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RESEARCH ON THE PERIODIC ORBIT OF NON-LINEAR DYNAMIC SYSTEMS USING CHEBYSHEV POLYNOMIALS

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In this paper, a new analysis method is presented to study the steady periodic solution of non-linear dynamical systems over one period. By using the good properties of Chebyshev polynomials, the state vectors appearing in the equations can be expanded in terms of Chebyshev polynomials over the principal period such that the original non-linear differential problem is simplified to a set of non-linear algebraic equations. Furthermore, all systems, including linear, weak non-linear and strong non-linear can be analyzed in the same way for no limitation of small parameter any more. It is also very efficient to get the asymptotic solution of periodical orbit even for high-dimensional dynamical systems. The numerical accuracy of the proposed technique is compared with that of the standard numerical Runge–Kutta method.

1. INTRODUCTION

Except for the transient motion, the steady motion of an object could be classified into periodic, quasi-periodic and chaotic motion. Although the interest of many researchers is in chaos, the study of periodic motion is of great importance in various fields of science and technology, since periodic motion is very common and has a close relationship with chaos. In the past, several methods have been used to study the periodic solution of non-linear systems. A comprehensive literature survey on periodic motion and its stability was provided by Ling [1]. Based on the trigonometric collocation method, the periodic solution in rotor systems was studied by Nataraj [2]. Using the power series method, Qaisi [3] investigated the forced undamped Duffing's oscillator. However, only the undamped systems could be analyzed. Wang [4] analyzed the piecewise linear model of a single-degree-of-freedom system with elastic friction damping. The solution was obtained by using the linear equation theory. The periodic solution of Mathieu oscillator was presented by Mahmoud [5]. The method was based on the generalized averaging method and could be used to analyze the strong non-linear systems by using a newly defined expansion small parameter. Kim [6] developed the multiple harmonic balance method for obtaining the aperiodic steady state solution. This was a generalization of the direct HBM to multiple time scales. Nevertheless, the Jacobian matrix and its inverse should be computed in each iteration. Therefore, this method was cumbersome for large order systems. Based on the two-point boundary problem, the shooting method [7] is used very commonly. However, it is purely a numerical method and cannot be used to get the analytical expression.

The theory of linear system cannot be used directly to solve the non-linear problems, because of the speciality and complexity of the latter. When dealing with non-linear analysis, the existing asymptotic methods have some inherent shortcomings. For example, the perturbation method depends on the assumption of small parameters and the KBM method is very tedious in getting the high order asymptotic solution. Fortunately, the Chebyshev polynomials method was suggested by Sinha [8, 9] in the study of linear systems with periodic parameters. Then, this method was combined with the Liapunov–Floquet transformation to design controllers of parametrically excited systems [10, 11]. The attractive feature of this technique is that it can reduce the original parametric excited differential system to a system of linear algebraic equations very conveniently. The method used in reference [12] is Picard iteration and a collocation procedure. The solutions are obtained by solving a set of linear algebraic equations at each stage and one does not have to solve a set of non-linear equations. However, this method could only be used to draw the orbit of dynamic systems and directly determine the periodic solution of parametrically excited systems with a given period T. It could not be used to obtain the analytical solution of the periodic orbit of autonomous system. In our paper, both autonomous and non-autonomous systems are analyzed in a similar way. Also, it is possible to get the number of periodic orbits by solving a set of non-linear equations only one time. This is the main difference between the two papers.

2. THEORY OF THE CHEBYSHEV POLYNOMIALS

The properties of the Chebyshev polynomials have been described in references [8, 9]. In this section, it is necessary to review certain properties of these polynomials.

The Chebyshev polynomials of the first kind are defined by the following relations:

$$T_n(s) = ((-1)^n 2^n n!/(2n)!)(1 - s^2)^{1/2}(\mathrm{d}/\mathrm{d}s)^n(1 - s^2)^{n-1/2}, \quad n = 0, 1, 2, 3, \ldots \tag{1}$$

and are orthogonal over the interval $[-1, 1]$ with respect to the weight function $w(s) = (1 - s^2)^{-1/2}$.

