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WAVELET-BASED STOCHASTIC SEISMIC RESPONSE OF A DUFFING OSCILLATOR

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This paper presents formulation for the non-stationary response of a Duffing oscillator under a seismic excitation process. The excitation process is assumed to be characterized through wavelet coefficients and the non-linear system is replaced by a stochastic equivalent linear system with time-varying parameters. An example ground motion process has been used to show that the proposed approach gives accurate response estimates in case of mildly non-linear systems.

1. INTRODUCTION

In case of severe excitations, such as earthquake-induced ground motions, the structural systems may behave non-linearly due to the inelastic excursions. Their behaviour may be modelled by assuming linear viscous damping and non-linear stiffness. Such modelling is also found to be applicable to the systems with skiny hysteretic characteristics. Duffing's oscillator is a commonly used oscillator with non-linearity in stiffness as it represents several cases of plate, shell and beam vibrations (see reference [1]). The response of Duffing oscillator to both deterministic and random excitations has been studied by researchers (see, e.g., references [1, 2]), and exact or approximate solutions are available in case of white-noise, stationary and amplitude-modulated non-stationary processes. However, the non-stationary response explicitly accounting for the frequency non-stationarity has not received much attention.

Among several approximate methods of analyzing non-linear systems, the use of equivalent linear techniques is quite common in engineering applications. These techniques are used to obtain preliminary design estimates and to help in providing qualitative insight into the nature of the non-linear response. The equivalent linearization was initially proposed by Krylov and Bogoliubov [3] in the form of method of averaging and was later developed by Caughey [4] for applying to random vibration problems. The existing random vibration approach, however, needs to be modified to suitably account for both amplitude and frequency non-stationarities which are inherent in a ground motion process. In contrast with the popular approach of using amplitude-modulating functions, these non-stationarities can be far more effectively handled by using the wavelet analytic tools.

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Wavelet analysis has emerged as a very powerful time–frequency analysis tool to tackle frequency non-stationarity in the earthquake ground motions. Recently, a wavelet-based linearization technique has been developed by Basu and Gupta [5] to obtain the response of a Duffing oscillator to deterministic excitations. Central to this formulation is the replacement of the non-linear system by a linear system with time-varying properties as also suggested by Mason [6]. This paper extends this idea to obtain the (stochastic) response of a Duffing oscillator under a seismic excitation process and proposes formulation for the instantaneous natural frequency of the equivalent linear system. The input–output relationship and the peak response statistics of the equivalent linear system are obtained by using the formulation of Basu and Gupta [7]. The proposed formulation has been validated via time–history simulations in case of an example ground motion process.

2. STOCHASTIC FORMULATION

2.1. GROUND MOTION

Let $\ddot{z}(t)$ be a zero-mean ground acceleration process with non-stationary Gaussian characteristics. This process can be characterized by the functionals of its wavelet coefficients, $E[W_\psi^2 \ddot{z}(a, b)]$ [7], where the wavelet transform, $W_\psi f(a, b)$, of any given square-integrable function, $f(t)$, and its inverse relationship are, respectively, given by

$$W_\psi f(a, b) = \int_{-\infty}^{\infty} f(t)\psi_{a,b}(t)\,\mathrm{d}t \tag{1}$$

and

$$f(t) = \frac{1}{2\pi C_\psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{a^2}\,W_\psi f(a, b)\psi_{a,b}(t)\,\mathrm{d}a\,\mathrm{d}b. \tag{2}$$

In equations (1) and (2),

$$\psi_{a,b}(t) = \frac{1}{|a|^{1/2}}\,\psi\left(\frac{t-b}{a}\right), \quad a, b \in \mathbf{R}^+ \tag{3}$$

is the translated and dilated form of a suitably chosen wavelet basis function, $\psi(t)$. Whereas the parameter, b, localizes the basis function at $t = b$ and its neighbourhood, the parameter, a, captures the contribution of $f(t)$ to the frequencies in the frequency band of $\psi_{a,b}(t)$. In equation (2),

$$C_\psi = \int_{-\infty}^{\infty}\frac{|\hat{\psi}(\omega)|^2}{|\omega|}\,\mathrm{d}\omega \tag{4}$$

is a finite quantity, where

$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \tag{5}$$

is the Fourier transform of $\psi(t)$.

