibit the dynamics with no periodicity. However, Lu [11] demonstrated, by using the metric defined in Hausdoff space [12] as an observatory tool, that although the response of an impact oscillator, a chaotic system at some parameter ranges, does not exhibit periodicity in Euclidean space, it does show periodicity in Hausdoff space. In order to explore the further possibility of periodicity embedded in chaotic motion, this study, by conducting simulations for six cases of Duffing, van der Pol and piecewise linear systems, succeeded in validating the periodicities of "averaged responses", which are obtained by averaging the responses over a pre-chosen domain in phase space.

In the process of acquiring "averaging responses", a single initial condition is first used to plot responses on phase plane in order to ensure that the system and its parameters used are in the range for generating chaotic responses. Secondly, a rectangular domain in phase plane is chosen and discretized into $N \times N$ grid points, i.e., N^2 initial conditions, which can be utilized to simulate N^2 responses. Averaging these N^2 responses, an "averaged response" is obtained. Through frequency analysis, the periodicity of this averaged response is

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identified and further quantified on the levels of periodicity. In the present study, the aforementioned process is conducted for six cases in the forms of three types of non-linear equations.

2. CASE STUDIES

2.1. DUFFING EQUATION

Consider the Duffing equation in the following:

$$\ddot{x} + \gamma\dot{x} - k_1 x + k_3 x^3 = f\cos(\omega_s t), \tag{1}$$

where $\gamma = 0{\cdot}3$, $k_1 = 1{\cdot}0$, $k_3 = 1{\cdot}0$, $\omega_s = 1{\cdot}2$. Figure 1(a) shows the bifurcation plot of sampled response amplitude $X(nT)$ by the method of Poincaré sections versus the excitation forcing amplitude f. From this figure, the parameter range of f corresponding to chaotic motions can be identified. For $f = 0{\cdot}45$ when the system undergoes chaotic motion, Figure 1(b) shows the phase portrait with the initial conditions $x(0) = 3{\cdot}0$, $\dot{x}(0) = 3{\cdot}0$, while Figure 1(c) shows the corresponding time responses. In Figure 1(c), the adjacent peak values are different; therefore, the response is non-periodic. Figure 1(d) shows the corresponding frequency spectrum, where it can be seen that the frequency contents, which include the excitation frequency, are continuously distributed over a finite range of frequencies below 4.

With the chaotic motion confirmed for $f = 0{\cdot}45$, simulations are next performed to obtain pre-defined "averaged responses". To achieve this goal, N grid points located evenly in a discretized rectangular domain $-3 \leqslant x \leqslant 3, -3 \leqslant \dot{x} \leqslant 3$ are first chosen as initial

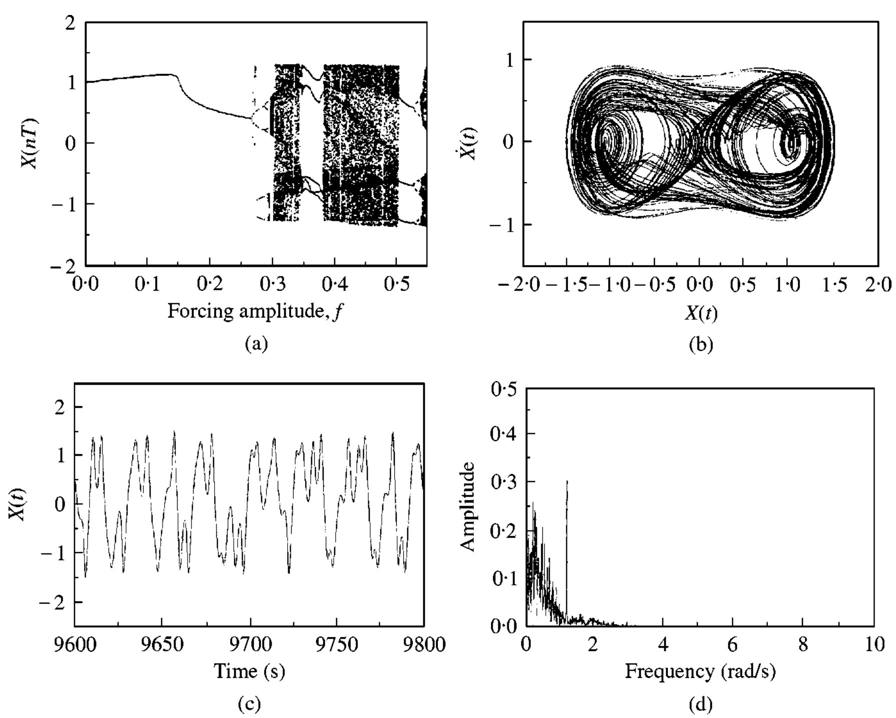

Figure 1. The first case of Duffing equation (a) bifurcation plot, (b) phase portrait, (c) time history, and (d) spectrum for initial condition (3, 3).

