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INVESTIGATION OF THE STABILITY AND STEADY STATE RESPONSE OF ASYMMETRIC ROTORS, USING FINITE ELEMENT FORMULATION

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The motion of a rotor of which both the bearings and the shaft cross-section are asymmetric is generally governed by ordinary differential equations with periodic coefficients. In this paper, modifications are made to incorporate the effect of shaft asymmetry into an existing finite element procedure developed for rotors with symmetric shaft. The model for the elastic asymmetric shaft takes into account rotary inertia and gyroscopic inertia. In addition, this paper describes an existing method of investigating the stability of a general system of differential equations with periodic coefficients, and evaluates its efficiency when applied to asymmetric rotors. The method described is based on Floquet's theory and involves the computation of a transfer matrix over one period of motion. The detailed presentation of the equations of motion, both in a rotating frame of reference and in a fixed one, is accompanied by an analysis of specific cases. The equations of motion for a simpler model are obtained by modal expansion. Numerical examples are given in order to show the finite element formulation and the transfer matrix method as applied to asymmetric rotors. Due to the use of linear equations, the results shown in this paper have limited practical utility, but they are useful tools in analyzing the behavior of periodic systems with weak non-linearities.

1. INTRODUCTION

It is important to distinguish between the asymmetry of the rotating part and the asymmetry of the fixed part of a rotor. If only one of the two is asymmetric, it is possible to establish a frame of reference in which the coefficients of the equations of motion are constant. If both parts are asymmetric, the system equations will, in the majority of cases, have periodic coefficients.

The principal methods used to investigate the stability of systems with periodic coefficients fall into three groups: perturbation methods [1], variants of Hill's infinite determinant method [2, section 7.6, 3], and methods combining the use of Floquet's theory [2, section 7.2] with numerical computation of the transfer matrix at the end of one period [4–6].

The main advantage of the methods from the third group, here called "transfer matrix methods", is their high degree of generality. It is not necessary for the equations of motion to satisfy restrictive conditions (as is the case with the perturbation methods, where the periodic coefficients need to be expressed in terms of a small parameter), nor are complex preparatory steps required before numerical procedures can be applied (as is the case with both the perturbation and the infinite determinant methods). The main disadvantage of

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these methods is the high computational effort required to obtain the transfer matrix over one period, while the stability of the motion can be evaluated for only one point in the parameter's space at a given time.

Many studies deal with rotors having partial asymmetry: either bearing asymmetry or disk and shaft asymmetry. Typical studies are by El-Marhomy [7], for bearing asymmetry, and Ardayfio and Frohrib [8], for shaft and disk asymmetry. In this case, it is possible to describe the movement of the rotor using a system of differential equations with constant coefficients.

Amongst the papers dealing with rotors having full asymmetry, the majority consider simple models, such as a massless shaft of uniform cross-section, a disk usually in plane motion, one or two identical elastic bearings, and environmental damping acting on the disk. Typical studies using Hill's infinite determinant method to analyze the motion stability are those by Brosens and Crandall [9], and Kotera and Yano [10]. Perturbation methods are used, for example, by Black and McTernan [11], and also Iwatsubo *et al.* [12]. The transfer matrix method is considered by Guilhen *et al.* [13], for a specific model with rigid bearings at the shaft ends, one elastic bearing and one rigid disk, attached to the shaft at different stations.

The adoption of simple models of asymmetric rotors facilitates the understanding of the behavior of such dynamic systems, but the practical use of these studies remains limited. Nelson and McVaugh [14] developed a finite element model for a rotor-bearing system with asymmetric bearings and symmetric shaft, which means a rotor with partial asymmetry. Their procedure is presented also in reference [15, chapter 3]. Inagaki *et al.* [16] studied a multi-disk fully asymmetric rotor with longitudinal variation of the shaft cross-section. The temporal equations of motion were obtained using the transfer matrix method ("transfer" between two stations of the shaft). The unbalance response is deduced by the harmonic balance method. Genta [17] applied the finite element method to fully asymmetric rotors, using complex co-ordinates, in order to reduce the size of the problem, and analyzed the stability of the motion using Hill's infinite determinant method. Kang *et al.* [18] also applied the finite element method to fully asymmetric rotors, studying the unbalanced response using the harmonic balance method. Neither of the two studies gives the elements of the matrices related to the shaft asymmetry.

In this paper, the finite element procedure for rotors with asymmetric bearings and symmetric shaft by Nelson and McVaugh [14] is extended to include shaft asymmetry. The motion stability and the unsteady response are analyzed using the method developed by Friedmann and associates [4, 5]. The elements of all matrices involved in the equations of motion are included in Appendix A.

2. TRANSFER MATRIX METHOD

Rotor systems are constrained by bearings, dampers, seals, etc., to very small lateral motion. Generally, these mechanisms are dynamic systems with weak non-linearities, and their motion is governed by ordinary differential equations with periodic coefficients, represented in a first order state variable form as follows:

$$\{\dot{x}\} = [A(t)]\{x\} + \{f(t)\} + \{N_1(\{x\}, t)\} + \{N_2(\{x\}, \{\dot{x}\}, t)\}, \tag{1}$$

with matrix $[A(t)]$ and vectors $\{f\}$, $\{N_1\}$ and $\{N_2\}$ periodic, of common period T.

An effective approach [5] to deal with the weak non-linearities combined in vectors $\{N_1\}$ and $\{N_2\}$ of system (1) is quasilinearization, consisting in an iterative evaluation of the periodic response of system (1), as a solution of linear periodic systems. This procedure is initiated with an initial estimation of the solution, obtained from system (1) after dropping the non-linear terms $\{N_1\}$ and $\{N_2\}$.