2.2. EQUIVALENT LINEAR PARAMETERS

Let us consider a non-linear oscillator with a non-linear restoring force, $g(x)$, and a viscous damping, c, per unit mass. The equation of motion for this oscillator, when subjected to the ground motion process, $\ddot{z}(t)$, may be expressed as

$$\ddot{x} + c\dot{x} + g(x) = -\ddot{z}, \tag{6}$$

where x is the displacement of the oscillator relative to the base and an overdot represents differentiation with respect to time, t. In the equivalent linearization technique, equation (6) is replaced by the equivalent linear equation, i.e.,

$$\ddot{x} + 2\zeta_{et}\omega_{et}\dot{x} + \omega_{et}^2 x = -\ddot{z}. \tag{7}$$

In equation (7), ω_{et} and ζ_{et}, respectively, represent the time-varying equivalent natural frequency and equivalent damping coefficient at the time instant, t. Since no non-linearity is assumed in the damping, the equivalent damping ratio is given by

$$\zeta_{et} = \frac{c}{2\omega_{et}}. \tag{8}$$

Wavelet transformation of equation (6) and integration by parts transforms $W_\psi \ddot{x}$ and $W_\psi \dot{x}$ to $W_{\ddot{\psi}} x/a^2$ and $-W_{\dot{\psi}} x/a$, respectively, where $W_\psi \ddot{x}$ and $W_\psi \dot{x}$ denote the wavelet transforms of $\ddot{x}(t)$ and $\dot{x}(t)$, respectively, with respect to the basis function, $\psi(t)$, and $W_{\ddot{\psi}} x$ and $W_{\dot{\psi}} x$ denote the wavelet transforms of $x(t)$ with respect to $\ddot{\psi}(t)$ and $\dot{\psi}(t)$ respectively.

These two terms can further be shown to be equal to $(\partial^2/\partial b^2) W_\psi x(a, b)$ and $(\partial/\partial b) W_\psi x(a, b)$, respectively, by applying the chain rule of differentiation, where $W_\psi x(a, b)$ denotes the wavelet transform of $x(t)$ with respect to $\psi(t)$. Thus, equation (6) leads to (see reference [8])

$$\frac{\partial^2}{\partial b^2} W_\psi x(a, b) + 2\zeta_{et}\omega_{et} \frac{\partial}{\partial b} W_\psi x(a, b) + W_\psi g(a, b) = -W_\psi \ddot{z}(a, b), \quad a, b \in \mathbf{R}^+, \tag{9}$$

where $W_\psi g(a, b)$ denotes the wavelet transform of the non-linear stiffness function, $g(x(t))$, with respect to $\psi(t)$. On performing similar operations and approximating $W_\psi(\omega_{et}^2 x(a, b))$ by $\omega_{et}^2 W_\psi x(a, b)$ (due to the localized nature of wavelet basis), equation (7) becomes

$$\frac{\partial^2}{\partial b^2} W_\psi x(a, b) + 2\zeta_{et}\omega_{et} \frac{\partial}{\partial b} W_\psi x(a, b) + \omega_{et}^2 W_\psi x(a, b) = -W_\psi \ddot{z}(a, b), \quad a, b \in \mathbf{R}^+. \tag{10}$$

For numerical calculations, we follow a scheme similar to that by Alkemede [9], and discretize a and b at $a_j = \sigma^j$ and $b_j = (j-1)\Delta b$, where σ and Δb are the discretization parameters. The step changes at $a = a_j$ and $b = b_j$, respectively, are defined as

$$\Delta b_j = [(b_{j+1} - b_j) + (b_j - b_{j-1})]/2 = \Delta b \tag{11}$$

and

$$\Delta a_j = [(a_{j+1} - a_j) + (a_j - a_{j-1})]/2 = \frac{a_j}{2}\left(\sigma - \frac{1}{\sigma}\right). \tag{12}$$

