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A NEW METHOD FOR MODEL ORDER SELECTION AND MODAL PARAMETER ESTIMATION IN TIME DOMAIN

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This paper mainly deals with the identification of structural modal parameters, from output accelerometers only, using a vector autoregressive moving average (VARMA(p, q)) model method. The problem of determining the order p of the AR part, or equivalently the number of modes in a frequency band, is also examined using a combination of the multivariate minimum description length (MDL) criterion and an overdetermined instrumental variable sequence. The AR coefficients are then obtained from this instrumental variable sequence. The companion matrix is formed and the modal parameters of the vibrating system deduced. Numerical and experimental results are treated to show the effectiveness of this new procedure.

1. INTRODUCTION

Modal parameter identification is the procedure used to identify parameters of the modal model describing the dynamic properties of a mechanical system in vibration. The methods for modal parameter identification can be categorized into two groups: frequency domain methods and time domain methods. The frequency domain analysis has been widely used for many years and has been proved to be efficient in many cases. This frequency method uses the fast Fourier transform (FFT) and curve fitting but is limited by the problem of leakage, interference and bias. To avoid the limitation of frequency domain analysis, time domain methods have been developed. These methods have generally been used for identification in the form of impulse or free decay responses, which are then considered as weighted expansions of participating modes in the vibration. These time domain methods have demonstrated the capability of identifying very closely spaced or pseudo-repeated frequencies and heavy damping ratios. Another advantage of these time domain methods is that they do not need to perform any domain change: they do not use FFT to estimate modal parameters or to apply any time window and problems such as leakage, bias and variance are avoided. This leads to good estimated modal parameters, including short data records. The principal time domain methods are the Ibrahim time domain method [1] and the eigensystem realization method of Juang [2]. The Ibrahim time domain method uses a set of free decay vibration measurements in a single analysis to identify simultaneously all parameters of the excited modes in a test. The eigensystem realization algorithm of Juang [2] uses concepts of the controllability theory and impulse responses. These methods consider both input and output data for the identification of modal parameters.

In this paper, a new method is presented to identify the mode shapes as well as the natural frequencies and damping ratios of a vibrating structure on the basis of a multivariate

ARMA process [3–6]; the mathematical model method uses an unmeasured white noise as excitation and treats random responses as the time series of a VARMA(p, q) process. The identification of modal parameters is realized here in the time domain from accelerometers output only. Using a VARMA(p, q) representation, we estimate the autoregressive AR coefficients and form the companion matrix. The spectral decomposition of this matrix brings us to the estimation of modal parameters. The first step in the estimation of AR coefficients is the determination of the number of these coefficients. This number is p and is also the order of the multivariate AR part. Once p has been estimated, the number of modes in a frequency band is derived. Note that in a mechanical structure, the number n of modes in a frequency band is related to the order p of the AR part and to the number of accelerometers m by the well known relation $p = 2n/m$.

Several information theoretical criteria have been proposed for this model order selection task. Akaike [7, 8] has provided two criteria. His first criterion is the final prediction error (FPE) criterion and selects the order so that the average error variance for a one-step prediction is minimized. Akaike also suggested another selection criterion using a maximum likelihood approach to derive a criterion, termed the Akaike information criterion (AIC). The AIC determines the model order by minimizing an information theoretical function. FPE and AIC are asymptotically equivalent, but do not yield consistent estimates of the model order; the result is a tendency to overestimate the order as the data record length increases [9]. In response to this, another effective criterion (the minimum description length (MDL) criterion) is proposed by Schwarz [10] and Rissanen [11]. The MDL criterion, also called the Bayesian information criterion (BIC), uses a penalty function which provides consistent estimation of the model order. All these methods are only applicable to scalar processes and a generalization to multivariate processes is established in this paper.

Using a combination of an overdetermined instrumental variable scheme [12] and the multivariate MDL criterion a new method for AR order determination of a vector-autoregressive moving average or VARMA(p, q) process is proposed. To determine p, the order of the multivariate AR part, an overdetermined instrumental variable product moment matrix is defined. Once p has been estimated, the AR coefficients are derived from the optimization of this MDL criterion.

This paper is organized as follows. A model of a vibrating structure and its VARMA formulation is given in section 2. The determination of the order p using the multivariate minimum description length is described in section 3. The estimation of AR coefficients is also treated in this section. The effectiveness of the method for model order selection, and modal parameter estimation, from output accelerometers only, is shown in section 4 with a numerical example and mechanical structures in laboratory. This paper is summarized briefly in section 5.

2. MODELLING A VIBRATING SYSTEM AND ITS VARMA REPRESENTATION

Consider a structure excited by an unknown random Gaussian force. The objective is to determine the number of modes excited in a frequency band from the time response delivered by the output of m sensors (accelerometers) and the modal parameters of this structure.

For an n-degree-of-freedom vibratory system, the equation of motion can be expressed as

$$\mathbf{M}_0\ddot{\boldsymbol{\xi}}(t) + \mathbf{C}_0\dot{\boldsymbol{\xi}}(t) + \mathbf{K}_0\boldsymbol{\xi}(t) = \boldsymbol{\eta}(t), \tag{1}$$

where $\mathbf{M}_0$, $\mathbf{C}_0$ and $\mathbf{K}_0$ are the system mass, damping and stiffness matrices $(n \times n)$ respectively; $\ddot{\boldsymbol{\xi}}(t)$, $\dot{\boldsymbol{\xi}}(t)$ and $\boldsymbol{\xi}(t)$ are $(n \times 1)$ vectors of acceleration, velocity and displacement and $\boldsymbol{\eta}(t)$ the $(n \times 1)$ unmeasured excitation vector, which is a random Gaussian force. A state space model can be formed in lieu of the model given by equation (1) as

$$\dot{\mathbf{x}}(t) = \tilde{\mathbf{A}}\mathbf{x}(t) + \tilde{\mathbf{B}}\boldsymbol{\eta}(t), \tag{2}$$

where $\mathbf{x}(t)$ is the $2n$ dimensional state vector

$$\mathbf{x}(t) = \begin{bmatrix} \boldsymbol{\xi}(t) \\ \dot{\boldsymbol{\xi}}(t) \end{bmatrix} \tag{3}$$

and $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$ are the $(2n \times 2n)$, $(2n \times n)$ matrices

$$\tilde{\mathbf{A}} = \begin{bmatrix} 0 & \mathrm{I} \\ -\mathbf{M}_0^{-1}\mathbf{K}_0 & -\mathbf{M}_0^{-1}\mathbf{C}_0 \end{bmatrix}; \quad \tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}_0^{-1} \end{bmatrix}. \tag{4}$$

The response of the dynamic system is measured by the m output quantities in the output $\mathbf{y}(t)$ using accelerometers. An $(m \times 1)$ vector output equation called the observation equation can be written as

$$\mathbf{y}(t) = \mathbf{H}_a\ddot{\boldsymbol{\xi}}(t) = \mathbf{H}_a\mathbf{M}_0^{-1}[-\mathbf{K}_0\boldsymbol{\xi}(t) - \mathbf{C}_0\dot{\boldsymbol{\xi}}(t) + \boldsymbol{\eta}(t)] \tag{5}$$

or

$$\mathbf{y}(t) = \mathbf{H}\mathbf{x}(t) + \mathbf{D}\boldsymbol{\eta}(t), \tag{6}$$

where $\mathbf{H}_a$ is the output influence matrix $(m \times n)$ for acceleration. This matrix specifies which points of the system are observed from accelerometers. $\mathbf{H}$ is the $(m \times 2n)$ output influence matrix for the state vector $\mathbf{x}(t)$

$$\mathbf{H} = \mathbf{H}_a\mathbf{M}_0^{-1}[-\mathbf{K}_0 - \mathbf{C}_0] \tag{7}$$

and $\mathbf{D}$ is an $(m \times n)$ direct transmission matrix

$$\mathbf{D} = \mathbf{H}_a\mathbf{M}_0^{-1}. \tag{8}$$

Equations (2) and (6) constitute a continuous time state-space model of a dynamic system. Note that the order of the system $2n$ is the dimension of the state matrix $\tilde{\mathbf{A}}$. The dynamic characteristics of the system governed by equation (1) are fully represented by the system matrix $\tilde{\mathbf{A}}$. In other words, if modal decomposition is desired, the modal parameters can be obtained by solving the following eigenvalue problem:

$$[\tilde{\mu}_i\mathbf{I} - \tilde{\mathbf{A}}]\tilde{\boldsymbol{\Omega}}_i = 0, \tag{9}$$

where $\tilde{\mu}_i$'s are the eigenvalues and $\tilde{\boldsymbol{\Omega}}_i$'s are the eigenvectors of $\tilde{\mathbf{A}}$. Note that the eigenvectors corresponding to the measurements are $\mathbf{H}_a\tilde{\boldsymbol{\Omega}}_i$.