In this paper, the discussion will center on n-degree-of-freedom dynamic systems governed by the linear system of $m = 2n$ first order differential equations and m unknown functions,

$$[\dot{x}\} = [A(t)]\{x\} + \{f(t)\}, \tag{2}$$

where $\{x(t)\}$ is the state variable vector, $[A(t)]$ is the system matrix, and vector $\{f(t)\}$ describes time-dependent forces, both $[A(t)]$ and $\{f(t)\}$ being periodic, of common period T.

It is of major interest to find the steady state response of system (2), i.e., the periodic solution, and to evaluate its stability.

2.1. STABILITY

The study of the stability of the steady state solution of system (2) can be reduced (see reference [2, Chapter 7]) to the study of the stability of the trivial solution of the associated homogeneous system:

$$\{\dot{x}\} = [A(t)]\{x\}. \tag{3}$$

It should be noted that this result is more general, being valid for system (2) replaced with a non-autonomous system

$$\{\dot{x}\} = \{X(\{x\}, t)\} \tag{2b}$$

with the functions X_i being periodic, of period T and system (3) replaced with the variational system attached to system (2b) and to its periodic solution $\{p(t)\}$,

$$\{\dot{y}\} = [a(t)]\{y\}, \tag{3b}$$

where $\{y(t)\}$ contains the small perturbations from $\{p(t)\}$, and matrix $[a(t)]$ is periodic, of period T.

For system (3), the time transfer matrix is defined as an $m \times m$ matrix $[\Phi(t)]$ with its columns consisting of a set of linearly independent solutions.

Since matrix $[A(t)]$ is periodic, of period T, an extension of Floquet's theory (see reference [2, Chapter 7]) shows that $[\Phi(t)]$ is fully known when its variation during one period T is known. Furthermore, it is shown that the stability of the trivial solution of equation (3) is fully defined by the eigenvalues of the transfer matrix over one period $[\Phi(T)]$, known as the characteristic multipliers of system (3). The trivial solution is asymptotically stable if the modulus of all m eigenvalues is less than one, and is unstable if the modulus of at least one of the eigenvalues is greater than one.

2.2. STEADY-STATE RESPONSE

For non-homogeneous system (2), without considering the periodicity of matrix $[A(t)]$ and vector $\{f(t)\}$, the general solution is given by (see reference [2, Chapter 6])

$$\{x(t)\} = [\Phi(t)]\{x(0)\} + \int_0^t [\Phi(t, s)]\{f(s)\}\,\mathrm{d}s, \tag{4}$$

in which $[\Phi(t)]$ is the transfer matrix of the homogeneous system satisfying the initial conditions $[\Phi(0)] = [I_m]$, and matrix $[\Phi(t, s)]$ is given by

$$[\Phi(t, s)] = [\Phi(t)]\,[\Phi(s)]^{-1}. \tag{5}$$

It is clear that the first term on the right-hand side of expression (4) is the general solution of the homogeneous system (3), denoted by $\{x_h(t)\}$. Matrix $[\Phi(t, s)]$ makes the transfer between the values at moments "t" and "s" of $\{x_h(t)\}$:

$$\{x_h(t)\} = [\Phi(t, s)]\,\{x_h(s)\}. \tag{6}$$

If matrix $[A(t)]$ and vector $\{f(t)\}$ are periodic, of common period T, and if the modulus of every characteristic multiplier of the corresponding homogeneous system (3) is different from one, system (2) has one and only one periodic solution [3]. This periodic solution is the steady state response of the periodic non-homogeneous system only if the associated periodic homogeneous system is stable, that is to say, only if the modulus of every characteristic multiplier is less than one.

To obtain the initial conditions corresponding to the steady state response, condition $\{x(T)\} = \{x(0)\}$ is substituted into equation (4). An algebraic system for $\{x(0)\}$ is derived:

$$([I_m] - [\Phi(T)])\{x(0)\} = \int_0^T [\Phi(t, s)]\{f(s)\}\,\mathrm{d}s. \tag{7}$$

The steady state response can be obtained by taking the initial conditions $\{x(0)\}$ given by equation (7) and numerically integrating system (2) over one period. If $\{x(T)\}$ is not close enough to $\{x(0)\}$, the integration is continued over another few periods, until the desired convergence is obtained.

In order to calculate the transfer matrix over one period $[\Phi(T)]$, Friedmann *et al.* [4] considered the division of the period T into a number of equal parts. By denoting the division points as

$$0 = t_0 < t_1 < \dots < t_{N-1} < t_N = T, \tag{8}$$

a useful relation can be derived from equation (6):

$$[\Phi(T, t_i)] = [\Phi(T, t_{i+1})]\,[\Phi(t_{i+1}, t_i)]. \tag{9}$$

Matrix $[\Phi(T)] = [\Phi(T, 0)]$ is generated by an iterative calculation, based on relation (9) and starting with $[\Phi(T, t_N)] = [I_m]$. The elementary transfer matrix $[\Phi(t_{i+1}, t_i)]$ is obtained by numerically integrating the homogeneous system (3) over the corresponding elementary interval. The iterative calculation not only gives matrix $[\Phi(T)]$, to be used in the stability evaluation, but also matrices $[\Phi(T, t_i)]$, appearing in the right-hand term of the algebraic system (7) for the initial condition $\{x(0)\}$, if an integration scheme using division points (8) is considered.

It has been shown in reference [4] that a very efficient way to obtain the elementary transfer matrix is by considering the fourth order Runge–Kutta scheme with Gill coefficients.

3. CONFIGURATION

The mathematical model (Figure 1) consists of a flexible horizontal shaft, one or more rigid disks and two or more flexible bearings.