On using this discretization scheme, the expected square of the difference between equations (9) and (10) may be expressed as

$$E[\varepsilon_{ij}^2] = E[\{\omega_{et}^2 W_\psi x(a_j, b_i) - W_\psi g(a_j, b_i)\}^2], \tag{13}$$

where $E[\cdot]$ denotes the expectation operator. This term, when summed over all j values along with a norm of $1/a_j$, represents the expected error in the instantaneous energy of the response at $t = b_i$. We minimize this with respect to ω_{et}^2, i.e.,

$$\frac{\partial}{\partial \omega_{et}^2} \sum_j \frac{1}{a_j} E[\varepsilon_{ij}^2] = 0. \tag{14}$$

On assuming the non-linear oscillator to be a Duffing oscillator with the non-linear stiffness per unit mass, $g(x) = \omega_n^2 x + \varepsilon\omega_n^2 x^3$ (ε controls the type and degree of non-linearity and ω_n is the natural frequency of the oscillator for $\varepsilon = 0$), and on substituting equation (13) into equation (14), the time-varying equivalent natural frequency squared at the time instant, $t = b_i$, is obtained as

$$\omega_{ei}^2 = \omega_n^2 \left[1 + \varepsilon \frac{\sum_j (1/a_j) E[W_\psi x^3(a_j, b_i) W_\psi x(a_j, b_i)]}{\sum_j (1/a_j) E[W_\psi^2 x(a_j, b_i)]} \right], \tag{15}$$

where $W_\psi x^3$ is the wavelet transform of $x^3(t)$ with respect to $\psi(t)$. It may be noted that since the equivalence between the original non-linear and the equivalent linear equation has been obtained in a statistical sense, the expression for the linear frequency in equation (15) contains terms involving the expectation operator. To simplify the second term in equation (15), the following result of wavelet analysis (see reference [10]) is used:

$$\frac{1}{2\pi C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{a^2} W_\psi f(a, b) W_\psi h(a, b) \, \mathrm{d}a \, \mathrm{d}b = \int_{-\infty}^{\infty} f(t) h(t) \, \mathrm{d}t. \tag{16}$$

Substituting $f = x$ and $h = x^3$ and taking expectation on both sides, equation (16) leads to

$$\frac{1}{2\pi C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[W_\psi x(a, b) W_\psi x^3(a, b)] \frac{\mathrm{d}a}{a^2} \mathrm{d}b = \int_{-\infty}^{\infty} E[x^4(t)] \, \mathrm{d}t. \tag{17}$$

From this equation, on using the time-localization property of the wavelets and on discretizing over a and b, the instantaneous value, $E[x^4(t)]|_{t=b_i}$, is obtained as

$$E[x^4(t)]|_{t=b_i} = \frac{K}{a_j} E[W_\psi x(a_j, b_i) \, W_\psi x^3(a_j, b_i)] \tag{18}$$

with

$$K = \frac{1}{4\pi C_\psi} \left(\sigma - \frac{1}{\sigma} \right). \tag{19}$$

Further, expression for the instantaneous mean-square value of the response function, $x(t)$, is obtained by using $f = h = x$ in the discretized version of equation (16), and thus,

$$E[x^2(t)]|_{t=b_i} = \frac{K}{a_j} E[W_\psi^2 x(a_j, b_i)]. \tag{20}$$

The assumption of non-stationary response to be locally Gaussian with zero mean leads to the following relationship between the fourth order and second order moments of the instantaneous response probability density function,

$$E[x^4(t)]|_{t=b_i} = 3(E[x^2(t)]|_{t=b_i})^2. \tag{21}$$

On using equations (18)–(21) in equation (15), the instantaneous natural frequency of the equivalent system is obtained as

$$\omega_{ei}^2 = \omega_n^2 \left[1 + 3\varepsilon \sum_j \frac{K}{a_j} E[W_\psi^2 x(a_j, b_i)] \right]. \tag{22}$$