After sampling with constant period Δt and transformation of the $2n$ first order differential equations (2) and (6) into a discrete time equation, the following discrete time state-space model and the discrete time observation equation are obtained:

$$\mathbf{x}_{t+1} = \mathbf{F}\mathbf{x}_t + \mathbf{v}_t, \tag{10}$$

$$\mathbf{y}_t = \mathbf{H}\mathbf{x}_t + \mathbf{D}\boldsymbol{\eta}_t, \tag{11}$$

where $\mathbf{x}_t$ represents the discrete unobserved state vector of dimension $2n$; $\mathbf{F} = e^{\tilde{\mathbf{A}}\Delta t}$ is the $(2n \times 2n)$ discrete time state-space matrix or discrete time transition matrix and $\mathbf{v}_t$ is given by

$$\mathbf{v}_t = \int_0^{\Delta t} e^{\tilde{\mathbf{A}}s}\, \tilde{\mathbf{B}}\boldsymbol{\eta}(t-s)\,ds. \tag{12}$$

The excitation $\boldsymbol{\eta}_t$ is a random process and is not constant over the sampling period Δt, so $\mathbf{v}_t$ cannot be obtained directly by integration of equation (11). A method to obtain $\mathbf{v}_t$ is described in reference [13].

The interest here is the mixed vector (or multivariate) autoregressive moving average representation for the $\{\mathbf{y}_t\}$ process. Consider the m-dimensional stationary processes $\mathbf{y}_t$ with VARMA (p, q) representation [2–6]

$$\mathbf{y}_t = \sum_{i=1}^{p} \mathbf{A}_i \mathbf{y}_{t-i} + \sum_{j=0}^{q} \mathbf{B}_j \mathbf{u}_{t-j}, \tag{13}$$

where $\mathbf{y}_t$ is the discrete observation vector and $\mathbf{u}_t$ is a white Gaussian noise with zero mean and unknown covariance matrix $\mathbf{Q}_u = E\{\mathbf{u}_t\mathbf{u}_t^{\mathrm{T}}\}$. This equation contains the AR part with AR matrix coefficients A_i and the MA part with MA matrix coefficients B_j. The order of the autoregressive part is p, which is theoretically $p = 2n/m$. The order of the MA part is q.

The modal parameters of the vibrating system are completely characterized by the eigenvalues and eigenvectors of the companion matrix $\mathbf{A}$ [2, 3] containing the AR coefficients of the VARMA representation (13).

$$\mathbf{A} = \begin{bmatrix} 0 & I & 0 & . & 0 \\ 0 & 0 & I & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & I \\ \mathbf{A}_P & \mathbf{A}_{P-1} & . & . & \mathbf{A}_1 \end{bmatrix}. \tag{14}$$

The eigenvalues $\tilde{\mu}_i$ of the state matrix $\tilde{\mathbf{A}}$ and the eigenvalues μ_i of the companion matrix $\mathbf{A}$ have the following relationship:

$$\mu_i = e^{\tilde{\mu}_i \Delta t}. \tag{15}$$

The eigenvectors are the same for the continuous and discrete time: $\mathbf{H}_a\tilde{\boldsymbol{\Omega}}_i$. Therefore, the global modal parameters, the natural frequencies f_i and damping ratios c_i of the vibrating system can be determined by [3]

$$f_i = \frac{1}{2\pi\Delta t}\sqrt{\frac{[\ln(\mu_i\mu_i^*)]^2}{4} + \left[\cos^{-1}\left(\frac{\mu_i + \mu_i^*}{2\sqrt{\mu_i\mu_i^*}}\right)\right]^2}, \tag{16}$$

$$c_i = \sqrt{\frac{[\ln(\mu_i\mu_i^*)]^2}{\left[\ln(\mu_i\mu_i^*)]^2 + 4\left[\cos^{-1}\left(\frac{\mu_i + \mu_i^*}{2\sqrt{\mu_i\mu_i^*}}\right)\right]^2\right.}}, \tag{17}$$

for $i = 1, 2, \ldots, n$.

The eigenvalue decomposition of the companion matrix give the modal parameters of the vibrating system. In this paper, a procedure is proposed to determine the number of AR coefficients A_i, from accelerometer outputs only, these AR coefficients and then the modal parameters and mode shapes of the vibrating structure.

3. DETERMINATION OF THE NUMBER OF AR COEFFICIENTS BY THE MULTIVARIATE MDL CRITERION

Since the true orders p and q are unknown, consider the general case of equation (13) with (p, q) replaced by unknown (p', q'). Equation (13) for $t = 0, \ldots, N-1$ is developed and the matrix system

$$[[\mathbf{A}_0] - [\mathbf{A}_1] - \cdots - [\mathbf{A}_{p'}]] \begin{bmatrix} \mathbf{y}_0 & \mathbf{y}_1 & \mathbf{y}_2 & \cdot & \mathbf{y}_{p'} & \mathbf{y}_{p'+1} & \cdot & \mathbf{y}_{N-1} \\ \mathbf{0} & \mathbf{y}_0 & \mathbf{y}_1 & \cdot & \mathbf{y}_{p'-1} & \mathbf{y}_{p'} & \cdot & \mathbf{y}_{N-2} \\ \mathbf{0} & \mathbf{0} & \mathbf{y}_0 & \cdot & \mathbf{y}_{p'-2} & \mathbf{y}_{p'-1} & \cdot & \mathbf{y}_{N-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdot & \mathbf{y}_0 & \mathbf{y}_1 & \cdot & \mathbf{y}_{N-1-p'} \end{bmatrix}$$

$$= [[\mathbf{B}_0][\mathbf{B}_1] \cdots [\mathbf{B}_{q'}]] \begin{bmatrix} \mathbf{u}_0 & \mathbf{u}_1 & \mathbf{u}_2 & \cdot & \mathbf{u}_{p'} & \mathbf{u}_{p'+1} & \cdot & \mathbf{u}_{N-1} \\ \mathbf{0} & \mathbf{u}_0 & \mathbf{u}_1 & \cdot & \mathbf{u}_{p'-1} & \mathbf{u}_{p'} & \cdot & \mathbf{u}_{N-2} \\ \mathbf{0} & \mathbf{0} & \mathbf{u}_0 & \cdot & \mathbf{u}_{p'-2} & \mathbf{u}_{p'-1} & \cdot & \mathbf{u}_{N-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdot & \mathbf{u}_{p'-q} & \mathbf{u}_{p'-q'+1} & \cdot & \mathbf{u}_{N-1-q'} \end{bmatrix} \quad (18)$$

is formed with N the data length and $[\mathbf{A}_0] = [\mathbf{I}_m]$.