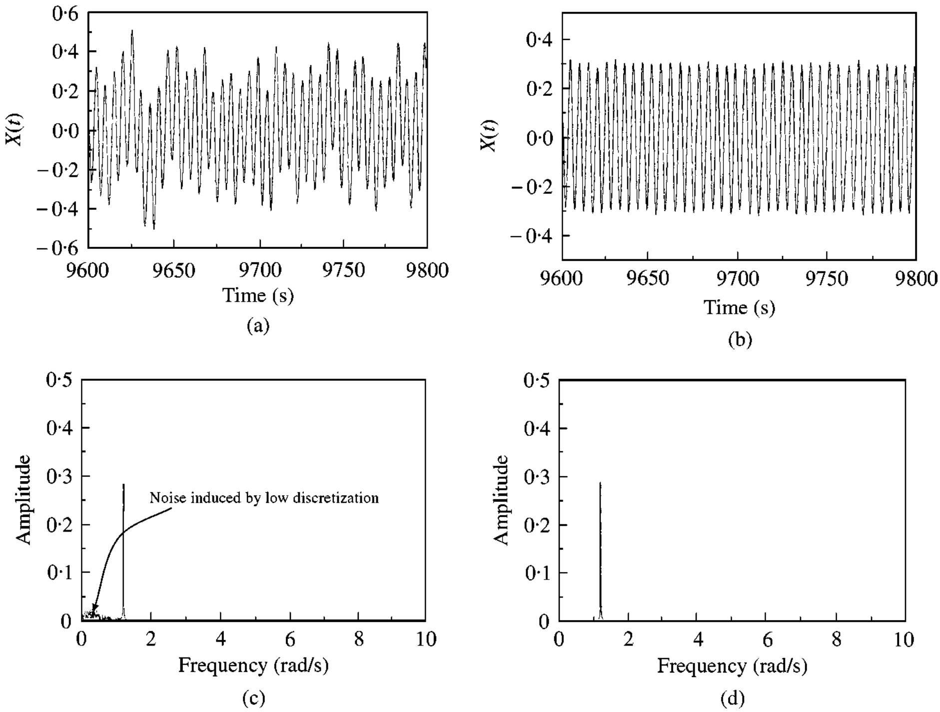

Figure 2. The first case for averaged responses of Duffing equation in initial condition domain $-3 \leqslant x \leqslant 3, -3 \leqslant \dot{x} \leqslant 3$: time history of (a) 100 points and (b) 10 000 points; spectrum of (c) 100 points and (d) 10 000 points.

conditions. The averaged response is then obtained by averaging the N sets of corresponding simulated time responses. Figure 2(a, b) show the averaged responses for $N = 100$ and 10 000. The corresponding frequency spectra are shown in Figure 2(c, d) respectively. It can be seen from Figure 2(a, b) that as N increases, the characteristic of the response gets closer to that of a perfect periodic response, that is, as the density of domain discretization for initial conditions increases, the periodicity becomes more evident. It is seen from Figure 2(c, d) that the location of fundamental frequency in spectra $\omega = 1{\cdot}2$ coincides with the excitation frequency ω_s, and the magnitude at $\omega = 1{\cdot}2$ is the same as that in Figure 1(d). It is also seen that in Figure 2(c) noise is present at low frequency region while in Figure 2(d) no noise is visible due to the smaller number of discretized initial conditions.

The above study is repeated to confirm the findings with different parameters $\gamma = 0{\cdot}2$, $k_1 = 0{\cdot}0$, $k_3 = 1{\cdot}0$, $\omega_s = 1{\cdot}0$. Figure 3(a) shows the bifurcation plot of sampled response amplitude $X(nT)$ by the method of Poincaré sections versus the excitation forcing amplitude f, where $f = 8{\cdot}0$ is identified as the parameter leading to chaos. Figure 3(b) shows the phase portrait with the initial conditions $x(0) = 3{\cdot}0$, $\dot{x}(0) = 3{\cdot}0$, while Figure 3(c) depicts the corresponding time responses and Figure 3(d) shows the corresponding frequency spectrum where three major harmonics at odd multiples of excitation frequencies, 1, 3, 5 are present. Figure 4(a, b) show the averaged responses for $N = 100$, 10 000. Figure 4(c, d) depict the corresponding frequency spectra, respectively, where the peaks at odd multiples 1, 3, 5 in both figures with the same magnitudes as those in Figure 3(d) are observed. Very small frequency components such as noise are found in Figure 4(c) compared to none in Figure 4(d), which is due to the small number of discretizing initial conditions in Figure 4(c). It can