For evaluation of $\sum_j (K/a_j) E[W_\psi^2 x(a_j, b_i)]$, the zeroth moment of the power spectral density function (PSDF) of the instantaneous response may be used (see reference [7]). If a modified form of the Littlewood–Paley basis function described by

$$\psi(t) = \frac{1}{\pi\sqrt{(\sigma-1)}} \frac{\sin \sigma\pi t - \sin \pi t}{t} \tag{23}$$

is used, the expression for this moment is obtained as

$$m_0|_{t=b_i} = \sum_j \frac{K}{a_j} E[W_\psi^2 x(a_j, b_i)] = \sum_j K' E[W_\psi^2 \ddot{z}(a_j, b_i)] I_{0,j} \tag{24}$$

with $K' = K/(\sigma - 1)\pi$ and

$$I_{0,j} = \int_{\pi/a_j}^{\sigma\pi/a_j} \frac{1}{(\omega^2 - \omega_{ei}^2)^2 + (2\zeta_{ei}\omega\omega_{ei})^2} \, \mathrm{d}\omega. \tag{25}$$

In equation (25), $\zeta_{ei}(= c/2\omega_{ei})$ is the damping ratio, ζ_{et}, at $t = b_i$. For solving equation (24), knowledge of ω_{ei} is essential and therefore, it is required to assume an initial value of ω_{ei} to solve equation (22) iteratively. For mildly non-linear systems, $\omega_{ei} = \omega_n$ may be a good initial value to start with. The iteration is continued till the required convergence is achieved in ω_{ei}.

2.3. RESPONSE STATISTICS

Once the values of ω_{ei} and ζ_{ei} are obtained, the nth moment of the PSDF of the instantaneous response may be computed as [7]

$$m_n|_{t=b_i} = \int_0^\infty \sum_j \frac{K' E[W_\psi^2 \ddot{z}(a_j, b_i)] \omega^n \chi_{[\pi/a_j, \sigma\pi/a_j]}(\omega)}{(\omega^2 - \omega_{ei}^2)^2 + (2\zeta_{ei}\omega\omega_{ei})^2} \, \mathrm{d}\omega, \tag{26}$$

where $\chi_{[\cdot]}$ denotes the indicator function which is equal to one on the interval, $[\cdot]$, and zero otherwise. Other relevant response parameters and statistics can also be obtained by using the moments of the PSDF of the instantaneous response. For example, the instantaneous rate of crossing, Ω_i, and bandwidth parameter, λ_i, respectively, are given by

$$\Omega_i = \sqrt{\frac{m_2|_{t=b_i}}{m_0|_{t=b_i}}} \tag{27}$$

and

$$\lambda_i = \sqrt{1 - \frac{m_1^2|_{t=b_i}}{m_0|_{t=b_i} m_2|_{t=b_i}}}. \tag{28}$$

These parameters can be used to obtain the largest peak statistics of the response process, $x(t)$, of duration, T. The probability that the process, $|x(t)|$, remains below the level, x, during the time interval, $(0, T)$, is given by [11]

$$P_T(x) = \exp\left[-\int_0^T \alpha(t)\,\mathrm{d}t\right] = \exp\left[-\sum_i \alpha(t)|_{t=b_i}\Delta b\right], \tag{29}$$

where

$$\alpha(t) = \frac{\Omega_i}{\pi} \mathrm{e}^{-x^2/2m_0|_{t=b_i}} \frac{1 - \exp(-\sqrt{\pi/2}\lambda_i^{1\cdot 2} x/\sqrt{m_0|_{t=b_i}})}{1 - \exp(-x^2/2m_0|_{t=b_i})}. \tag{30}$$

3. NUMERICAL STUDY

To illustrate the above formulation, an excitation process corresponding to the recorded ground motion at the Pacoima dam site in case of the 1971 San Fernando earthquake has been considered. An ensemble of 20 accelerograms has been generated for this process by using the SYNACC program [12] as in References [7, 13]. The values of the discretization parameters have been assumed as $\sigma = 2^{1/4}$ and $\Delta b = 0{\cdot}02$ (as in references [7, 13]) and the ensemble-averaged values of $W_\psi^2 \ddot{z}(a_j, b_i)$ have been determined for $i = 1$–2047 and $j = -17$–4 to characterize the input process. The non-linear Duffing oscillators taken for the illustration are assumed to have the linear viscous damping parameter, $c/2\omega_n = 0{\cdot}02$. Stochastic responses have been estimated from the proposed (wavelet-based) formulation as