The previous system can be written as:

$$\boldsymbol{\Psi}_{p'} \mathbf{Y}_{p'} = \boldsymbol{\Omega}_{p'} \quad (19)$$

with $\mathbf{Y}_{p'}$ the $m(p'+1) \times N$ matrix of data, $\boldsymbol{\Omega}_{p'}$ the $(m \times N)$ matrix of coefficients MA and white Gaussian noise $\mathbf{u}_t$ and $\boldsymbol{\Psi}_{p'}$ the $m \times m(p'+1)$ matrix which contains the AR parameters:

$$\boldsymbol{\Psi}_{p'} = [\mathbf{A}_0 \mathbf{A}_1 \cdots \mathbf{A}_{p'}]. \quad (20)$$

In order to develop a multivariate MDL criterion, an extended instrumental variable matrix [12] $\mathbf{Z}_k$ of dimension $m(k+1) \times N$ is introduced:

$$\mathbf{Z}_k = \begin{bmatrix} \mathbf{z}_0 & \mathbf{z}_1 & \cdots & \mathbf{z}_k & \cdots & \mathbf{z}_{N-1} \\ \mathbf{0} & \mathbf{z}_0 & \cdots & \mathbf{z}_{k-1} & \cdots & \mathbf{z}_{N-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{z}_0 & \cdots & \mathbf{z}_{N-k-1} \end{bmatrix} \quad (21)$$

in which $\mathbf{z}_i$ is a vectorial instrumental variable sequence. The instrumental variable sequence is highly correlated with the observed sequence $\mathbf{y}_i$ but completely uncorrelated with the noise. If the case of $k = p'$ is considered, then

$$\lim_{N\to\infty} \frac{1}{N}\mathbf{Z}_{p'}\mathbf{Y}_{p'}^{\mathrm{T}} = \mathbf{R}; \quad \lim_{N\to\infty} \frac{1}{N}\mathbf{Z}_{p'}\boldsymbol{\Omega}_{p'}^{\mathrm{T}} = 0. \tag{22}$$

Note that the dimension of $\mathbf{Z}_k\mathbf{Y}_{p'}^{\mathrm{T}}$ is $m(k+1)\times m(p'+1)$ and increasing the number of rows in $\mathbf{Z}_k\mathbf{Y}_{p'}^{\mathrm{T}}$ implies the use of more information. An overdetermined instrumental variable method is then preferable. Pre-multiplying equation (19) by $(1/N)\mathbf{Z}_k^{\mathrm{T}}$ gives

$$\frac{1}{N}\boldsymbol{\Psi}_{p'}\mathbf{Y}_{p'}\mathbf{Z}_k^{\mathrm{T}} = \frac{1}{N}\boldsymbol{\Omega}_{p'}\mathbf{Z}_k^{\mathrm{T}} \tag{23}$$

Putting $\mathbf{W}_{p'} = \frac{1}{N}\mathrm{Y}_{p'}\mathbf{Z}_k^{\mathrm{T}}$ matrix $m(p'+1)\times m(k+1)$ and $\mathbf{V}_k = \frac{1}{N}\boldsymbol{\Omega}_{p'}\mathbf{Z}_k^{\mathrm{T}}$ matrix $m\times m(k+1)$ gives

$$\boldsymbol{\Psi}_{p'}\mathbf{W}_{p'} = \mathbf{V}_k. \tag{24}$$

Furthermore, the $m(p'+1)\times m(p'+1)$ overdetermined instrumental variable product moment matrix can be defined by

$$\boldsymbol{\Gamma}_{p'} = \mathbf{W}_{p'}\mathbf{W}_{p'}^{\mathrm{T}} = \frac{1}{N^2}\mathbf{Y}_{p'}\mathbf{Z}_k^{\mathrm{T}}\mathbf{Z}_k\mathbf{Y}_{p'}^{\mathrm{T}}. \tag{25}$$

Note that $\boldsymbol{\Gamma}_{p'}$ is a symmetric positive semidefinite matrix and both the matrix $\boldsymbol{\Gamma}_{p'}$ and the multivariate minimum description length (MDL) criterion give the order p of the AR part of the VARMA process. Following Schwarz [10] and Rissanen [11], the MDL criterion is equal to the sum of the log-likelihood function of the maximum likelihood estimator of the model parameters and a function that penalizes the use of a large number of model parameters. In the multivariate case, the number of free adjusted parameters in the AR part is $m^2(p'+1)$. Denote $\mathbf{V}_k = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{m(k+1)}]$.

This matrix consists of $m(k+1)$ independent, m-dimensional, normal random vectors $\mathbf{v}_i$, with zero mean and unknown covariance matrix $Q_v = E[\mathbf{v}_i\mathbf{v}_i^{\mathrm{T}}]$. Thus, the MDL criterion is given by

$$J_{MDL}(p') = -\log f(\mathbf{v}_1, \ldots, \mathbf{v}_{m(k+1)}) + \frac{1}{2}m^2(p'+1)\log(m(k+1)), \tag{26}$$

where $f(\mathbf{v}_1, \ldots, \mathbf{v}_{m(k+1)})$ denotes the probability density function of $\{\mathbf{v}_i\}$. For a multivariate normal model

$$f(\mathbf{v}_1, \ldots, \mathbf{v}_{m(k+1)}) = (2\pi)^{-m^2(k+1)/2}\times\frac{1}{\det(\mathbf{Q}_v)^{m(k+1)/2}}\exp\left(\frac{1}{2}\mathrm{tr}(\mathbf{Q}_{\mathrm{v}}^{-1}\boldsymbol{\Psi}_{p'}\boldsymbol{\Gamma}_{p'}\boldsymbol{\Psi}_{p'}^{\mathrm{T}})\right). \tag{27}$$

By substituting $f(\mathbf{v}_1, \ldots, \mathbf{v}_{m(k+1)})$ into equation (26), $J_{MDL}(p')$ reduces to

$$J_{MDL}(p', \boldsymbol{\Psi}_{p'}) = m^2\left(\frac{k+1}{2}\right)\log(2\pi)$$
$$+ m\left(\frac{k+1}{2}\right)\log(\det(\mathbf{Q}_v)) + \frac{1}{2}\mathrm{tr}(\mathbf{Q}_{\mathrm{v}}^{-1}\boldsymbol{\Psi}_{p'}\boldsymbol{\Gamma}_{p'}\boldsymbol{\Psi}_{p'}^{\mathrm{T}}) + \frac{1}{2}m^2(p'+1)\log(m(k+1)). \tag{28}$$

For fixed p', the matrix $\mathbf{Q}_v$ that minimizes criterion (28) is $\mathbf{Q}_v = \mathbf{\Psi}_{p'}\mathbf{\Gamma}_{p'}\mathbf{\Psi}_{p'}^{\mathrm{T}}$ and the minimum value of $\det(\mathbf{Q}_v)$ is obtained by the eigenvalue decomposition of $\mathbf{\Gamma}_{p'}$

$$\mathbf{\Gamma}_{p'} = [\mathbf{S}_{\Lambda} \quad \mathbf{S}_{\varepsilon}] \begin{bmatrix} \mathbf{\Lambda} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\varepsilon} \end{bmatrix} \begin{bmatrix} \mathbf{S}_{\Lambda}^{\mathrm{T}} \\ \mathbf{S}_{\varepsilon}^{\mathrm{T}} \end{bmatrix}, \tag{29}$$