The shifted Chebyshev polynomials of the first kind are orthogonal over the interval $[0, 1]$ with respect to the weight function $w(s) = (s - s^2)^{-1/2}$, and are given by

$$T^*_n(s) = T_n(2s - 1), \quad s \in [0, 1]. \tag{2}$$

From equation (2), the recurrence relations could be written as

$$T^*_{n+1}(s) = 2(2s - 1)T^*_n(s) - T^*_{n-1}(s). \tag{3}$$

Meanwhile, the orthogonality relationships are given by

$$\int_0^1 T^*_n(s)T^*_k(s)w(s)\,\mathrm{d}s = \begin{cases} 0, & n \neq k, \\ \pi/2, & n = k \neq 0, \\ \pi, & n = k = 0. \end{cases} \tag{4}$$

Any continuous function can be expanded in terms of these polynomials as

$$f(s) = \sum_{n=0}^{\infty} a_n T_n^*(s). \tag{5}$$

From the orthogonality relation, the Chebyshev coefficients a_n can be expressed as

$$a_n = \frac{1}{\delta}\int_0^1 w(s) f(s)\, T_n^*(s)\, \mathrm{d}s, \quad n = 0, 1, 2, 3, \ldots, \tag{6}$$

where

$$\delta = \begin{cases} \pi/2, & n \neq 0, \\ \pi, & n = 0 \end{cases}$$

for the shifted Chebyshev polynomials of the first kind.

The integration matrix associated with the polynomials can be expressed in the form

$$\int_0^s \{T^*(\tau)\}^{\mathrm{T}}\, \mathrm{d}\tau = \{T^*(s)\}^{\mathrm{T}} [G], \tag{7}$$

where

$$\{T^*(s)\} = \{T_0^*(s), T_1^*(s), T_2^*(s), \ldots, T_{m-1}^*(s)\}^{\mathrm{T}} \tag{8}$$

is an $m \times 1$ vector of the shifted Chebyshev polynomials of the first kind and G is an $m \times m$ integration matrix defined by

$$G^{\mathrm{T}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & & & 0 \\ -\frac{1}{8} & 0 & \frac{1}{8} & 0 & & & 0 \\ -\frac{1}{6} & -\frac{1}{4} & 0 & \frac{1}{12} & & & 0 \\ \frac{1}{16} & 0 & \frac{1}{12} & 0 & & \ddots & 0 \\ \vdots & \vdots & \vdots & & \ddots & 0 & \frac{1}{4(m-1)} \\ \frac{(-1)^m}{2m(m-2)} & 0 & 0 & 0 & 0 & \frac{-1}{4(m-2)} & 0 \end{bmatrix}. \tag{9}$$

For two arbitrary functions

$$f(s) = \sum_{n=0}^{m-1} a_n T_n^*(s) \quad \text{and} \quad g(s) = \sum_{n=0}^{m-1} b_n T_n^*(s), \tag{10}$$

the multiplication can be expressed as

$$f(s)g(s) = \{T^*(s)\}[Q]\{b\}, \tag{11}$$

where Q is an $m \times m$ product matrix defined by

$$Q = \begin{bmatrix} a_0 & \frac{a_1}{2} & \frac{a_2}{2} & \frac{a_3}{2} & \cdots & \frac{a_{m-1}}{2} \\ a_1 & a_0 + \frac{a_2}{2} & \frac{a_1 + a_3}{2} & \frac{a_2 + a_4}{2} & \cdots & \frac{a_{m-2} + a_m}{2} \\ a_2 & \frac{a_1 + a_3}{2} & a_0 + \frac{a_4}{2} & \frac{a_1 + a_5}{2} & \cdots & \frac{a_{m-3} + a_{m-1}}{2} \\ a_3 & \frac{a_2 + a_4}{2} & \frac{a_1 + a_5}{2} & a_0 + \frac{a_6}{2} & \cdots & \frac{a_{m-4} + a_{m-2}}{2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m-1} & \frac{a_{m-2} + a_m}{2} & \frac{a_{m-3} + a_{m-1}}{2} & \frac{a_{m-4} + a_{m-2}}{2} & \cdots & a_0 + \frac{a_{2m-2}}{2} \end{bmatrix} \tag{12}$$

and $\{b\} = \{b_0 \, b_1 \, b_2 \cdots b_{m-1}\}^{\mathrm{T}}$.