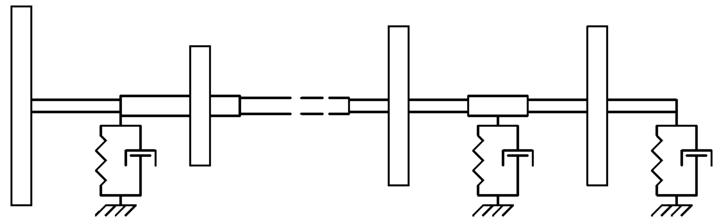

Figure 1. General model of the rotor.

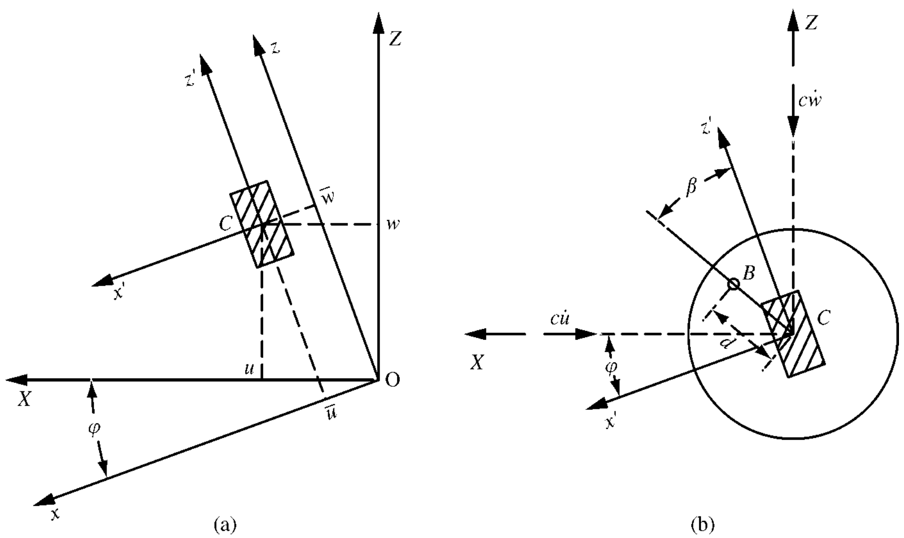

Figure 2. (a) Co-ordinate systems: OXYZ and Oxyz, fixed and rotating frames of reference; $Cx'y'z'$, system of principal axes of inertia of shaft cross-section. (b) Mass unbalance and external damping.

The shaft cross-section is asymmetric, having different principal moments of inertia, and varies step-by-step along the longitudinal axis (Figure 2(a)). However, the principal directions of inertia of the cross-section do not vary along the shaft.

The mathematical model for a bearing is a massless spring–damper system (Figure 3). Its characteristics (stiffness and damping) in horizontal and vertical directions are different. The cross-coupled characteristics in the horizontal and vertical directions are also considered. This linear and anisotropic model with eight coefficients (four for stiffness and four for viscous damping) is the same as in reference [14].

We analyze the unbalance response using concentrated masses placed on disks (point B in Figure 2(b)) and we consider an environmental viscous damping acting on the disks (Figure 2(b)).

Two co-ordinate systems can be seen in Figure 2(a). On the fixed system OXYZ, the Y-axis is along the undeflected horizontal shaft and the Z-axis is oriented vertically upward. The rotating system Oxyz has the y-axis coincidental with the Y-axis of the fixed system, and x- and z-axis parallel with the principal axes x' and z' of the shaft cross-section (if the angular deflections are neglected).

The finite element procedure for rotors with symmetric shaft, developed by Nelson and McVaugh [14], will be considered. Modifications will be made to accommodate the effect of shaft asymmetry. The shaft model includes the effect of rotary inertia and the gyroscopic effect. Shear of the cross-section and internal damping will be neglected.

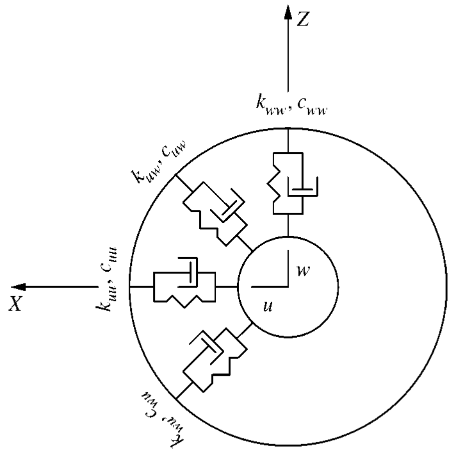

Figure 3. Bearing stiffness and damping.

4. KINEMATICS

We consider (Figure 4) two frames with the origin at the center of the shaft cross-section, $CXYZ$, having the axes parallel with the axes of the fixed frame OXYZ, and $Cx'y'z'$, whose axes are the principal directions of inertia of the cross-section. The link between the two frames is made through the set of Euler's angles φ, θ and ψ. To bring the shaft cross-section from its undeflected position to the current one, three rotations are to be applied successively: one of angle ψ about the Z-axis, one of angle θ about the new axis x, denoted as x_1, and one of angle φ about the new axis z, denoted as z_2.

Assuming the angles θ and ψ to be small and the rotational speed $\dot{\varphi}$ constant and denoted as Ω, the instantaneous angular velocity of the shaft cross-section has, in the $x'y'z'$ frame, the components

$$\omega_x = \dot{\theta}\cos\varphi - \dot{\psi}\sin\varphi, \qquad \omega_y = \Omega + \dot{\psi}\theta, \qquad \omega_z = \dot{\theta}\sin\varphi + \dot{\psi}\cos\varphi. \tag{10}$$

See reference [15, Chapter 1], for the general expressions.