where $\mathbf{\Lambda}$ is the diagonal matrix containing the mp' largest eigenvalues in decreasing order and the columns of $\mathbf{S}_{\Lambda}$ are the corresponding eigenvectors; $\boldsymbol{\varepsilon}$ is the diagonal matrix which contains the m smallest eigenvalues and the columns of $\mathbf{S}_{\varepsilon}$ are the corresponding eigenvectors. Constraining $\mathbf{\Psi}_{p'}$ to be orthogonal, the choice of $\mathbf{\Psi}_{p'}$ that minimizes criterion (28) is found to be the matrix of eigenvectors associated with the minimum eigenvalues of $\mathbf{\Gamma}_{p'}$. With this choice, $\mathbf{\Psi}_{p'} = \mathbf{S}_{\varepsilon}^{\mathrm{T}}$ and

$$\mathbf{Q}_{\mathrm{v}} = \mathbf{S}_{\varepsilon}^{\mathrm{T}}[\mathbf{S}_{\Lambda}\mathbf{\Lambda}\mathbf{S}_{\Lambda}^{\mathrm{T}} + \mathbf{S}_{\varepsilon}\boldsymbol{\varepsilon}\mathbf{S}_{\varepsilon}^{\mathrm{T}}]\mathbf{S}_{\varepsilon} = \mathbf{S}_{\varepsilon}^{\mathrm{T}}\mathbf{S}_{\Lambda}\mathbf{\Lambda}\mathbf{S}_{\Lambda}^{\mathrm{T}}\mathbf{S}_{\varepsilon} + \mathbf{S}_{\varepsilon}^{\mathrm{T}}\mathbf{S}_{\varepsilon}\boldsymbol{\varepsilon}\mathbf{S}_{\varepsilon}^{\mathrm{T}}\mathbf{S}_{\varepsilon}. \tag{30}$$

Since the eigenvectors are orthonormal $\mathbf{Q}_v = \boldsymbol{\varepsilon}$ and $\det(\mathbf{Q}_v) = \prod_{i=1}^{m} \varepsilon_i$, where ε_i are the smallest eigenvalues of $\mathbf{\Gamma}_{p'}$ (the diagonal elements of ε). Substituting into equation (28) and dropping all terms that do not depend on $\mathbf{p}'$ or $\mathbf{\Psi}_{p'}$,

$$J_{MDL}(\mathbf{p}') = m\left(\frac{k+1}{2}\right)\log\left(\prod_{i=1}^{m} \boldsymbol{\varepsilon}_i\right) + \frac{1}{2}m^2\mathbf{p}'\log(m(k+1)). \tag{31}$$

The $\mathbf{\Psi}_{p'}$ in the argument of J_{MDL} has been dropped, since the explicit $\mathbf{\Psi}_{p'}$ dependence is suppressed; it has been incorporated into the product $\prod_{i=1}^{m} \boldsymbol{\varepsilon}_i$ term. Multiplying both sides of equation (31) by $2/m(k+1)$ and combining terms gives

$$J_{MDL}(p')\frac{2}{m(k+1)} = \log\left(\prod_{i=1}^{m} \varepsilon_i m(k+1)^{mp'/(k+1)}\right). \tag{32}$$

Since the function log(.) is a monotonically increasing function a different criterion can be formed that contains exactly the same information as $J_{MDL}(p')$; the new criterion has its minimum value at the same point as $J_{MDL}(p')$. The new criterion chosen is

$$J_{MDL}(p') = \left(\prod_{i=1}^{m} \varepsilon_i\right)m(k+1)^{mp'/(k+1)}. \tag{33}$$

Therefore, examining the m minimum eigenvalues of the overdetermined instrumental variable product moment matrix $\mathbf{\Gamma}_{p'}$ is equivalent to examining the m minimum singular values of the matrix $\mathbf{W}_{p'}$, for different values of p'. Then the new multivariate minimum description length criterion (33) can be formed and an abrupt change in this criterion can be sought for different values of p' or equivalently $J(p')/J(p'-1)$ can be computed and search its minimum sought. Hence, the method for multivariate AR order determination is to select p' associated with the minimum of the quotient $J(p')/J(p'-1)$.

There are several ways to choose the instruments and the matrix $\mathbf{Z}_k$ satisfying conditions (22). In this paper, only the special case where $\mathbf{z}_t = \mathbf{y}_{t-h}$ is considered. Then, it can be seen from equation (13) that $h > q$ should be taken so that $\mathbf{z}_t$ is highly correlated with $\mathbf{y}_t$ but completely uncorrelated with $\mathbf{u}_t$. For the selection of instruments different values of h are considered. Selecting $\mathbf{z}_t = \mathbf{y}_{t-h}$ gives

$$\mathbf{W}_{p'} \frac{1}{N} \mathbf{Y}_{p'} \mathbf{Z}_k^{\mathrm{T}} = \begin{bmatrix} \hat{\mathbf{R}}(h) & \hat{\mathbf{R}}(h+1) & . & \hat{\mathbf{R}}(h+k) \\ \hat{\mathbf{R}}(h-1) & \hat{\mathbf{R}}(h) & . & \hat{\mathbf{R}}(h+k-1) \\ . & . & . & . \\ \hat{\mathbf{R}}(h-p') & \hat{\mathbf{R}}(h-p'+1) & . & \hat{\mathbf{R}}(h+k-p') \end{bmatrix}, \tag{34}$$

where $\hat{\mathbf{R}}(i)$'s are the sample covariance matrices $(m \times m)$ of $\mathbf{y}_t$ given by

$$\hat{\mathbf{R}}(i) = \frac{1}{N} \sum_{t=i}^{N-1} \mathbf{y}_t \, \mathbf{y}_{t-i}^{\mathrm{T}}, \quad i \geqslant 0 \tag{35}$$

and $\hat{\mathbf{R}}(-i) = \hat{\mathbf{R}}(i)^{\mathrm{T}}$.

It is important to note that this minimization problem under constraints gives

$$\boldsymbol{\Psi}_{p'} = \mathbf{S}_{\varepsilon}^{\mathrm{T}} \tag{36}$$

So, from this relation the AR coefficients can be determined, the companion matrix (14) formed and the modal parameters of the vibrating system extracted. This method to obtain AR coefficients is called the Overdetermined Instrumental Variable method (ODIV method). A numerical example and tests in laboratory are now presented.

4. EXAMPLES

4.1. A NUMERICAL EXAMPLE

The procedure for multivariate AR order determination and modal parameter estimation is now applied to an elementary system of three degrees of freedom constituted of masses and springs. This system is excited by a random force and corresponds to the situation where inputs cannot be measured. Only the responses are used. For the selection of the instrumental variables $\mathbf{z}_t = \mathbf{y}_{t-h}$ different values of h have been considered and it has been found that all these selections could give the satisfactory results of the product of the m minimum eigenvalues of the overdetermined instrumental variable product moment matrix $\Gamma_{p'}$ for different values of p'. This is not surprising because the delayed output y_{t-h} with h somewhat smaller than the MA order q are weakly correlated with the MA part given in equation (13); hence (22) holds approximately. In theory, all $\mathbf{z}_t = \mathbf{y}_{t-h}$ with $h > q$ automatically satisfy (22) and are very good instruments. However, it should be noted that the larger the h the weaker is $\mathbf{y}_{t-h}$ correlated with $\mathbf{y}_t$ and poor sample covariance matrices $\hat{R}(h)$ are obtained. Therefore, h should take a smaller value for a set of data, even if it is smaller than the unknown MA order q.

In the numerical example, arbitrarily the matrix of masses $\mathbf{M}_0 = [0{,}13\ 0\ 0;\ 0\ 0{,}\ 2\ 0;\ 0\ 0\ 0{,}\ 15]$, the matrix of stiffness $\mathbf{K}_0 = [30\ -10\ 0;\ -10\ 15\ -5;\ 0\ -15\ 12]$ and the matrix of damping $\mathbf{C} = 0{,}1\mathbf{M}_0 + 0{,}01\mathbf{K}_0$ have been chosen. The number of data points is $N = 1000$ and $\Delta t = 0{,}15$ s. In this numerical example, all excited masses are considered and $m = 1$ (one assumed sensor), which corresponds to the scalar case. Using the MDL criterion developed in the paper, or more precisely the quotient $J(p')/J(p'-1)$ for different values of p' and h with $h = 5$, statistics for one simulation run are shown in Table 1.