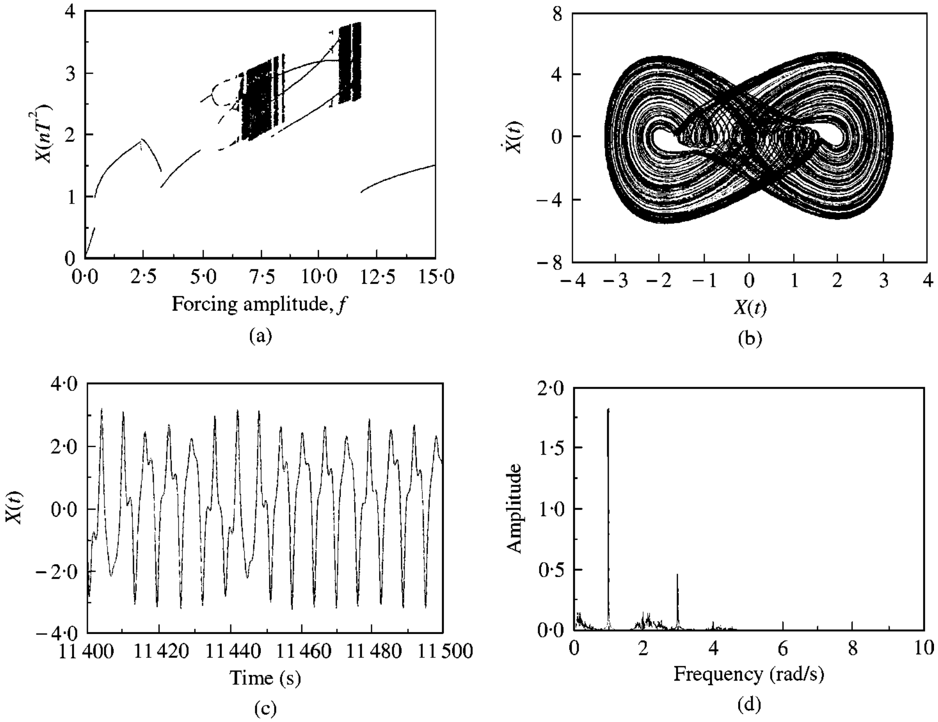

Figure 3. The second case of Duffing equation (a) bifurcation plot, (b) phase portrait, (c) time history, and (d) spectrum for initial condition (3, 3).

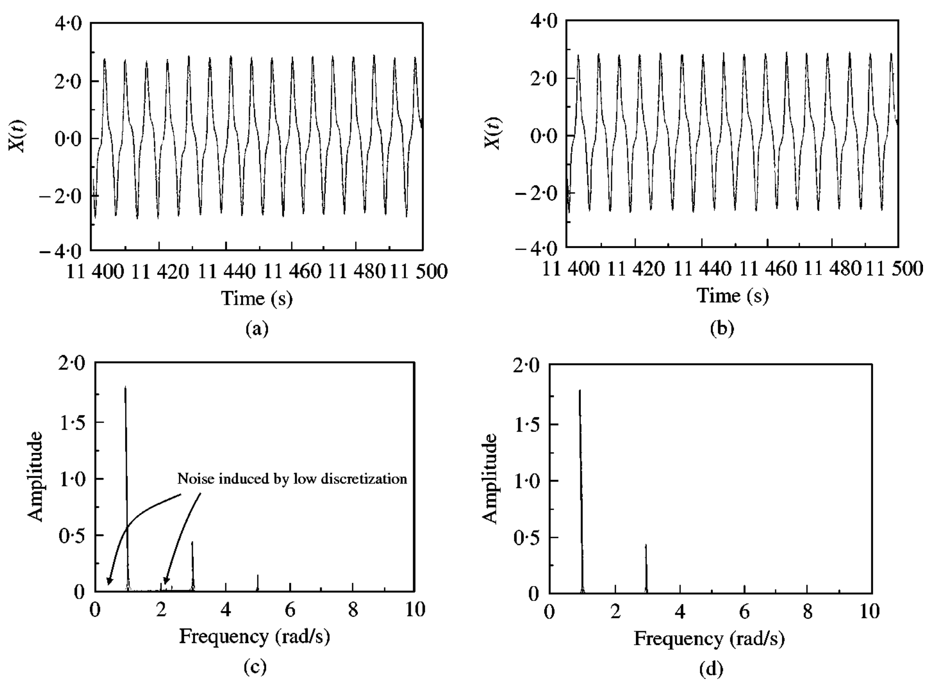

Figure 4. The second case for averaged responses of Duffing equation in initial condition domain $-3 \leqslant x \leqslant 3, -3 \leqslant \dot{x} \leqslant 3$: time history of (a) 100 points and (b) 10 000 points; spectrum of (c) 100 points and (d) 10 000 points.

be easily shown, based on Figure 4(a, d) that the findings obtained for the first set of system parameters still hold.

Based on the results from Figures 2 and 4, we find that the "averaged response" of a chaotic Duffing system exhibits a strong near-periodicity with a fundamental frequency the same as that of the excitation. Furthermore, this near-periodicity approaches a perfect periodicity as the number of initial conditions in a chosen rectangular domain increases.

2.2. VAN DER POL EQUATION

Consider the van der Pol equation in the following, which includes a non-linear damping term:

$$\ddot{x} + \gamma(x^2 - 1)\dot{x} + k_1 x = f\cos(\omega_s t), \tag{2}$$

where $\gamma = 5{\cdot}0$, $k_1 = 1{\cdot}0$, $\omega_s = 2{\cdot}46$. Figure 5(a) shows the bifurcation plot of sampled response amplitude $X(nT)$ by the method of Poincaré sections versus the excitation forcing amplitude f. For $f = 4{\cdot}8$ when the system undergoes chaotic motion, Figure 5(b) shows the phase portrait with the initial conditions $x(0) = -3{\cdot}0$, $\dot{x}(0) = 0{\cdot}0$, while Figure 5(c) is the corresponding time responses. In Figure 5(c, d), it is observed that the response is non-periodic.