In the fixed frame OXYZ (Figure 2), the position of the shaft cross-section is defined by the displacements of its elastic center u, w, and by the angles ψ, θ. The angles θ and ψ are approximately equal to the angular deflections collinear with the X- and Z-axis, respectively (Figure 5):

$$\psi = -\frac{\partial}{\partial y}u, \qquad \theta = \frac{\partial}{\partial y}w. \tag{11}$$

In the rotating frame Oxyz (Figure 2), the position of the shaft cross-section is defined by the displacements $\bar{u}$, $\bar{w}$ and the slopes $\bar{\psi}$, $\bar{\theta}$. The transformation of displacements and slopes from the fixed frame to the moving frame are defined by

$$\begin{Bmatrix} u \\ w \end{Bmatrix} = [T_2]\begin{Bmatrix} \bar{u} \\ \bar{w} \end{Bmatrix}, \qquad \begin{Bmatrix} \psi \\ \theta \end{Bmatrix} = [T_2]^{\mathrm{T}}\begin{Bmatrix} \bar{\psi} \\ \bar{\theta} \end{Bmatrix}, \tag{12}$$

with matrix $[T_2]$ given in Appendix A.

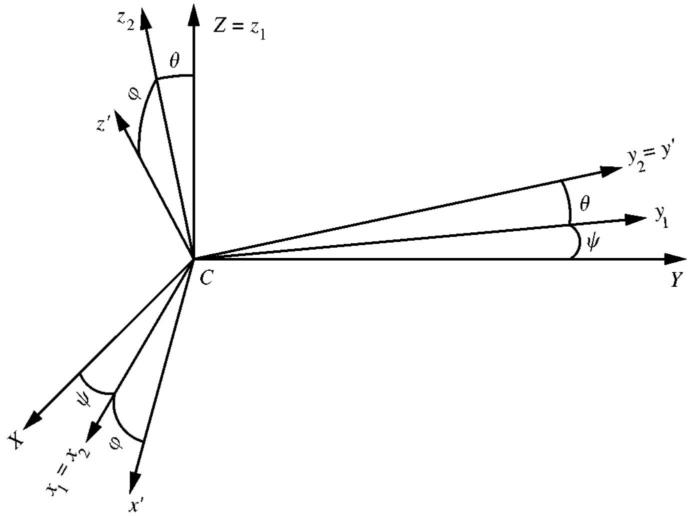

Figure 4. Euler's angles.

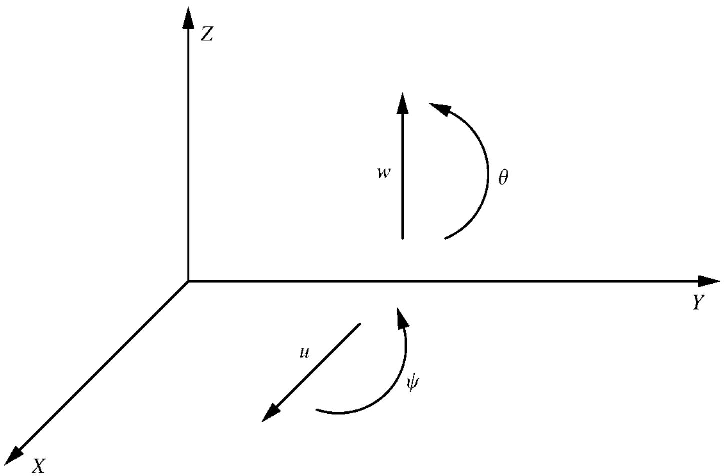

Figure 5. Displacements of the shaft elastic axis.

5. EQUATIONS OF MOTION OF ROTOR ELEMENTS

5.1. DISKS

The center of mass of the rigid disk coincides with the elastic center of the shaft cross-section. The nodal displacements vector of the disk in fixed co-ordinates is given by

$$\{\delta_0\} = \{u_0 \ \ w_0 \ \ \psi_0 \ \ \theta_0\}^{\mathrm{T}},$$

its components being the displacements of the shaft at disk attachment.

5.1.1. *Equations of motion of disk in fixed frame*

The kinetic energy of the disk has the expression

$$T = \tfrac{1}{2} m(\dot{u}_0^2 + \dot{w}_0^2) + \tfrac{1}{2}(J_x \omega_x^2 + J_y \omega_y^2 + J_z \omega_z^2), \qquad (13)$$

where ω_x, ω_y and ω_z are the angular velocities given by equation (10), while m, $J_x = J_z$ and J_y are the mass and the moments of inertia of the disk.

By substituting equation (10) into equation (13), and also considering the effect of the unbalance, we obtain

$$T_D = \tfrac{1}{2}m(\dot{u}_0^2 + \dot{w}_0^2) + \tfrac{1}{2}J_x(\dot{\theta}_0^2 + \dot{\psi}_0^2) + J_y\Omega\dot{\psi}_0\theta_0$$
$$+ m_u \mathrm{d}\Omega[\dot{u}_0\cos(\Omega t + \beta) - \dot{w}_0\sin(\Omega t + \beta)], \tag{14}$$

where Ω is the angular speed of the rotor, m_u is the unbalance mass (assumed to be small if compared with m), d and β are the radius and the angle defining the location of the unbalance.

The four terms in equation (14) give, in order, the translating inertia effect, the rotary inertia effect, the gyroscopic effect, and the unbalance effect.

The virtual work of the disk weight and of damping forces acting on the disk can be written as

$$\delta L_D = -\, mg\delta w_0 - c\dot{u}_0\delta u_0 - c\dot{w}_0\delta w_0, \tag{15}$$

where c is the damping coefficient, m is the mass of the disk, and g is the gravitational acceleration.