The minimum is obtained for $p' = 6$. The order of the AR part is $p = 6$. Thus, the number of scalar AR coefficients used to form the companion matrix is 6.

TABLE 1

Statistics of $J(p')/J(p'-1)$ with $m = 1$ and $h = 5$ for the simulated system

	$p' = 1$	$p' = 2$	$p' = 3$	$p' = 4$	$p' = 5$	$p' = 6$	$p' = 7$	$p' = 8$	$p' = 9$
$k = 7$	0·72	0·34	1·07	0·56	1·22	0·048	0·19	6·27	1·84
$k = 9$	0·71	0·25	1·06	0·64	1·14	0·050	1·04	2·06	1·74
$k = 11$	0·69	0·26	1·11	0·61	1·21	0·12	0·47	2·17	2·68
$k = 13$	0·64	0·25	1·12	0·59	1·16	0·12	1·13	1·53	2·54
$k = 15$	0·66	0·24	1·09	0·59	1·14	0·13	1·08	1·59	2·85

TABLE 2

Statistics of $J(p')/J(p'-1)$ with $m = 3$ and $h = 5$ for the simulated system

	$p' = 1$	$p' = 2$	$p' = 3$	$p' = 4$	$p' = 5$	$p' = 6$	$p' = 7$	$p' = 8$	$p' = 9$
$k = 7$	5×10^{-4}	3×10^{-6}	0·02	0·15	0·08	0·11	0·02	0·27	0·31
$k = 9$	5×10^{-4}	10^{-5}	0·05	0·11	0·21	0·64	0·09	0·27	0·28
$k = 11$	4×10^{-4}	2×10^{-5}	0·04	0·18	0·23	0·41	0·68	0·74	0·87
$k = 13$	3×10^{-4}	4×10^{-5}	0·05	0·15	0·29	0·46	0·75	0·85	0·82
$k = 15$	3×10^{-4}	7×10^{-5}	0·04	0·22	0·23	0·56	0·49	0·67	0·71

TABLE 3

Estimated natural frequencies and damping coefficients of the simulated system of three degrees of freedom with $m = 3$ and $h = 5$

	f_1	f_2	f_3	c_1	c_2	c_3
$k = 7$	0·957	1·558	2·495	0·042	0·056	0·084
$k = 9$	0·956	1·556	2·499	0·041	0·057	0·080
$k = 11$	0·955	1·557	2·501	0·042	0·057	0·082
$k = 13$	0·953	1·558	2·516	0·040	0·058	0·070
$k = 15$	0·952	1·560	2·508	0·042	0·058	0·086

If the multivariate case with $m = 3$ (three assumed sensors) and $h = 5$ is considered, statistics for one simulation run different values of $J(p')/J(p'-1)$ are shown in Table 2.

The minimum is obtained for $p' = 2$. The order of the multivariate AR part is $p = 2$. So, the number of AR matrices (3×3) used to form the companion matrix is 2. It is easy to verify that in these two cases, $p = 2n/m$.

Once the order has been estimated the AR coefficients can be determined from (36) and the companion matrix formed. The exact natural frequencies and damping coefficients of the simulated system are $f_i = \{0{\cdot}957;\ 1{\cdot}564;\ 2{\cdot}531\}$ and $c_i = \{0{\cdot}038;\ 0{\cdot}054;\ 0{\cdot}082\}$. The estimated natural frequencies, and damping coefficients of this vibrating system, obtained from multi-output data only, with $m = 3$; $h = 5$ and for different values of k are given in Table 3.

This method gives satisfactory results in frequency and damping coefficients estimation.

Figure 1 shows the mode shapes for the simulated system. The proposed method yields accurate estimates of the mode shapes.

Note that these results are obtained from only one simulation run and improved results can be obtained using the ensemble averaged over several independent realizations. Improved results can also be obtained if the number of data points N increase.

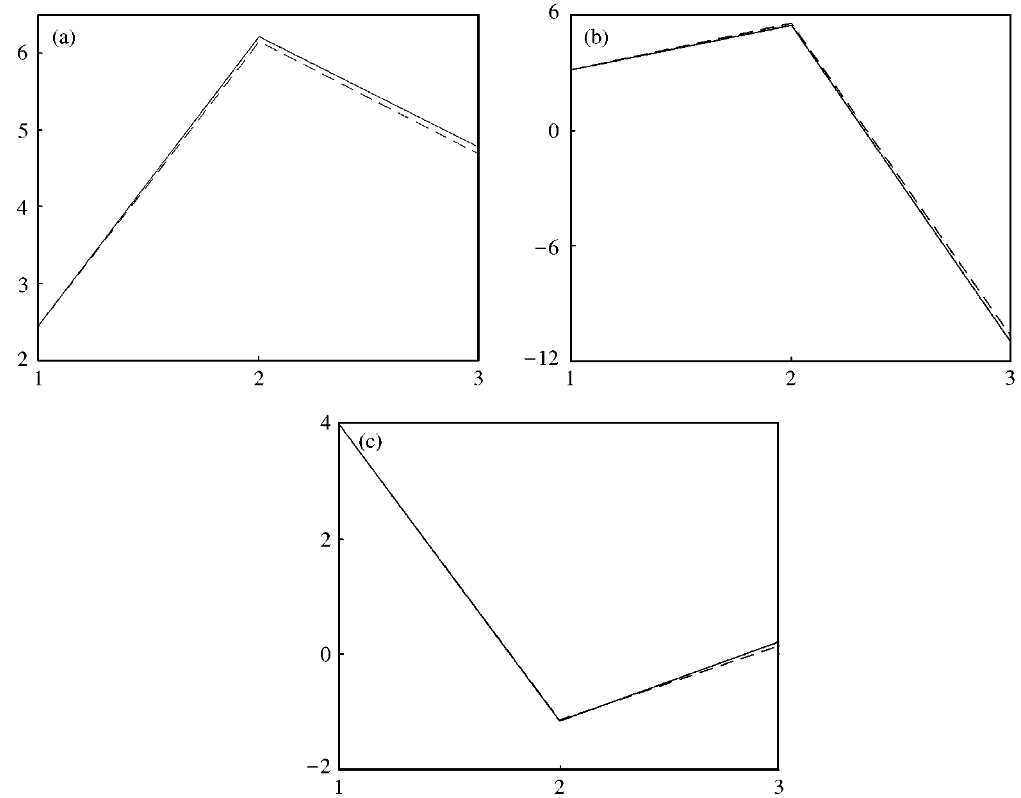

Figure 1. Comparison of exact (continuous line) and identified (dashed line) mode shapes for the simulated system of masses and springs: (a) first mode; (b) second mode; (c) third mode.

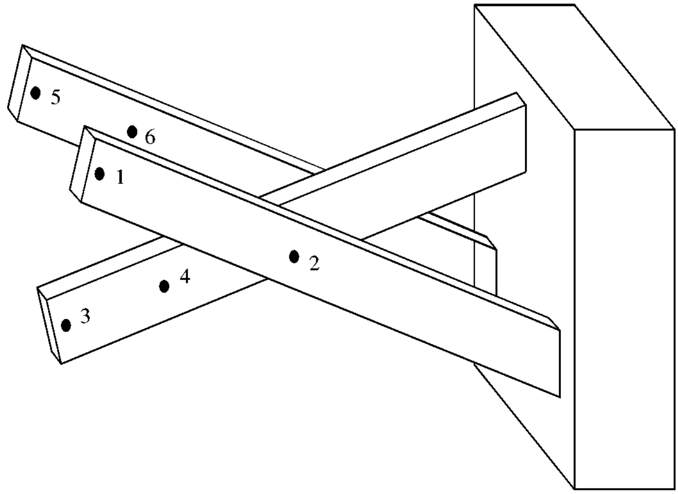

Figure 2. Experimental X beam.