With the chaotic response confirmed for $f = 4{\cdot}8$, the same procedure for simulations are next performed to obtain pre-defined "averaged responses". The rectangular domain

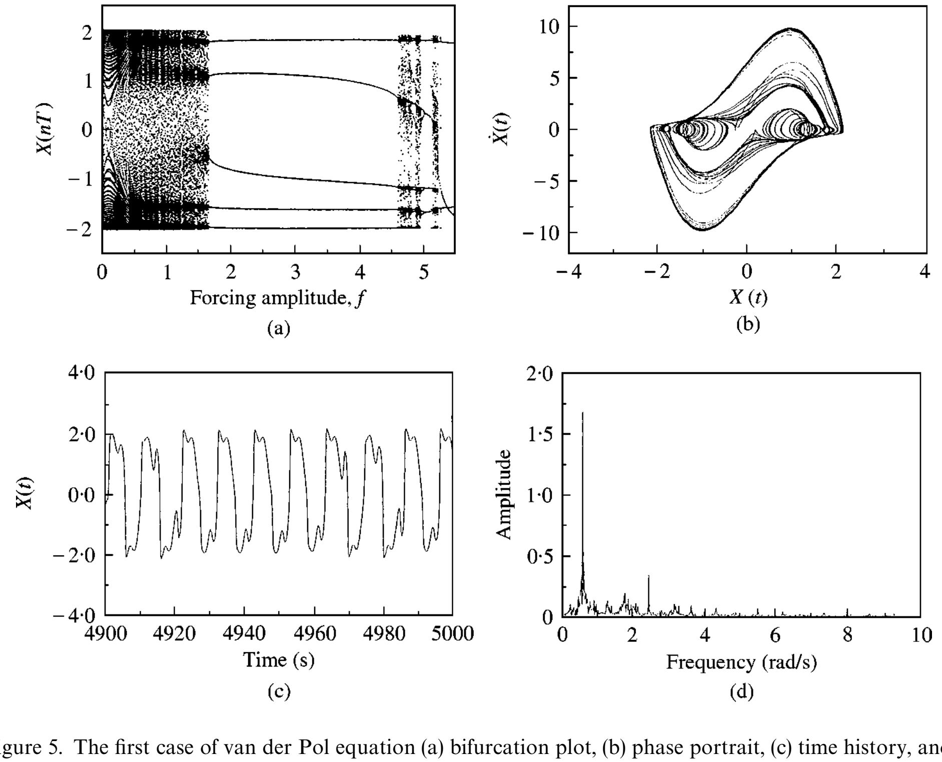

Figure 5. The first case of van der Pol equation (a) bifurcation plot, (b) phase portrait, (c) time history, and (d) spectrum for initial condition (3, 3).

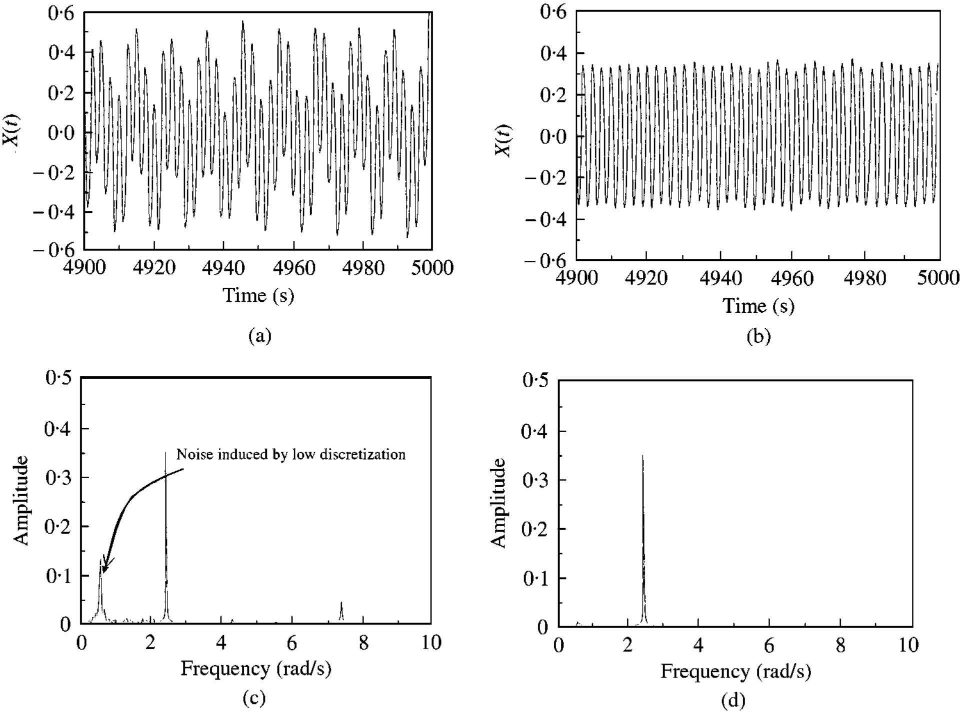

Figure 6. The first case for averaged responses of van der Pol equation in initial condition domain $-3 \leqslant x \leqslant 3, -3 \leqslant \dot{x} \leqslant 3$: time history of (a) 400 points and (b) 10 000 points; spectrum of (c) 400 points and (d) 10 000 points.

considered is $-3 \leqslant x \leqslant 3, -3 \leqslant \dot{x} \leqslant 3$, and $N = 400$, 10 000. Figure 6(a, b) show the averaged responses respectively. Compared to Figure 5(c), the averaged response in Figure 6(a, b) show higher levels of periodicity. The corresponding frequency spectra with $N = 400$ and 10 000 are shown in Figure 6(c, d), respectively, where it is seen that major frequency $\omega = 2{\cdot}46$ coincides with the excitation frequency. Small frequency contents with a low peak appear around 0·6 in Figure 6(c), which, compared to Figure 6(d) are identified as noise due to insufficient number of discretizing initial conditions.