The application of Lagrange's equations for the disk only gives

$$[M_D]\{\ddot{\delta}_0\} + [C_D]\{\dot{\delta}_0\} = \{Q_D\} + \{Q_L\}, \tag{16}$$

where $[M_D]$ and $[C_D]$ are the mass and damping matrices of the disk, $\{Q_D\}$ is the load vector and $\{Q_L\}$ is the vector of liaison forces which will disappear at the assembly of the elementary matrices.

Matrices $[M_D]$ and $[C_D]$, and also vector $\{Q_D\}$ are given in Appendix A.

5.1.2. *Equations of motion of disk in rotating frame*

The nodal displacements in fixed co-ordinates are related to those in rotating co-ordinates by the transformation equation

$$\{\delta_0\} = [T_4]\{\bar{\delta}_0\}, \tag{17}$$

with the matrix $[T_4]$ given in Appendix A. The substitution of equation (17) into equation (16), yields the equations of motion in rotating co-ordinates as

$$[\bar{M}_D]\{\ddot{\bar{\delta}}_0\} + [\bar{C}_D]\{\dot{\bar{\delta}}_0\} + [\bar{K}_D]\{\bar{\delta}_0\} = \{\bar{Q}_D\} + \{\bar{Q}_L\}, \tag{18}$$

where $[\bar{M}_D] = [M_D]$, $[\bar{C}_D] = 2\Omega[M_D][H_4] + [T_4]^{\mathrm{T}}[C_D][T_4]$, $[\bar{K}_D] = -\,\Omega^2[M_D] + \Omega[T_4]^{\mathrm{T}}[C_D][T_4][H_4]$ and $\{\bar{Q}_D\} = [T_4]^{\mathrm{T}}\{Q_D\}$, with matrices $[T_4]$ and $[H_4]$ given in Appendix A. $\{\bar{Q}_L\}$ contains liaison forces that will disappear at the assembly of the elementary matrices. Matrices $[\bar{K}_D]$ and $[\bar{C}_D]$, and also vector $\{\bar{Q}_D\}$ are given in Appendix A.

5.2. THE SHAFT

While the shaft element has two nodes, the nodal displacement vector includes four displacements and four slopes. Its expression in fixed co-ordinates is $\{\delta_e\} =$

$\{u_1\ w_1\ \psi_1\ \theta_1\ u_2\ w_2\ \psi_2\ \theta_2\}^{\mathrm{T}}$. The nodal displacements in fixed co-ordinates are related to those in rotating co-ordinates by the transformation equation:

$$\{\delta_e\} = [T_g]\{\bar{\delta}_e\}. \tag{19}$$

with matrix $[T_g]$ given in Appendix A.

In fixed co-ordinates, the displacements and the slopes along the shaft elements are represented by shape functions as

$$\begin{Bmatrix} u \\ w \end{Bmatrix} = [N(y)]\{\delta_e\} \tag{20}$$

and

$$\begin{Bmatrix} \psi \\ \theta \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial}{\partial y} u \\ \dfrac{\partial}{\partial y} w \end{Bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} [D(y)]\{\delta_e\}, \tag{21}$$

where $[N]$ is the matrix of shape functions

$$[N] = \begin{bmatrix} N_1 & 0 & -N_2 & 0 & N_3 & 0 & -N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \quad \text{and} \quad [D] = \left[\frac{\partial}{\partial y} N\right].$$

$N_i(y)$ are the typical displacement functions of a beam in bending:

$$N_1(y) = 1 - 3\bar{y}^2 + 2\bar{y}^3, \qquad N_2(y) = L\bar{y}(1 - 2\bar{y} + \bar{y}^2), \qquad N_3(y) = 3\bar{y}^2 - 2\bar{y}^3,$$

$$N_4(y) = L(-\bar{y}^2 + \bar{y}^3), \text{ with } \bar{y} = y/L. \tag{22}$$

5.2.1. *Kinetic energy*

The kinetic energy for the shaft element has a form similar to the one derived for the disk and is

$$T_A = \frac{1}{2}\rho A \int_0^L (\dot{u}^2 + \dot{w}^2)\,\mathrm{d}y + \tfrac{1}{2}\rho \int_0^L (I_x\omega_x^2 + I_p\omega_y^2 + I_z\omega_z^2)\,\mathrm{d}y, \tag{23}$$

where ω_x, ω_y and ω_z are the angular velocities given by equation (10), I_x and I_z are the second moments of area about principal axes x' and z' of the shaft, $I_p = I_x + I_z$ is the polar moment of area, A, L and ρ are the cross-section, the length, and the density of the shaft element.

By introducing equation (10) in equation (23), we obtain

$$T_A = T_t + T_r + T_{d,c}\cos(2\Omega t) + T_{d,s}\sin(2\Omega t) + T_g, \tag{24}$$

where $T_t = \frac{1}{2}\rho A \int_0^L(\dot{u}^2 + \dot{w}^2)\,\mathrm{d}y$, is the translating term, $T_r = \frac{1}{2}\rho I_m \int_0^L(\dot{\theta}^2 + \dot{\psi}^2)\,\mathrm{d}y$ is the rotatory term), $T_{d,c} = \frac{1}{2}\rho I_d \int_0^L(\dot{\psi}^2 + \dot{\theta}^2)\,\mathrm{d}y$ is the deviatory-cosine term, $T_{d,s} = \rho I_d \int_0^L \dot{\psi}\dot{\theta}\,\mathrm{d}y$ is the deviatory-sine term and $T_g = \rho I_p \Omega \int_0^L \dot{\psi}\theta\,\mathrm{d}y$ is the gyroscopic term, with $I_m = \frac{1}{2}(I_z + I_x)$, $I_d = \frac{1}{2}(I_z - I_x)$ and $I_p = 2I_m$.