4.2. A FIRST EXPERIMENTAL TEST: AN X BEAM

A first experimental test of three beams, called X beam, is treated (Figure 2). Three random excitations are applied on points 1, 3 and 5. Three accelerometers on points 2 and

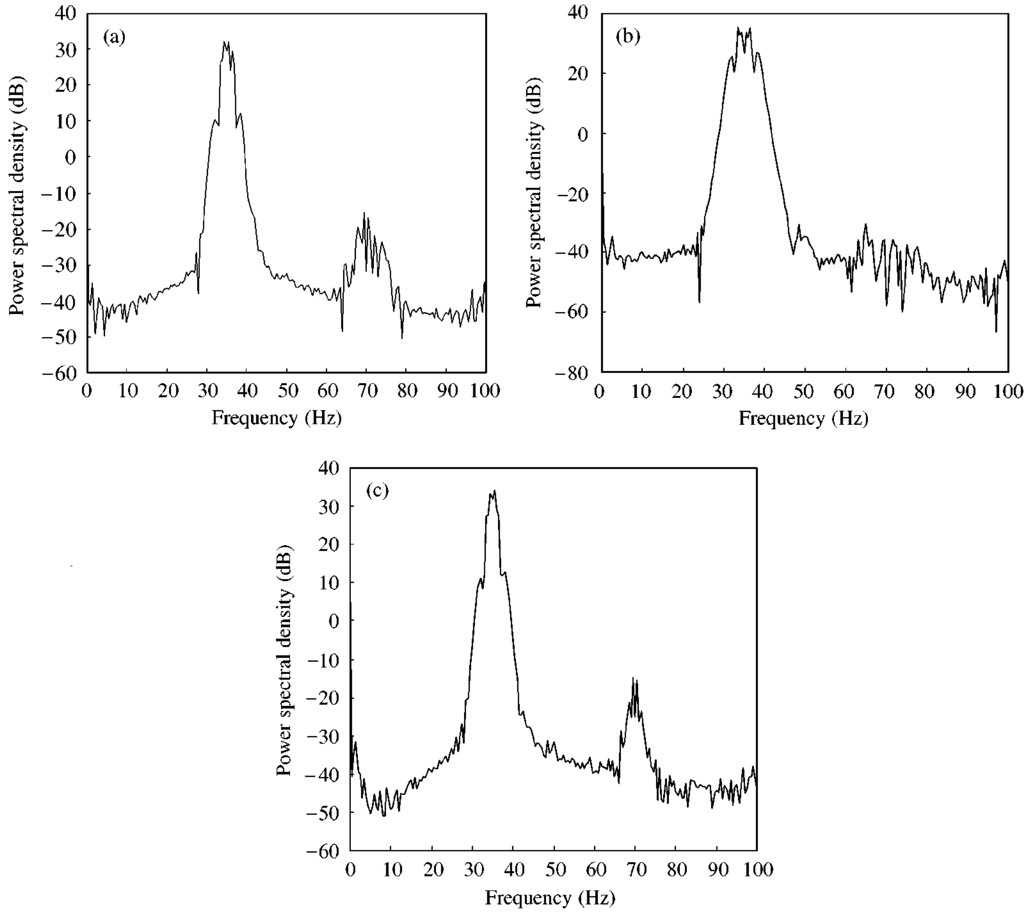

Figure 3. Frequency response of accelerometers: (a) on point 2; (b) on point 4; (c) on point 6.

4 and 6 and only their time responses are used. The sampling frequency is 512 Hz and 1500 data points are collected for each channel.

The purpose is to determine the order of the multivariate AR part (the number of AR matrices to form the companion matrix), or equivalently, the number of modes in the frequency band [0; 100 Hz], the eigenfrequencies and damping coefficients of this mechanical system, using data from output accelerometers only. Figure 3 shows the power spectral density function of accelerometers on points 2, 4 and 6 respectively. Note that the power spectral density is a measurement of the signal energy at various frequencies of interest.

It is impossible to determine the number of modes in the frequency band [0; 100 Hz] by counting the number of peaks of resonance from these power spectral density plots.

To determine the number of modes, consider the criterion presented in the paper, and compute for different values of p' the quotient $J(p')/J(p'-1)$ and seek its minimum. The value p of p' for which the minimum is reached is related to the number of modes n by $p = 2n/m$. In this case, consider $m = 3$; $h = 5$ and the statistics shown in Table 4 are seen.

The minimum is always obtained for $p' = 2$, the number of AR coefficients is 2 and the number of modes of the mechanical system is 3. This new criterion is very effective to determine the number of modes from output accelerometers only.

TABLE 4

Statistics of $J(p')/J(p'-1)$ with $m = 3$ and $h = 5$ for the X beam

	$p' = 1$	$p' = 2$	$p' = 3$	$p' = 4$	$p' = 5$	$p' = 6$	$p' = 7$	$p' = 8$	$p' = 9$
$k = 7$	2×10^{-5}	10^{-8}	4×10^{-4}	0·08	0·06	0·15	0·06	0·12	0·16
$k = 8$	10^{-5}	10^{-8}	3×10^{-4}	0·09	0·11	0·47	0·26	0·38	0·41
$k = 9$	10^{-5}	10^{-8}	2×10^{-4}	0·14	0·10	0·56	0·91	0·94	0·88
$k = 10$	10^{-5}	10^{-8}	10^{-4}	0·19	0·12	0·63	0·83	0·87	0·96
$k = 11$	10^{-5}	10^{-8}	8×10^{-5}	0·16	0·14	0·61	0·70	0·71	0·73

TABLE 5

Estimated natural frequencies and damping coefficients of the X beam with $m = 3$ and $h = 5$

	f_1	f_2	f_3	c_1	c_2	c_3
$k = 7$	34·33	35·76	36·74	0·75	0·80	0·20
$k = 9$	34·34	35·75	36·73	0·80	0·86	0·18
$k = 11$	34·34	35·75	36·74	0·81	0·87	0·19
$k = 13$	34·32	35·73	36·74	0·85	0·87	0·20
$k = 15$	34·31	35·69	36·74	0·94	0·90	0·17