The above study is repeated to confirm the findings with different parameters $\gamma = 5{\cdot}0$, $k_1 = 1{\cdot}0$, $\omega_s = 5{\cdot}1$. Figure 7(a) shows the bifurcation plot of sampled response amplitude $X(nT)$ by the method of Poincaré sections versus the excitation forcing amplitude f, where $f = 39$ is identified as the parameter leading to chaos. Figure 7(b) shows the phase portrait with the initial conditions $x(0) = 3{\cdot}0$, $\dot{x}(0) = 3{\cdot}0$, while Figure 7(c) depicts the corresponding time responses and Figure 7(d) shows the corresponding frequency spectrum. Figure 8(a, b) show the averaged responses for $N = 400$, 10 000 and Figure 8(c, d) depict the corresponding frequency spectra respectively. It can be easily shown, based on Figure 8(a, d) that the findings obtained for the first set of parameters still hold.

2.3. PIECEWISE LINEAR EQUATION

Consider the piecewise linear equations in the following:

$$\ddot{x} + \gamma\dot{x} + g(x) = f\sin(\omega_s t), \tag{3}$$

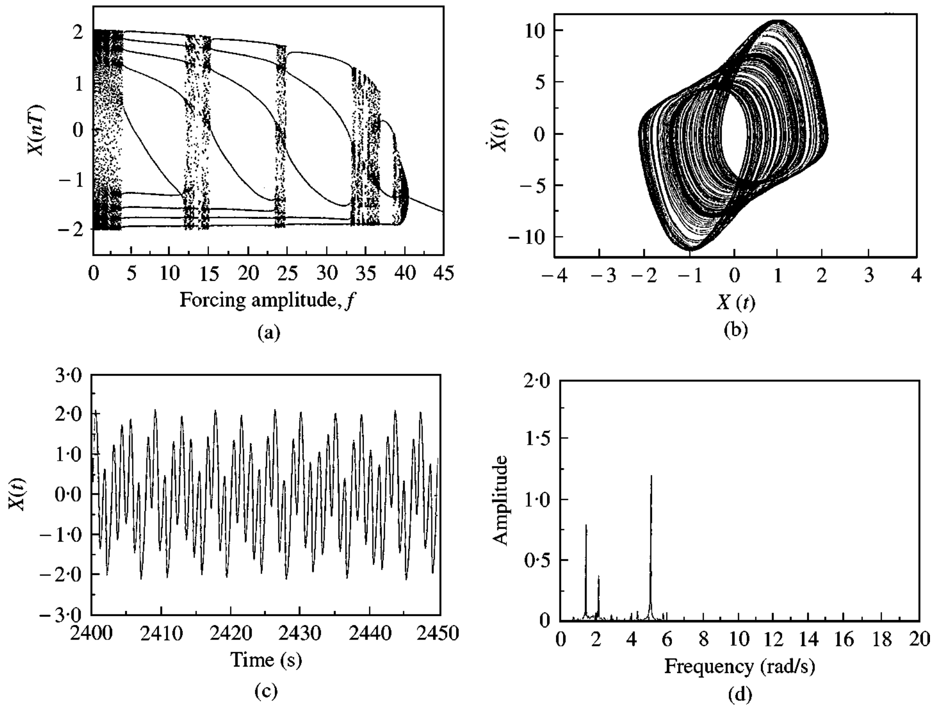

Figure 7. The second case of van der Pol equation (a) bifurcation plot, (b) phase portrait, (c) time history, and (d) spectrum for initial condition (3, 3).