5.2.2. *Strain energy*

If the shear deformations are neglected, the strain energy of the shaft element is

$$U = \frac{1}{2}\int_0^L [EI_z(\bar{u}'')^2 + EI_x(\bar{w}'')^2]\,\mathrm{d}y, \tag{25}$$

where $\bar{u}$ and $\bar{w}$ are the displacements of the center of the cross-section in rotating co-ordinates, I_x and I_z are the second moments of area about principal axes, E is Young's modulus, L is the length of the shaft element, $(\cdot)'' = (\partial^2/\partial y^2)(\cdot)$, and $\bar{u}''$ and $\bar{w}''$ are the deflections of bending in the directions of the principal axes.

Substituting the transformation equation (12) into equation (25), the strain energy can be expressed as

$$U = \frac{1}{2}\int_0^L EI_m[(u'')^2 + (w'')^2]\,\mathrm{d}y + \frac{1}{2}\int_0^L EI_d\langle[(u'')^2 - (w'')^2]\cos(2\Omega t) - 2u''w''\sin(2\Omega t)\rangle\,\mathrm{d}y, \tag{26}$$

where u and w are the displacements of the center of the cross-section in fixed co-ordinates, I_m and I_d are the mean and the deviatory area moments, and Ω is the angular speed of the rotor.

5.2.3. *Virtual work of the shaft weight*

The virtual work of the shaft weight has the expression

$$\begin{aligned}\delta L_A &= -g\int_0^L (\rho A\delta w)\,\mathrm{d}y \\ &= -\rho g A\left(\delta w_1\int_0^L N_1\,\mathrm{d}y + \delta\theta_1\int_0^L N_2\,\mathrm{d}y + \delta w_2\int_0^L N_3\,\mathrm{d}y + \delta\theta_2\int_0^L N_4\,\mathrm{d}y\right),\end{aligned} \tag{27}$$

where $N_k(y)$ are the shape functions given by equation (22), A, L and ρ are the cross-section, the length and the density of the shaft element, and g is the gravitational acceleration.

5.2.4. *Equations of motion of shaft in fixed frame*

Upon substituting equations (20) and (21) into the expressions (24), (26) and (27), Lagrange's equations provide the system

$$[M_A(t)]\{\ddot{\delta}_e\} + [C_A]\{\dot{\delta}_e\} + [K_A(t)]\{\delta_e\} = \{Q_A\} + \{Q_L\}, \tag{28}$$

with $[M_A(t)]$ and $[K_A(t)]$ being periodic matrices of period π/Ω, and $[C_A]$ being a constant matrix, given by

$$[M_A(t)] = [M_t] + [M_r] + [M_{d,c}]\cos(2\Omega t) + [M_{d,s}]\sin(2\Omega t), \tag{29}$$

$$[C_A] = \Omega[G], \qquad [K_A(t)] = [K_m] + \lfloor K_{d,c}\rfloor\cos(2\Omega t) + \lfloor K_{d,s}\rfloor\sin(2\Omega t). \tag{30, 31}$$

In equation (28), vector $\{Q_A\}$ represents the shaft weight, while vector $\{Q_L\}$ contains the liaison forces that will disappear when the elementary matrices are assembled.

The gyroscopic matrix, mass matrices and stiffness matrices of equations (29)–(31) may be expressed as

$$[G] = \tfrac{1}{2}\rho I_p \int_0^L [D]^T \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} [D]\,dy,$$

$$[M_t] = \rho A \int_0^L [N]^T [N]\,dy, \qquad [M_r] = \rho I_m \int_0^L [D]^T [D]\,dy,$$

$$[M_{d,c}] = \rho I_d \int_0^L [D]^T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} [D]\,dy, \qquad [M_{d,s}] = -\rho I_d \int_0^L [D]^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} [D]\,dy, \quad (32)$$

$$[K_m] = EI_m \int_0^L [B]^T [B]\,dy, \qquad [K_{d,c}] = EI_d \int_0^L [B]^T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} [B]\,dy,$$

$$[K_{d,s}] = EI_d \int_0^L [B]^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} [B]\,dy,$$

with $[N]$, the matrix of shape functions, $[D] = [(\partial/\partial y)N]$ and $[B] = (\partial^2/\partial y^2)[N]$.

The elements of the matrices defined by the above equations, and also of vector $\{Q_A\}$ are shown in Appendix A.

Matrices $[M_{d,c}]$, $[M_{d,s}]$, $[K_{d,c}]$ and $[K_{d,s}]$ are proportional to the deviatory moment of area of the shaft I_d, and will vanish for a symmetric shaft. Matrices $[M_t]$, $[M_r]$, $[K_m]$ and $[G]$, specific for a symmetric shaft, can be found also in reference [14].

5.2.5. *Equations of motion of shaft in rotating frame*

Substituting transformation equation (19) into equation (28), we obtain

$$[\bar{M}_A]\{\ddot{\bar{\delta}}_e\} + [\bar{C}_A]\{\dot{\bar{\delta}}_e\} + [\bar{K}_A]\{\bar{\delta}_e\} = \{\bar{Q}_A(t)\} + \{\bar{Q}_L\}, \quad (33)$$

where $[\bar{M}_A]$, $[\bar{C}_A]$ and $[\bar{K}_A]$ are constant matrices given by

$$[\bar{M}_A] = [M_t] + [M_r] + [M_d], \qquad [\bar{C}_A] = 2\Omega[\bar{M}_A][H_8] + \Omega[G], \quad (34, 35)$$

$$[\bar{K}_A] = [K_m] + [K_d] - \Omega^2[\bar{M}_A] + \Omega^2[G][H_8], \quad (36)$$

with matrix $[H_8]$ given in Appendix A.