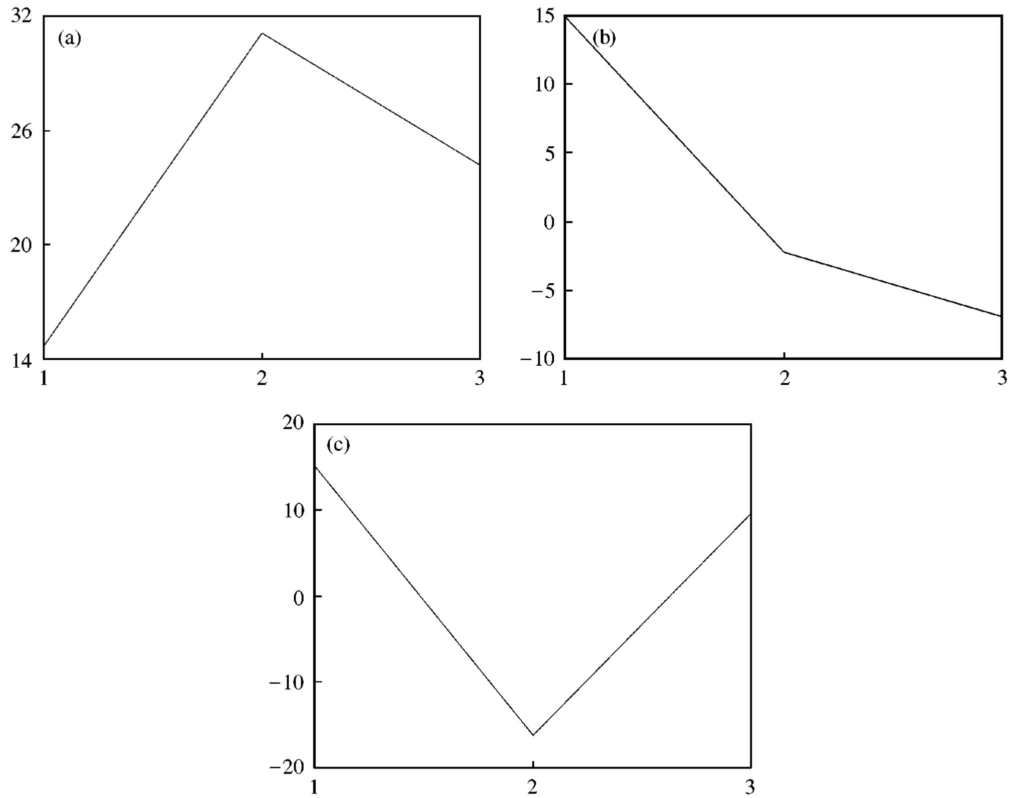

Figure 4. Mode shapes for the X beam: (a) first mode; (b) second mode; (c) third mode.

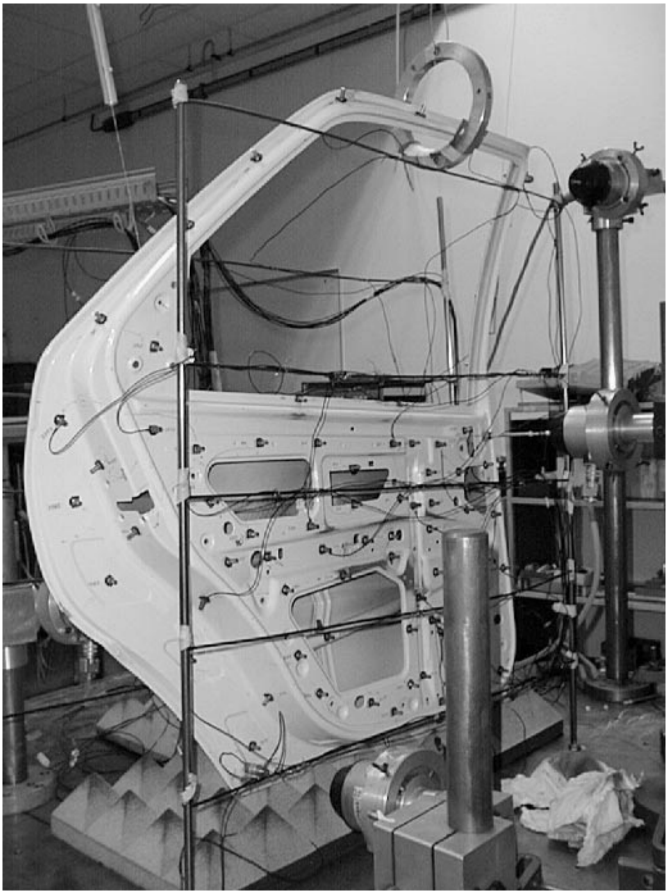

Figure 5. Experimental car door.

To obtain the modal parameters, determine the AR matrices coefficients A_1 and A_2 from equation (36), and form the companion matrix and its spectral decomposition. The estimated eigenfrequencies, in Hz, and damping coefficients, in percent, of this system, obtained from multi-output data only, with $m = 3$; $h = 5$ and for different values of k are given in Table 5.

Figure 4 shows the mode shapes for the X beam using the time series procedure presented here, from output accelerometers only, without measurement of the input force.

The method proposed here is very effective in determining modal parameters of vibrating systems with close natural eigenfrequencies.

4.3. A SECOND EXPERIMENTAL TEST: A CAR DOOR

A second experimental test of a car door is treated (Figure 5). A random and unmeasured excitation is applied on the car door and eight accelerometers are considered to estimate the modal parameters of this mechanical system. The sampling frequency is 400 Hz and 4096 data points are collected for each channel.

The purpose is to determine the order of the multivariate AR part, or equivalently, the number of modes in the frequency band [0; 200 Hz], the eigenfrequencies and damping coefficients of this car door from output accelerometers only. Figure 6 shows the power spectral density function obtained from three different accelerometers using an FFT. It is impossible to determine the number of modes in the frequency band [0; 200 Hz] by counting the number of peaks of resonance from these power spectral density plots.

To determine the order p of the AR part consider the criterion presented in the paper; compute for different values of p' the quotient $J(p')/J(p'-1)$ and search its minimum. Once the order p has been obtained the number of modes in the frequency band [0; 200 Hz] can be determined. In this case, consider $m = 8$; $h = 20$ and p is an integer of $(2n/m)$, which gives the statistics shown in Table 6.

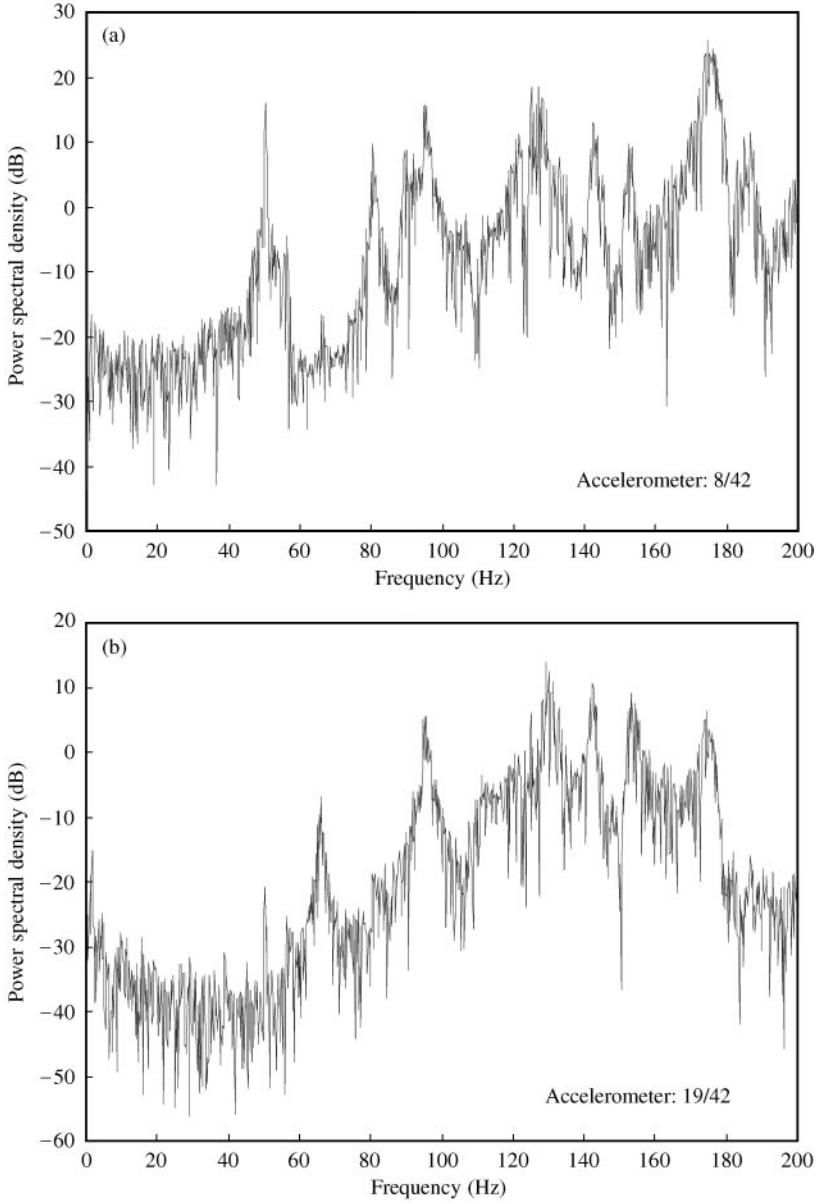

Figure 6. Frequency response from different accelerometers.