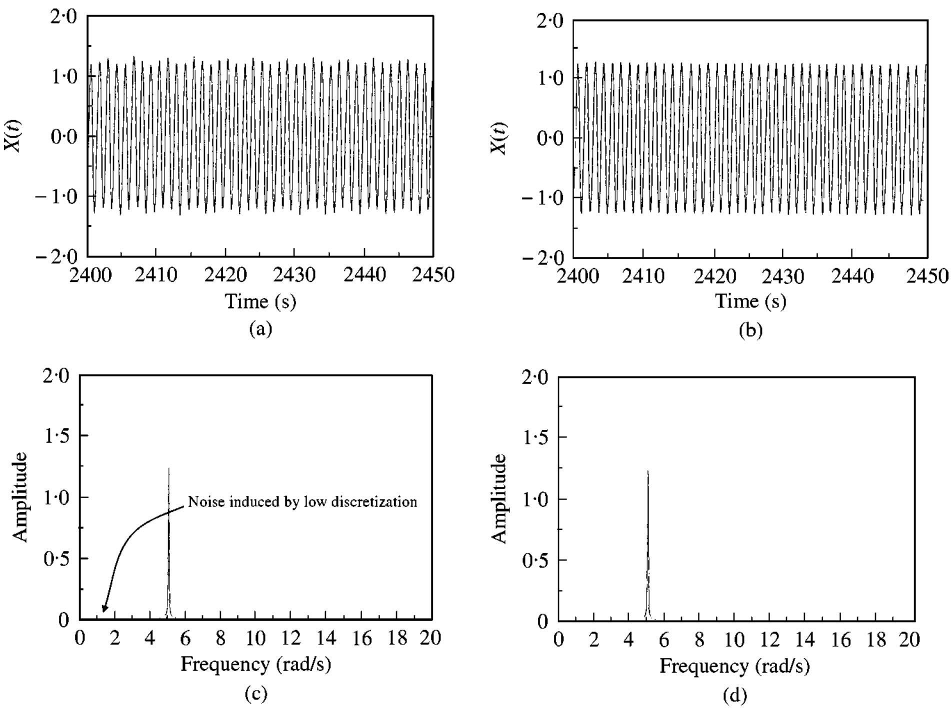

Figure 8. The second case for averaged responses of van der Pol equation in initial condition domain $-3 \leqslant x \leqslant 3$, $-3 \leqslant \dot{x} \leqslant 3$: time history of (a) 400 points and (b) 10 000 points; spectrum of (c) 400 points and (d) 10 000 points.

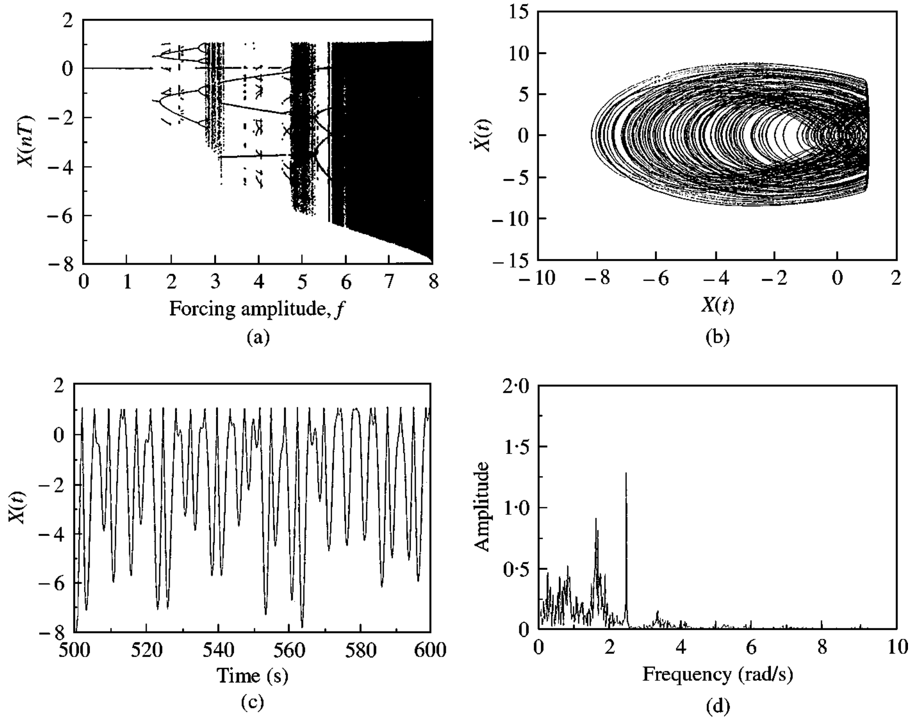

Figure 9. The first case of piecewise linear equation (a) bifurcation plot, (b) phase portrait, (c) time history, and (d) spectrum for initial condition (0, 0).

where

$$g(x) = \begin{cases} x & \text{for } x < d, \\ \eta^2 x + (1 - \eta^2)d & \text{for } x > d, \end{cases}$$

where $\gamma = 0{\cdot}02$, $\eta = 80$, $d = 1{\cdot}0$, $\omega_s = 2{\cdot}5$. Note that the piecewise linear equations are often used to describe the dynamics to incorporate gear backlash, corrosion, frictions and the clearance of suspension systems, etc. Figure 9(a) shows the bifurcation plot of sampled response amplitude $X(nT)$ by the method of Poincaré sections versus the excitation forcing amplitude f. For $f = 6{\cdot}25$ when the system undergoes chaotic motion, Figure 9(b) shows the phase portrait with the initial conditions $x(0) = 0{\cdot}0$, $\dot{x}(0) = 0{\cdot}0$, while Figure 9(c) shows the corresponding time responses. In Figure 9(c), it can be observed that the response is non-periodic. Figure 9(d) shows the corresponding frequency spectrum.