3. METHOD OF ANALYSIS

Consider the following no-linear dynamic system,

$$\mathrm{d}X/\mathrm{d}t = [A(X)]X + \{C(t)\}, \tag{13}$$

where $X(t)$ is an $n \times 1$ vector $\{x_1 \, x_2 \cdots x_n\}^{\mathrm{T}}$. Since there are some small differences in analyzing the autonomous systems and non-autonomous systems, we should discuss them separately.

3.1. AUTONOMOUS SYSTEMS

For the autonomous systems, equation (13) should be written as

$$\mathrm{d}X/\mathrm{d}t = [A(X)]X + \{C\}, \tag{14}$$

where $\{C\}$ is a constant vector. Suppose that the non-linear system has the principal period $\tilde{T}$. Note that the real principal period T of the system is not clear now and $\tilde{T}$ may not be equal to it. It is rational to make the transformation $t = \tilde{T} \times s$, because the shifted Chebyshev polynomials of the first kind are orthogonal over the interval [0,1]. The physical meaning of this change is that we regard the period of the period orbit as 1. Therefore, the good properties of the shifted Chebyshev polynomials could be used. Then, equation (14) should be rewritten as

$$\mathrm{d}X/\mathrm{d}s = [A'(X)]X + \{C'\}, \quad s \in [0,1], \tag{15}$$

where

$$[A'(X)] = \tilde{T} \cdot [A(X)] \quad \text{and} \quad \{C'\} = \tilde{T} \cdot \{C\}.$$

The solution vector $X(s)$ can be expanded in terms of the shifted Chebyshev polynomials in the interval [0,1] as

$$x_i(s) = \sum_{j=0}^{m-1} b_j^i T_j^*(s) = \{T^*(s)\}^{\mathrm{T}} b^i, \quad i = 1, 2, \ldots, n, \tag{16}$$

where

$$b^i = \{b_0^i, b_1^i, \ldots, b_{m-1}^i\}^{\mathrm{T}}, \quad b_j^i = 0 \quad (j > m-1).$$

The elements in matrix $[A'(X)]$ and vector $\{C'\}$ are expanded in terms of the shifted Chebyshev polynomials in the interval [0,1] as

$$A'_{ij}(s) = \{T^*(s)\}^{\mathrm{T}} d^{ij}, \quad C'_i = \{T^*(s)\}^{\mathrm{T}} c', \quad i, j = 1,2, \ldots, n, \tag{17}$$

where

$$d^{ij} = \{d_0^{ij}, d_1^{ij}, \ldots, d_{m-1}^{ij}\}^{\mathrm{T}}, \quad c' = \{c'_0, c'_1, \ldots, c'_{m-1}\}^{\mathrm{T}}.$$

and $\{T^*(s)\}$ is defined in equation (8). The solution x_i in the matrix $[A'(X)]$ can be dealt with as other elements in it.

Now, for convenience in algebraic manipulation, define $n \times nm$ matrix

$$[\bar{T}(s)]^{\mathrm{T}} = [I] \otimes \{T^*(s)\}^{\mathrm{T}}, \quad [\bar{G}] = [I] \otimes [G], \tag{18}$$

where I is an $n \times n$ identity matrix, and $\otimes$ represents the Kronecker product.