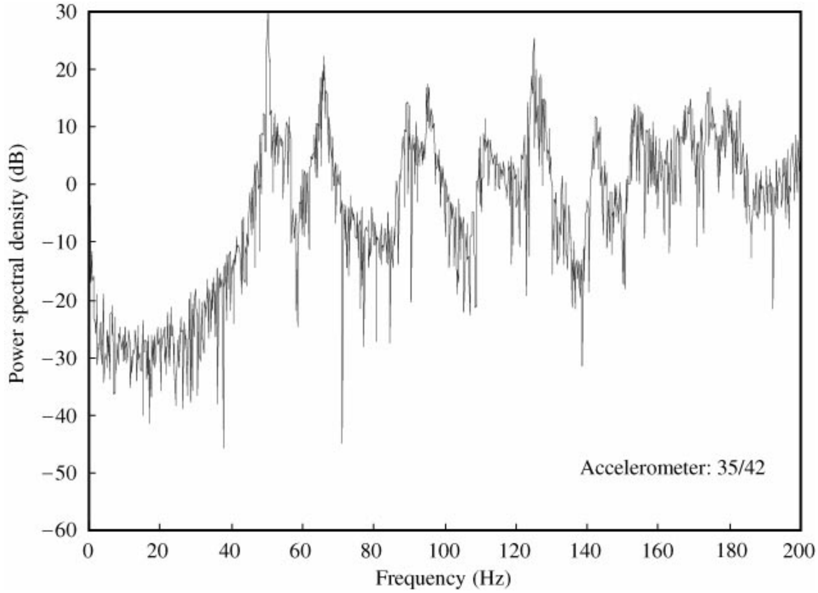

Figure 6. Continued.

TABLE 6

Statistics of $J(p')/J(p'-1)$ with $m=8$ and $h=20$ for the car door

	$p'=2$	$p'=3$	$p'=4$	$p'=5$	$p'=6$	$p'=7$	$p'=8$	$p'=9$
$k=10$	0·85	0·39	0·03	$4{\cdot}0\times10^{-4}$	1·64	1·26	0·81	0·65
$k=20$	0·22	0·59	0·51	$3{\cdot}5\times10^{-3}$	0·84	0·73	0·41	0·22
$k=30$	0·43	0·02	0·06	$5{\cdot}2\times10^{-3}$	1·66	0·94	0·31	0·44

The minimum value of $J_{MDL}(p')/J_{MDL}(p'-1)$ is always obtained for $p'=5$, the number of AR coefficients is 5 and the number of modes in the frequency band [0; 200 Hz] of this car door is 20. To form the companion matrix (14) 5 AR coefficient matrices A_i (8×8) are required. The estimated eigenfrequencies, in Hz, and damping coefficients, in percent, of this car door, obtained from accelerometers only, with $m=8$; $h=20$ and for different values of k are given in Table 7.

The method proposed in this paper is effective in determining the order of the AR part, the number of modes in a frequency band and modal parameters of this car door excited by a random force. Similar results for these modal parameters are obtained using the frequency domain by FRF where the excitation force or the input (sinus) is known and combined with the accelerometers output or using the Stochastic Realization Algorithm (SRA) [2, 4]. Here, only the outputs of the accelerometers are used without knowledge of the excitation.

5. CONCLUSION

A new approach for modal parameter estimation of a randomly excited structural system with an unmeasured input is proposed in this paper. Initially, a time domain procedure for

TABLE 7

Estimated natural frequencies and damping coefficients of the car door with the overdetermined instrumental variable method ($m = 8$; $h = 20$ and $k = 20$) and with SRA method

Mode	Frequencies (Hz) ODIV method	Damping (%) ODIV method	Frequencies (Hz) SRA method	Damping (%) SRA method
1	50·27	0·53	50·42	—
2	68·94	0·22	68·90	0·30
3	70·49	0·88	70·33	0·83
4	83·22	0·38	83·28	0·45
5	95·05	0·44	95·06	0·51
6	104·06	0·97	104·15	1·11
7	109·34	1·32	109·65	1·23
8	125·26	1·01	125·34	0·92
9	132·44	0·71	132·18	0·65
10	135·70	0·75	135·69	0·74
11	139·58	0·84	139·65	0·68
12	145·67	0·73	145·71	0·72
13	151·81	0·41	151·80	0·38
14	158·80	1·41	158·69	1·21
15	164·72	1·34	164·40	1·10
16	169·26	0·54	169·44	0·34
17	173·77	0·81	175·09	0·91
18	18·74	0·56	182·45	0·90
19	188·97	0·73	190·11	0·71
20	200·88	0·58	200·55	0·39

the determination of the number of modes in a frequency band, using only output accelerometers, has been developed. Based on a combination of the multivariate minimum description length theory and the overdetermined instrumental variable scheme, an efficient method for AR order determination of a multivariate ARMA model has been developed. This method is based on the product of the smallest eigenvalues of an overdetermined instrumental variable product moment matrix and the use of a new criterion. The AR coefficients are then derived from a minimization problem under constraints. The modal parameters are then obtained by the spectral decomposition of the companion matrix. A numerical example and experimental tests in laboratory have been presented. They have shown the effectiveness of the method in model order estimation and modal parameter determination. It may interesting to study other instrumental variable selections and to generalize this method to large industrial structures.

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APPENDIX A. NOMENCLATURE

n	degree of freedom of the vibration system
m	number of output measurements
p	order of the multivariate AR part
q	order of the multivariate MA part
f_i	natural frequency of the ith mode
c_i	damping ratio of the ith mode
N	number of data points
Δt	sampling period
$J_{MDL}(p')$	M.D.L. criterion
$\boldsymbol{\xi}(t)$	$(n \times 1)$ vector of displacements
$\boldsymbol{\eta}(t)$	$(n \times 1)$ white noise vector of unmeasured input
$\mathbf{y}(t)$	$(m \times 1)$ observation vector
$\mathbf{M}_0$	$(n \times n)$ mass matrix of the system
$\mathbf{C}_0$	$(n \times n)$ damping matrix of the system
$\mathbf{K}_0$	$(n \times n)$ stiffness matrix of the system
$\mathbf{F}$	$(2n \times 2n)$ transition matrix
$\mathbf{A}_i$	$(m \times m)$ AR coefficient matrix
$\mathbf{B}_i$	$(m \times m)$ MA coefficient matrix
$\mathbf{A}$	$(mp \times mp)$ companion matrix
$\boldsymbol{\Psi}_{p'}$	$(m \times m)$ $(p' + 1)$ AR coefficient matrices
$\boldsymbol{\Omega}_{p'}$	$(m \times N)$ matrix of coefficients MA and white noise
$\mathbf{Y}_{p'}$	$m(p' + 1) \times N$ matrix of data and zeros
$\mathbf{z}_i$	$(m \times 1)$ an instrumental variable sequence
$\mathbf{Z}_k$	$m(k + 1) \times N$ extended instrumental variable matrix
$\hat{\mathbf{R}}(i)$	$m \times m$ estimated covariance matrix