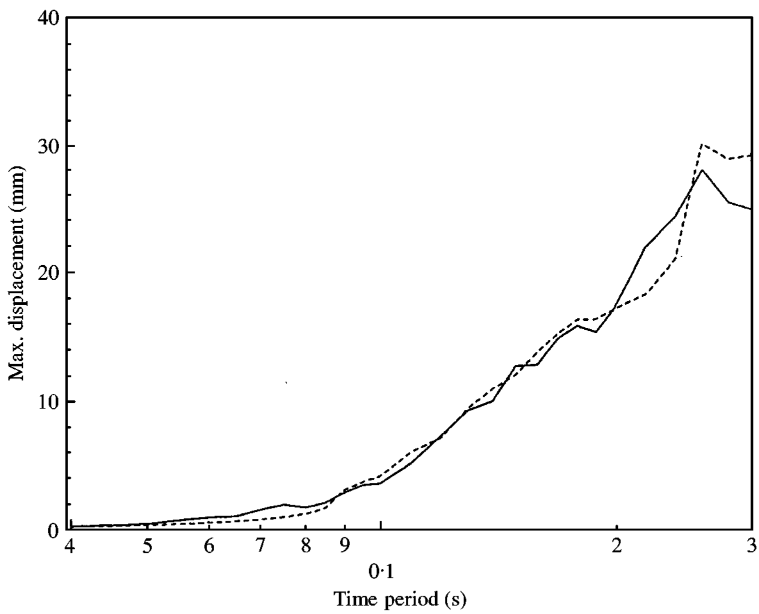

Figure 1. Comparison of maximum displacement spectra from simulation and wavelet-based approach in case of a hard Duffing oscillator with $\varepsilon = 0{\cdot}1$: ——, simulation; ----------, wavelet.

well as from the time–history simulations based on the fourth order Runge–Kutta algorithm. One step iteration has been found to be sufficient for calculating ω_{ei} in the present study.

Figure 1 shows the comparison of the expected maximum displacement response of a set of hard Duffing oscillators with $\varepsilon = 0{\cdot}1$ and with the linear part of the period, $2\pi/\omega_n$,

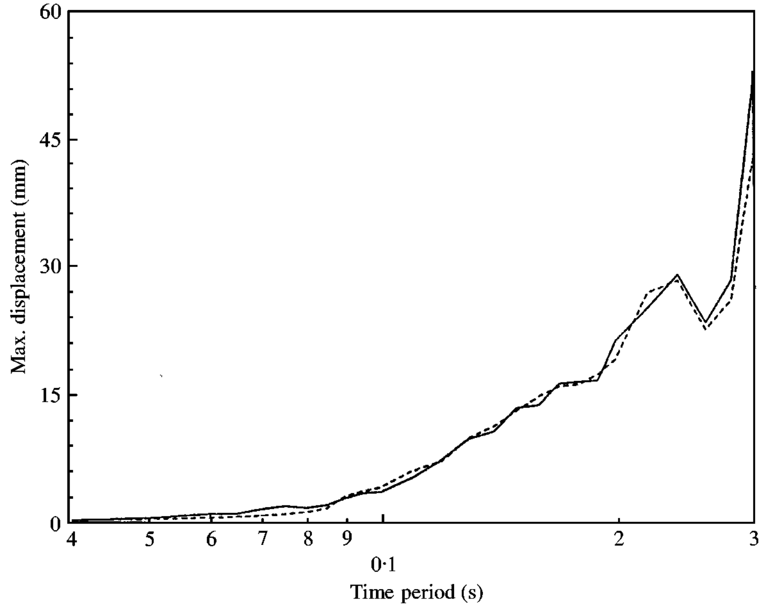

Figure 2. Comparison of maximum displacement spectra from simulation and wavelet-based approach in case of a soft Duffing oscillator with $\varepsilon = -0{\cdot}01$: ——, simulation; ----------, wavelet.