In equation (33), the weight vector $\{\bar{Q}_A(t)\}$ is periodic, of period $2\pi/\Omega$. Its elements are shown in Appendix A. Vector $\{\bar{Q}_L\}$ contains the liaison forces.

In equations (34)–(36), matrices $[M_t]$, $[M_r]$, $[K_m]$ and $[G]$ are as defined by equation (32), while matrices $[M_d]$ and $[K_d]$ are given by $[M_d] = [M_{d,c}]$ and $[K_d] = [K_{d,c}]$, with $[M_{d,c}]$ and $[K_{d,c}]$ as defined by equation (32).

5.3. BEARINGS

The nodal displacement vector of the bearing in fixed co-ordinates is given by

$$\{\delta_p\} = \{u_p \;\; w_p\}^{\mathrm{T}},$$

its components being the displacements of the shaft center at bearing attachment.

5.3.1. *Equations of motion of bearing in fixed frame*

The virtual work of the forces acting on the shaft can be written as

$$\delta L_p = -k_{uu}u_p\delta u_p - k_{uw}w_p\delta u_p - k_{wu}u_p\delta w_p - k_{ww}w_p\delta w_p$$
$$- c_{uu}\dot{u}_p\delta u_p - c_{uw}\dot{w}_p\delta u_p - c_{wu}\dot{u}_p\delta w_p - c_{ww}\dot{w}_p\delta w_p, \quad (37)$$

where k_{uu}, k_{uw}, k_{wu} and k_{ww} are the stiffness coefficients, and c_{uu}, c_{uw}, c_{wu} and c_{ww} are the damping coefficients.

The application of Lagrange's equations gives

$$[C_P]\{\dot{\delta}_p\} + [K_P]\{\delta_p\} = \{Q_L\}, \quad (38)$$

where $[K_P]$ and $[C_P]$ are the stiffness and damping matrices of the bearing, and $\{Q_L\}$ is a vector containing the liaison forces between bearing and shaft. Matrices $[K_P]$ and $[C_P]$ are given in Appendix A.

5.3.2. *Equations of motion of bearing in rotating frame*

The nodal displacements in fixed co-ordinates are related to those in rotating co-ordinates by the transformation equation

$$\{\delta_p\} = [T_2]\{\bar{\delta}_p\} \quad (39)$$

with the matrix $[T_2]$ given in Appendix A.

Substituting equation (39) into equation (38), we obtain

$$[\bar{C}_P(t)]\{\dot{\bar{\delta}}_p\} + [\bar{K}_P(t)]\{\bar{\delta}_p\} = \{\bar{Q}_L\}, \quad (40)$$

where $[\bar{C}_P(t)]$ and $[\bar{K}_P(t)]$ are periodic matrices, of period π/Ω, given by

$$[\bar{C}_P] = [T_2]^{\mathrm{T}}[C_P][T_2], \qquad [\bar{K}_P] = [T_2]^{\mathrm{T}}[K_P][T_2] + \Omega[\bar{C}_P][H_2], \quad (41, 42)$$

with matrices $[T_2]$ and $[H_2]$ given in Appendix A. The vector $\{\bar{Q}_L\}$ contains the liaison forces between bearing and shaft. Matrices $[\bar{C}_P(t)]$ and $[\bar{K}_P(t)]$ are given in Appendix A. There are particular cases when these two matrices are constant, the most general case being defined by the conditions

$$k_{uu} = k_{ww}, \qquad k_{wu} = -k_{uw}, \qquad c_{uu} = c_{ww} = 0 \quad \text{and} \quad c_{wu} = -c_{uw}.$$

6. GLOBAL EQUATIONS OF MOTION

6.1. GENERAL CASE

By assembling the elementary matrices of shaft elements, disks and bearings, we obtain a system of $4N_n$ second order differential equations and $4N_n$ unknown functions, where N_n is the number of nodes of the shaft partition. The global equations of motion in fixed co-ordinates are

$$[M(t)]\{\ddot{\delta}\} + [C]\{\dot{\delta}\} + [K(t)]\{\delta\} = \{Q_u(t)\} + \{Q_w\}, \tag{43}$$

where $[M(t)]$ and $[K(t)]$ are periodic matrices of period $T_1 = \pi/\Omega$, for which the time dependency is due to the shaft asymmetry, matrix $[C]$ is constant, $\{Q_u(t)\}$, the unbalance vector, is periodic, of period $T_2 = 2\pi/\Omega$, and $\{Q_w\}$, the weight vector, is constant.

In equation (43), $\{\delta\}$ is the vector of global DOF in fixed reference frame, given as

$$\{\delta\} = \{u_1 \quad w_1 \;\; \psi_1 \;\; \varphi_1 \;\; \cdots \;\; u_{N_n} \;\; w_{N_n} \;\; \psi_{N_n} \;\; \varphi_{N_n}\}^{\mathrm{T}}.$$

In rotating co-ordinates, the assembled global equations are

$$[\bar{M}]\{\ddot{\bar{\delta}}\} + [\bar{C}(t)]\{\dot{\bar{\delta}}\} + [\bar{K}(t)]\{\bar{\delta}\} = \{\bar{Q}_u\} + \{\bar{Q}_w(t)\}, \tag{44}$$

where $[\bar{M}]$ is a constant matrix, $[\bar{C}(t)]$ and $[\bar{K}(t)]$ are periodic matrices of period $T_1 = \pi/\Omega$, for which the time dependency is due to bearing asymmetry, $\{\bar{Q}_u\}$, the unbalance vector, is constant, and $\{\bar{Q}_w(t)\}$, the weight vector, is periodic of period $T_2 = 2\pi/\Omega$.