Integrating equation (15) with respect to s gives

$$[\bar{T}(s)]^{\mathrm{T}}\{B\} - [\bar{T}(s)]^{\mathrm{T}}\{X(0)\} = [\bar{T}(s)]^{\mathrm{T}}[\hat{A}]\{B\} + [\bar{T}(s)]^{\mathrm{T}}[\bar{G}]\{\hat{C}\}, \tag{19}$$

where

$$[\hat{A}]_{2nm\times 2nm} = \begin{bmatrix} A^{1,1} & A^{1,2} & \cdots & A^{1,2n} \\ A^{2,1} & A^{2,2} & \cdots & A^{2,2n} \\ \vdots & \vdots & \ddots & \vdots \\ A^{2n,1} & A^{2n,2} & \cdots & A^{2n,2n} \end{bmatrix}, \quad [A^{ij}]_{m\times m} = [GQ(d^{ij})]_{m\times m},$$

$\{\hat{C}\} = \{c^1, c^2, \ldots, c^n\}^{\mathrm{T}}$ and $\{B\} = \{b_0^1, b_1^1, \ldots, b_{m-1}^1, \ldots, b_0^n, b_1^n, \ldots, b_{m-1}^n\}^{\mathrm{T}}$ is an $nm \times 1$ unknown vector and $\{X(0)\}$ is an arbitrary starting point.

From equation (19), a set of non-linear algebraic equations can thus be obtained as

$$[I - \hat{A}]\{B\} = \{X(0)\} + [\bar{G}]\{\hat{C}\}. \tag{20}$$

Once the above equations are solved, the approximate analytical solution over the supposed periodic $\tilde{T}$ could be obtained. Then, the following steps are needed.

First, change the end point of the computed orbit into a new starting point. Second, solve equation (20) again and repeat these two steps enough times in order to make sure that the last computed orbit is a part of the real periodic orbit.

Finally, the following judgement should be made. If the starting point coincides with the end point, the supposed periodic $\tilde{T}$ is equal to the unknown real principal period T and the analytical solution of the periodic orbit can be obtained sequentially. If the computed orbit passes the starting point $\{X(0)\}$, it means that $\tilde{T} > T$ and the real T can be obtained by comparison, since the analytical expression of the orbit has been obtained now. That is to say, we have known the analytical expression of an orbit with time t from 0 to $\tilde{T}$. We could find $t = T$ $(0 < T \leqslant \tilde{T})$ satisfying $X(0) = X(T)$ by comparison i.e., by solving this equation. Otherwise, change $\tilde{T}$ into $k\tilde{T}\,(k > 1)$ and solve equation (20) again. Then repeat the last step. In our experience, it is better to choose k in the range of $[1, 2]$.

Once having determined the main period T and any one point on the orbit by using another method such as the shooting method, the asymptotic analytical solution of the periodic orbit can be obtained by solving equation (20) only one time.

For example, consider the three-dimensional Rossler's equation [13]

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3, \\ \dot{x}_2 &= x_1 + ax_2, \\ \dot{x}_3 &= b + x_3x_1 - cx_3, \end{aligned} \tag{21}$$

where $a = 0{\cdot}15$, $b = 0{\cdot}2$ and c is a variable parameter.

When $c = 3{\cdot}5$, the Rossler system has only one stable period 1 orbit. The principal period T is 5·92030065 and (2·7002161609, 3·4723025491, 3·0) is a point on the orbit.

If the accurate T is unknown, we could suppose that $\tilde{T}$ equals any real number. However, if $\tilde{T} > 8$, the convergence is not good. By using the method suggested before, computation can be started from any starting point such as (0,0,0). It could be seen that this approach is extremely accurate and effective by comparing the results with those obtained from Runge–Kutta integration, as shown in Figure 1.

If the period T and one point on the periodic orbit are known, the conclusion is just the same as those shown in Figure 1.