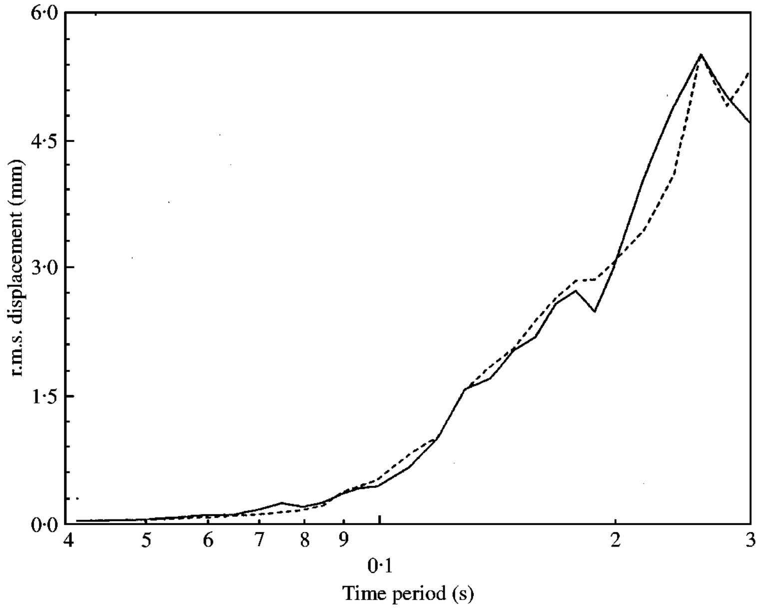

Figure 3. Comparison of r.m.s. displacement spectra from simulation and wavelet-based approach in case of a hard Duffing oscillator with $\varepsilon = 0{\cdot}1$: ——, simulation; ----------, wavelet.

ranging between 0·04 and 0·3 s. The results from the proposed formulation are seen to match very well with those from the simulation. Similar observations hold for Figure 2, where soft Duffing systems with $\varepsilon = -0{\cdot}01$ have been compared. Figures 3 and 4 show this agreement for the temporal root-mean-square (r.m.s.) displacement response in case of the hard ($\varepsilon = 0{\cdot}1$) and soft ($\varepsilon = -0{\cdot}01$) oscillators respectively. To compare the results of the proposed approach in greater detail with those from the simulation, first 20 s of the

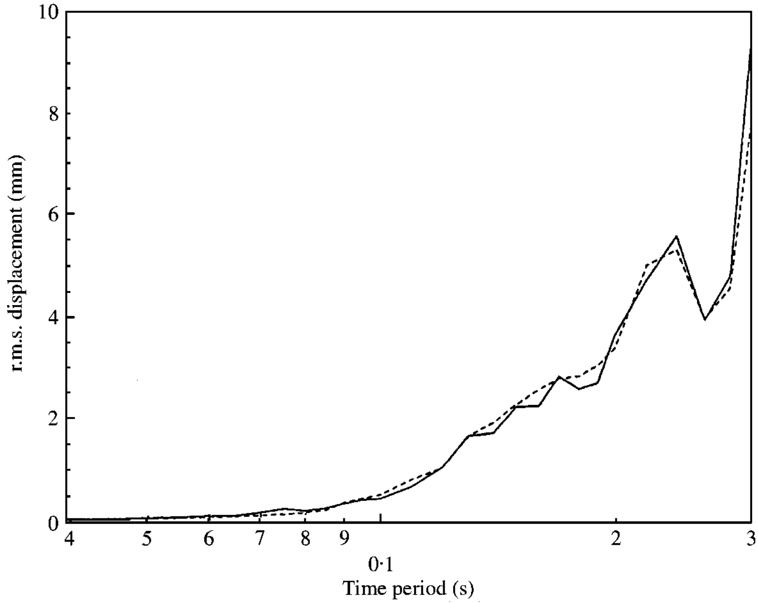

Figure 4. Comparison of r.m.s. displacement spectra from simulation and wavelet-based approach in case of a soft Duffing oscillator with $\varepsilon = -0{\cdot}01$: ——, simulation; ----------, wavelet.