With the chaotic response confirmed for $f = 6{\cdot}25$, the same procedure for simulations are next performed to obtain pre-defined "averaged responses". The rectangular domain considered is $-0{\cdot}01 \leqslant x \leqslant 0, -0{\cdot}01 \leqslant \dot{x} \leqslant 0$, and $N = 400$, $10\,000$. Figure 10(a, b) show the averaged responses respectively. Compared to Figure 9(c), the averaged response in Figure 10(a, b) shows higher levels of periodicity. The corresponding frequency spectra are shown in Figure 10(c, d). It can be also seen from Figure 10(c, d) that the peaks are located at the fundamental frequency $\omega = 2{\cdot}5$ which coincides with the excitation frequency. Small frequency contents, below 2·0 are observed in Figure 10(c), which, compared to Figure 6(d), are identified as noise due to insufficient number of discretizing initial conditions.

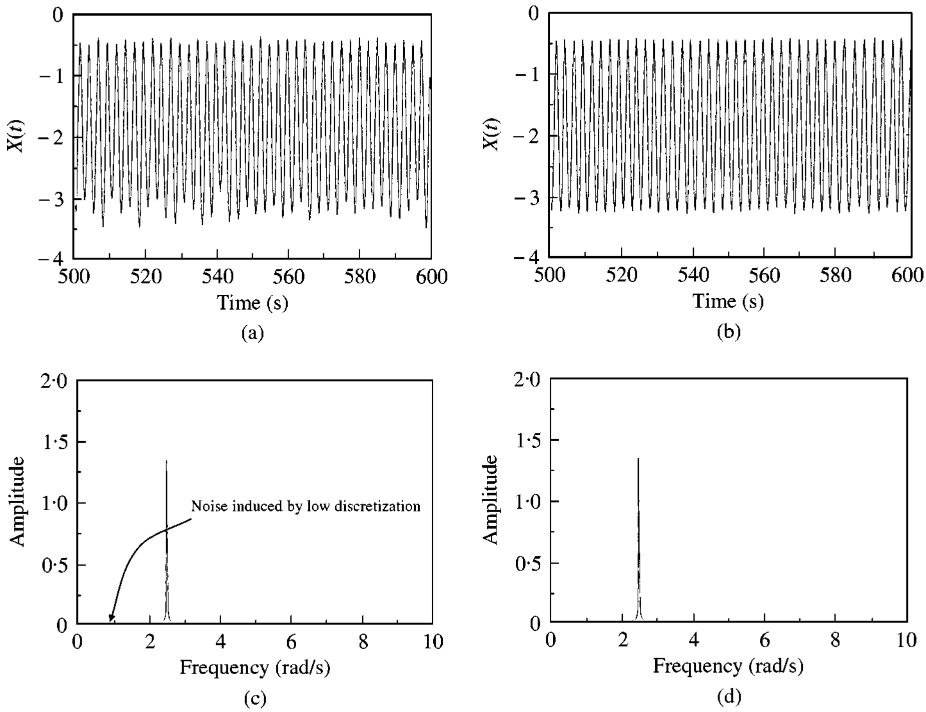

Figure 10. The first case for averaged responses of piecewise linear equation in initial condition domain $-0{\cdot}01 \leqslant x \leqslant 0$, $-0{\cdot}01 \leqslant \dot{x} \leqslant 0$: time history of (a) 400 points and (b) 10 000 points; spectrum of (c) 400 points and (d) 10 000 points.

The above study is repeated to confirm the findings with different parameters $\gamma = 0{\cdot}1$, $\eta = 10$, $d = 0{\cdot}05$, $\omega_s = 0{\cdot}7$ and $N = 100$, 2500. Figure 11(a) shows the bifurcation plot of sampled response amplitude $X(nT)$ by the method of Poincaré sections versus the excitation forcing amplitude f, where $f = 0{\cdot}49$ is identified as the parameter leading to chaos. Figure 11(b) shows the phase portrait with the initial conditions $x(0) = 0{\cdot}0$, $\dot{x}(0) = 0{\cdot}0$, while Figure 11(c) depicts the corresponding time responses and Figure 11(d) shows the frequency spectrum. It is observed that the time interval between two adjacent peaks is twice that of the excitation period. This phenomenon is due to the period-double bifurcation effects. Figure 12(a, b) show the averaged responses for $N = 100, 2500$ and Figure 12(c, d) depict the corresponding frequency spectra respectively. It is seen that the fundamental frequency of the averaged response $\omega = 0{\cdot}35$ is half the excitation frequency $\omega_s = 0{\cdot}7$ due to period-doubling effects. Despite this difference from the first case studied, it can still be seen that the averaged response exhibits a strong near-periodicity.

3. CONCLUSION

Three forms of non-linear chaotic systems namely: Duffing equation, van der Pol equation and piecewise linear equation were considered to validate the periodicity of the "averaged responses". For each form, two sets of system parameters were employed for case study; therefore, totally six case studies were conducted. For each case, discretization on a closed domain in phase plane was first performed to specify N^2 initial conditions and then simulations were conducted for the N^2 initial conditions to obtain N^2 responses. Averaging