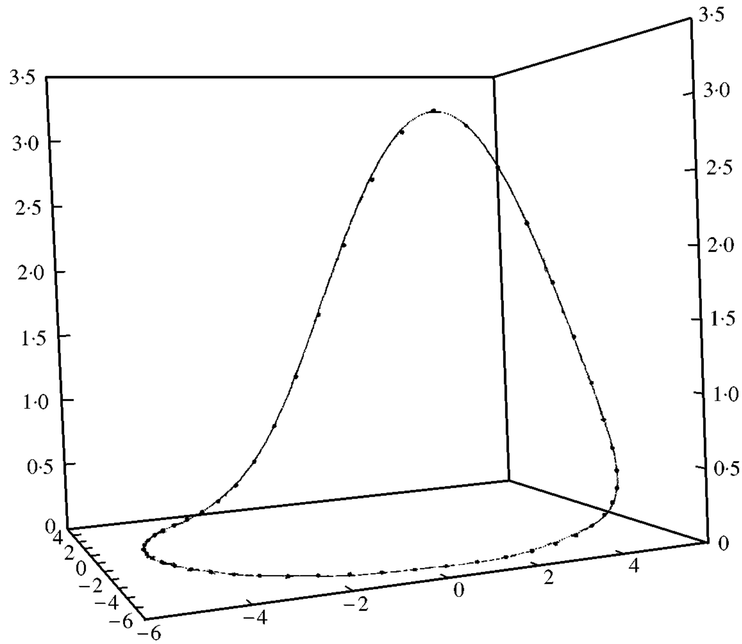

Figure 1. Three-dimensional phase portrait of Rossler's equation: ——, Chebyshev; —•—•—, Runge–Kutta.

As for high-dimensional dynamical systems, the suggested method can also be used directly. For example, consider the six dimensional coupled Rossler's equation [14]

$$\begin{aligned}
\dot{x}_1 &= -\,w_1 x_2 - x_3 + c(x_4 - x_1),\\
\dot{x}_2 &= w_1 x_1 + 0{\cdot}15 x_2,\\
\dot{x}_3 &= 0{\cdot}2 + x_3(x_1 - 3{\cdot}5),\\
\dot{x}_4 &= -\,w_2 x_5 - x_6 + c(x_1 - x_4),\\
\dot{x}_5 &= w_2 x_4 + 0{\cdot}15 x_5,\\
\dot{x}_6 &= 0{\cdot}2 + x_6(x_4 - 3{\cdot}5),
\end{aligned} \tag{22}$$

where $w_1 = 1{\cdot}03$, $w_2 = 0{\cdot}97$ and $c = 0{\cdot}13$.

The computed periodic orbit is shown in Figures 2–4. It can be concluded that the accuracy of this approach is very good even for high-dimensional dynamical systems.

3.2. NON-AUTONOMOUS SYSTEMS

For the non-autonomous systems, the main procedure is similar to those discussed before. We only need to change $\{C\}$ into $\{C(t)\}$ in corresponding equations in Section 3.1.

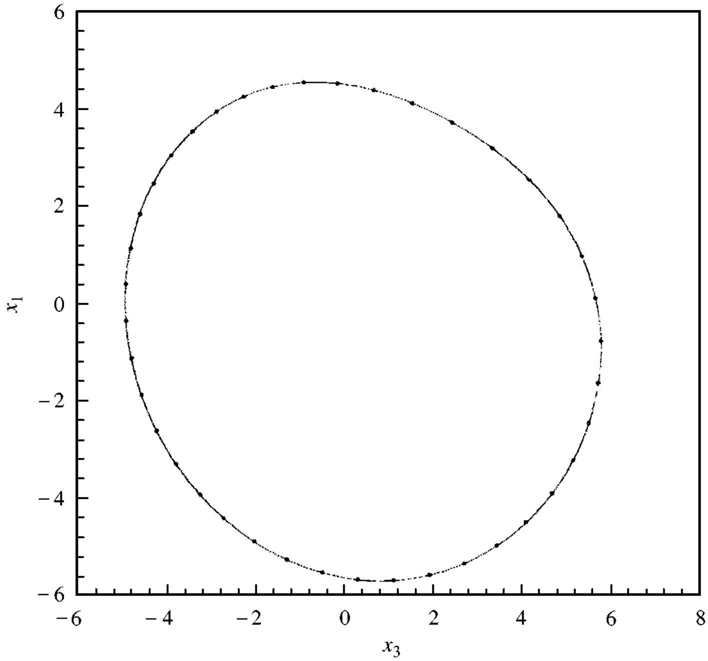

Figure 2. Phase portrait in x_1–x_2 plane of coupled Rossler's equation: ——, Chebyshev; —•—•—, Runge–Kutta.