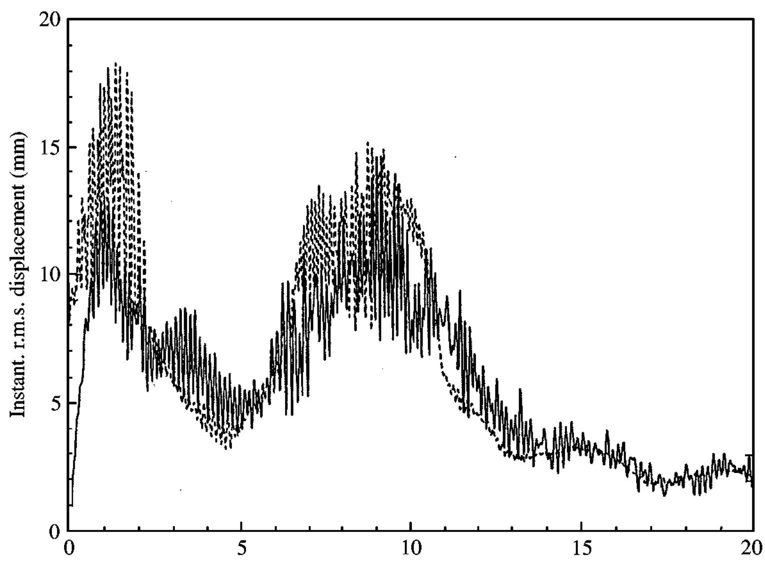

Figure 5. Comparison of instantaneous r.m.s. displacement time-histories from simulation and wavelet-based approach in case of a hard Duffing oscillator with $\varepsilon = 0{\cdot}1$ and linear period = 0·3 s: ——, simulation; ----------, wavelet.

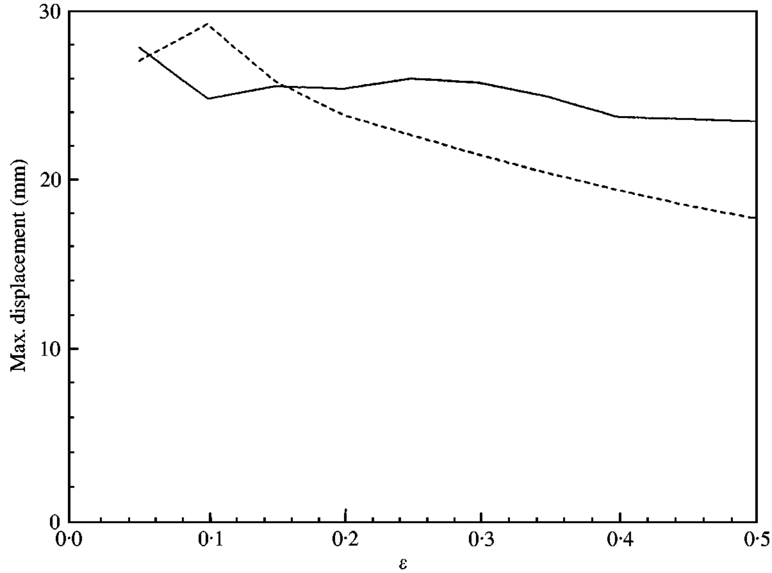

Figure 6. Comparison of maximum displacement responses from simulation and wavelet-based approach for different values of ε in case of hard Duffing oscillators with linear period = 0·3 s: ——, simulation; ---------, wavelet.

instantaneous r.m.s. displacement response time history of a hard Duffing oscillator with $\varepsilon = 0{\cdot}1$ and linear period of 0·3 s have been considered. Figure 5 shows that excellent agreement is obtained between the two time histories. Further, to see how good the proposed approach is for different values of ε, a set of hard Duffing oscillators with linear period of 0·3 s and ε ranging between 0·05 and 0·5 have been considered. Figure 6 shows that the two curves for expected maximum displacement response compare reasonably well up to a value of ε about 0·25. Beyond this, the wavelet-based estimates become poorer. It is thus implied that the proposed formulation is applicable only for mildly non-linear systems with low ε values.

4. CONCLUSIONS

A stochastic approach based on wavelet transform has been formulated to obtain the seismic response of a non-linear oscillator with non-linearity in stiffness only. A Duffing oscillator has been considered to obtain the wavelet-based stochastic equivalent linear parameters. These parameters vary in time and thus, the given non-linear system is replaced by a time-varying linear system. The proposed formulation accounts for the non-stationarity, in both amplitude and frequency, in the ground motion process. A numerical study based on comparison with simulation results shows that this formulation may be used to obtain reliable response estimates in case of mildly non-linear systems. Though the proposed formulation is obtained for base-excited oscillators, it may be easily used for other types of forcing functions also.

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