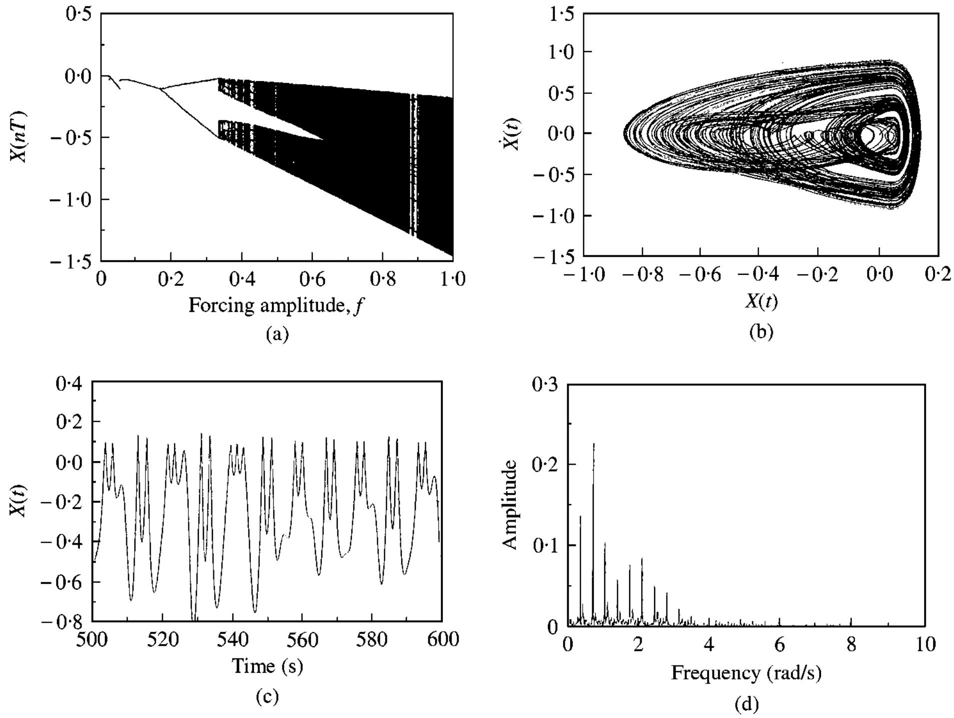

Figure 11. The second case of piecewise linear equation (a) bifurcation plot, (b) phase portrait, (c) time history, and (d) spectrum for initial condition (0, 0).

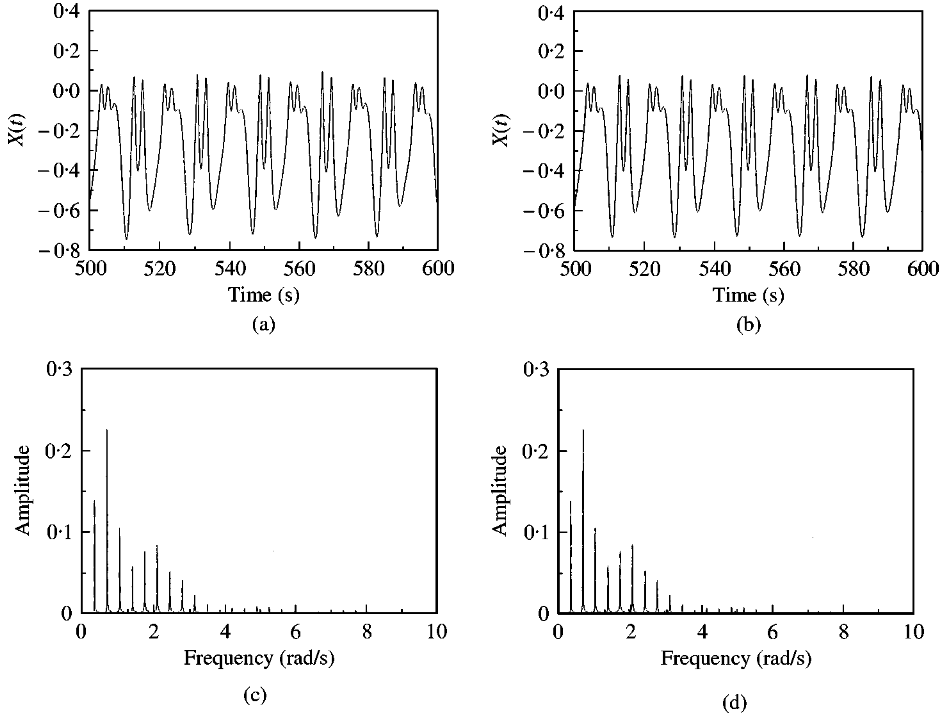

Figure 12. The second case for averaged responses of piecewise linear equation in initial condition domain $-0{\cdot}01 \leqslant x \leqslant 0$, $-0{\cdot}01 \leqslant \dot{x} \leqslant 0$: time history of (a) 100 points and (b) 2500 points; spectrum of (c) 100 points and (d) 2500 points.

these N^2 responses, the "averaged response" was then attained for analysis. Based on the results by the averaged responses presented from Figures 1–12 it is concluded that the "averaged responses" of chaotic Duffing, van der Pol and piecewise linear equations exhibit strong near-periodicity with fundamental frequencies the same as those of the excitations. Furthermore, the near-periodicity approaches a perfect periodicity as the number of initial conditions in the chosen rectangular domain increases.

For the piecewise linear equations, in the second case with certain system parameters, large frequency components are found in the middle of the spectrum (due to period-doubling bifurcation) with each multiple of the excitation frequency and these components weakening as the frequency increases.

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