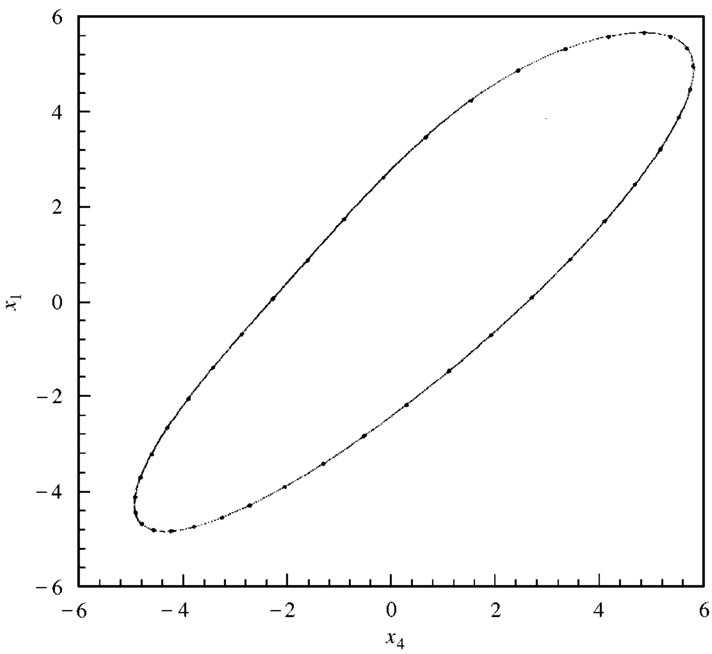

Figure 3. Phase portrait in x_1–x_4 plane of coupled Rossler's equation: ——, Chebyshev; —•—•—, Runge–Kutta.

Now, the principal period is known. Compute from any starting point and the final stable periodic orbit can be obtained. However, it is impossible to get the stable periodic orbit by solving equation (20) only one time despite the determination of one point on the orbit in advance. This is the main difference between the non-autonomous systems and the

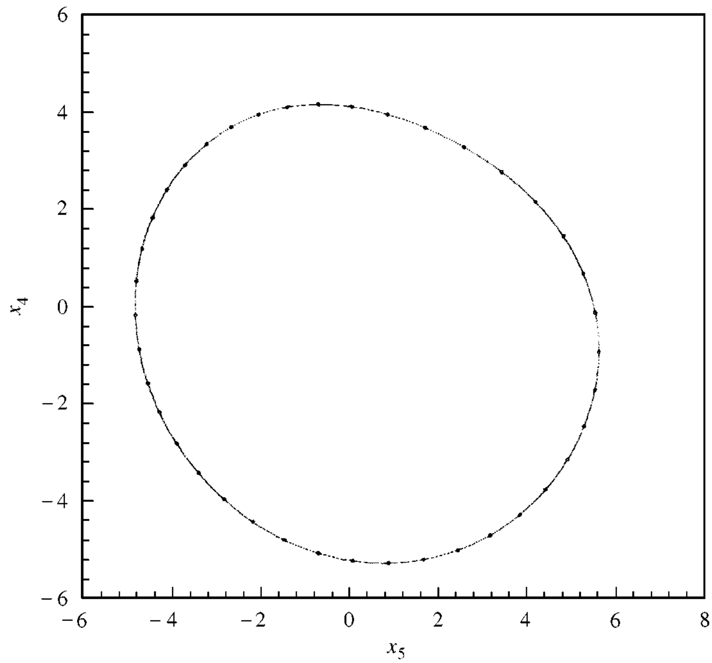

Figure 4. Phase portrait in x_4–x_5 plane of coupled Rossler's equation: ——, Chebyshev; —•—•—, Runge–Kutta.

autonomous systems. The reason for this difference is that, for non-autonomous systems, we use the condition that t varies from 0 to $\tilde{T}$ when we get the Chebyshev coefficients of $\{C(t)\}$. However, we, do not use it when we analyze the autonomous systems. Therefore, the solution we obtained satisfied the autonomous systems at any time, while it only satisfied the non-autonomous systems at time $[0,\tilde{T}]$. Consequently, it is possible for us to get the stable periodic orbit of autonomous systems by solving equation (20) only one time, while merely obtaining the transient response of non-autonomous systems.

For example, consider the two-dimensional Duffing's equation [15]

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -ax_1 - bx_2 - cx_1^3 + d\cos(2t), \end{aligned} \tag{23}$$

where $a = 0{\cdot}02$, $b = 0{\cdot}25$, $c = 0{\cdot}05$ and $d = 8{\cdot}5$.

If it starts from an arbitrary point, such as (0,0), the final result is shown in Figure 5. However, even if the starting point is on the stable orbit, such as (1·3889551, 3·4712163), the one-step computation result, i.e., solving equation (20) one time, is shown in Figure 6. It could be seen that the computed orbit and the stable period orbit do not coincide with each other. If we see the result of the first 5 steps, it can be found that the evolution orbit coincides with the numerical integration orbit, which gives the transient response orbit, as shown in Figure 7.

4. DISCUSSION

Now, we have discussed the new method to get the asymptotic polynomial solution of stable periodic orbit by using the Chebyshev polynomials with a given starting point. Can

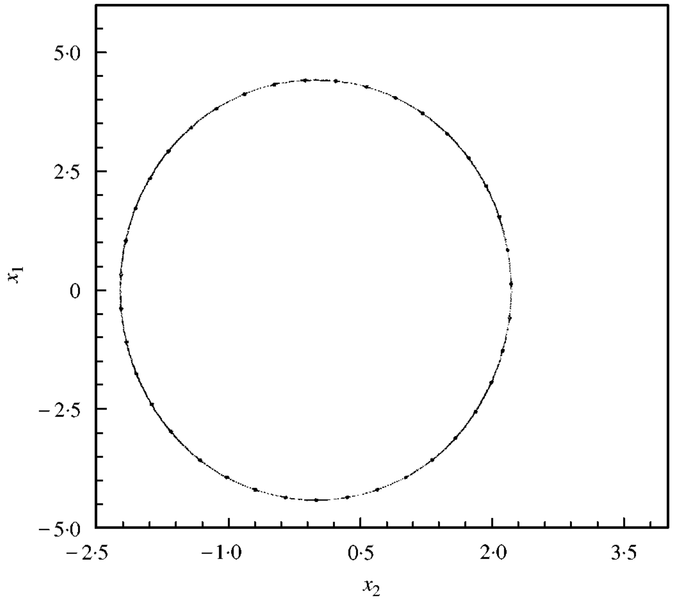

Figure 5. Phase portrait in x_1–x_2 plane of Duffing's equation: ——, Chebyshev; —•—•—, Runge–Kutta.

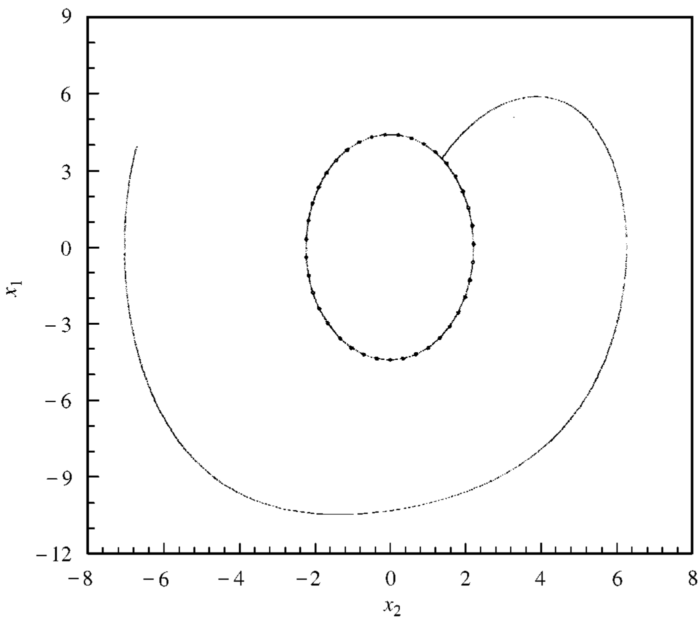

Figure 6. One-step evolution in x_1–x_2 plane: —•—•—, Stable orbit; ——, One-step orbit.

this method be used to analyze the unstable periodic orbit and the number of periodic orbit? The answer is yes; nevertheless, only the autonomous systems can be analyzed because of the possibility of getting the periodic solution of this kind of systems by only one step.

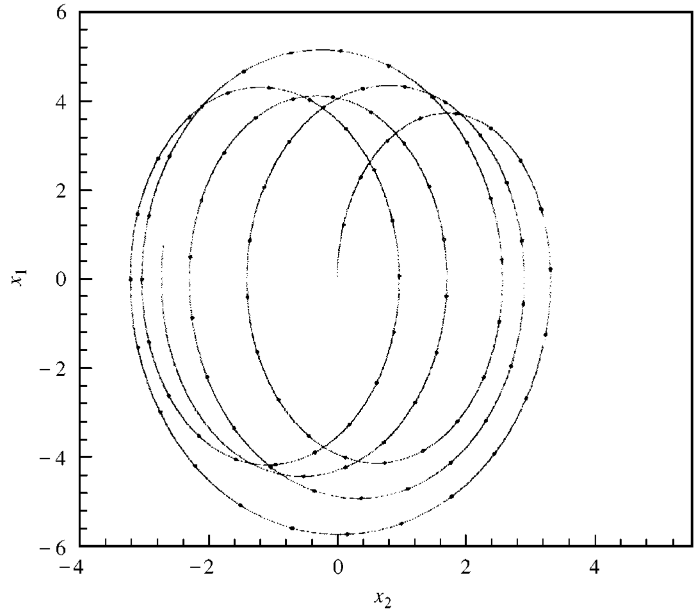

Figure 7. Comparison of transient response: ——, Chebyshev; —•—•—, Runge–Kutta.

One should have noticed that equation (20) is a non-linear algebraic equation. The number of all of the real roots should be more than one. However, we get only one real root, since the solving method we used is quasi-Newton iteration. If we get another real root for the different starting point chosen, it means the possibility of coexistence of periodic orbit. If the starting point is regarded as variable and some equations satisfying the periodic conditions are increased, the solving of equation (20) by quasi-Newton iteration will only result in zero root, which is obviously not the root we want to get. In this case, the Wu method [16] can be used to get all the real roots of this kind of non-linear algebraic equation. Thus, it is possible to obtain all the stable and unstable periodic orbits. This has a very important meaning in the study of non-linear dynamics. Since the Wu method is only suggested in theory and has not been commonly used, we could not give such an example in this paper.

5. CONCLUSIONS

In this paper, a new method for the analysis of the periodic orbit of non-linear dynamic systems is developed. The strategy is based on the fact that the state vector of either linear or non-linear systems can be expanded in terms of Chebyshev polynomials over the principal period. Such an expansion reduces the original problem to a set of non-linear algebraic equations from which the solution in the interval of one period can be obtained. It has been shown that the proposed analysis technique is virtually free of the small parameter limitations. From the example studied in section 3, the suggested method provides extremely convergent solutions and correct behaviors of the non-linear systems when compared with the Runge–Kutta numerical scheme. To conclude, the authors would like to state that for the first time the Chebyshev polynomials are used to analyze the periodic orbit of autonomous strong non-linear dynamic systems. It is anticipated that the suggested method would serve as a new tool in the study of co-existence of the periodic orbits. The stability of the periodic orbit so obtained will be studied in the further research.

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