Each of the above systems can be transformed into a first order differential system, with $8N_n$ equations and $8N_n$ unknown functions. From a numerical point of view, it is more convenient to consider the equations of motion in rotating co-ordinates, because mass matrix $[\bar{M}]$ is constant. Substituting also $\varphi = \Omega t$ for the variable t, system (44) can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\begin{Bmatrix}\{\delta\}\\ \{\delta'\}\end{Bmatrix} = [A(\varphi)]\begin{Bmatrix}\{\delta\}\\ \{\delta'\}\end{Bmatrix} + \{f(\varphi)\}, \tag{45}$$

with

$$[A(\varphi)] = \begin{bmatrix} [0] & [1] \\ -\dfrac{1}{\Omega^2}[\bar{M}]^{-1}[\bar{K}(\varphi)] & -\dfrac{1}{\Omega}[\bar{M}]^{-1}[\bar{C}(\varphi)] \end{bmatrix}, \qquad \{f(\varphi)\} = \begin{Bmatrix} \{0\} \\ \dfrac{1}{\Omega^2}[\bar{M}]^{-1}\{\bar{Q}(\varphi)\} \end{Bmatrix}.$$

Matrix $[A(\varphi)]$ has a periodic variation with frequency of π, due to bearing asymmetry, while vector $\{f(\varphi)\}$ has a periodic variation with frequency of 2π, due to rotor weight.

To study only the stability of the motion, the transfer matrix $[\Phi(T)]$, with $T = \pi$, is calculated using the method described in section 2, and its $8N_n$ eigenvalues are evaluated.

In order to study the stability and to obtain the steady state response at the same time, the period to be considered is $T = 2\pi$. Calculated values of matrix $[\Phi(T, \varphi)]$ at the division points of the period T are used to build the algebraic system of type (7) for the initial conditions $\{x(0)\}$. If the periodic non-homogeneous system (45) is integrated using Simpson's rule, the number of intervals per period N must be even.

6.2. MASSLESS SHAFT, ASSOCIATED WITH UNDAMPED BEARINGS

If the shaft mass is ignored, matrix $[M]$ in equation (43) becomes constant. Reordering the equations of system (43) and also the elements of the global DOF vector $\{\delta\}$, we can highlight the contribution of the N_d disks, as follows:

$$\begin{bmatrix} [M_{dd}] & [0_{dr}] \\ [0_{rd}] & [0_{rr}] \end{bmatrix} \begin{Bmatrix} \{\ddot{\delta}_d\} \\ \{\ddot{\delta}_r\} \end{Bmatrix} + \begin{bmatrix} [C_{dd}] & [0_{dr}] \\ [0_{rd}] & [C_{rr}] \end{bmatrix} \begin{Bmatrix} \{\dot{\delta}_d\} \\ \{\dot{\delta}_r\} \end{Bmatrix} + \begin{bmatrix} [K_{dd}] & [K_{dr}] \\ [K_{rd}\} & \{K_{rr}] \end{bmatrix} \begin{Bmatrix} \{\delta_d\} \\ \{\delta_r\} \end{Bmatrix} = \begin{Bmatrix} \{Q_d\} \\ \{0_r\} \end{Bmatrix}, \tag{46}$$

where index "d" refers to disks, and index "r" refers to the rest of the rotor. Vector $\{\delta_d\}$ contains the $4N_d$ displacements associated with the N_d disks.

If the damping of the bearings is also ignored, we have $[C_{rr}] = [0_{rr}]$ and the last $4(N_n - N_d)$ equations of system (46) give

$$\{\delta_r\} = -[K_{rr}]^{-1}[K_{rd}]\{\delta_d\}. \tag{47}$$

Consequently, the first $4N_d$ equations (46) become

$$[M_{dd}]\{\ddot{\delta}_d\} + [C_{dd}]\{\dot{\delta}_d\} + [K^*_{dd}(t)]\{\delta_d\} = \{Q_d\}, \tag{48}$$

with

$$[K^*_{dd}(t)] = [K_{dd}] - [K_{dr}(t)]\,[K_{rr}(t)]^{-1}[K_{rd}(t)]. \tag{49}$$

In equations (48) and (49), the variable matrices are periodic, of period $T_1 = \pi/\Omega$, and the time dependency is due to the shaft asymmetry.

System (48) can be transformed into a first order differential system of type (45), with $8N_d$ equations and $8N_d$ unknown functions. The advantage of this formulation is the capability to refine the finite element partition of the shaft, without increasing the size of the motion equations system.

6.3. SYMMETRIC SHAFT

If the shaft cross-section is symmetric, matrices $[M]$ and $[K]$ from equations (43) are constant, and the motion is governed by a linear system with constant coefficients, defined as

$$[M]\{\ddot{\delta}\} + [C]\{\dot{\delta}\} + [K]\{\delta\} = \{Q_u(t)\} + \{Q_w\}. \tag{50}$$

To evaluate the motion stability, we have only to analyze the eigenvalues of a constant matrix. The evaluation of the steady state response, with its components, the weight response and the unbalance response, is given below. The weight response is the solution of the algebraic system defined as

$$[K]\{\delta_w\} = \{Q_w\}. \tag{51}$$

To obtain the unbalance response, we express the unbalance vector from equation (50) as $\{Q_u(t)\} = \mathrm{Re}(\{V\}e^{i(\Omega t + \beta)})$, where $\{V\}$ is a constant vector with complex elements. It is easy to show that the unbalance response will be obtained as

$$\{\delta_u(t)\} = \mathrm{Re}(\{z_0\}\,\mathrm{e}^{\mathrm{i}\Omega t}), \tag{52}$$

where $\{z_0\}$ is a constant vector with complex elements, defined as the solution of the algebraic system

$$([K] - \Omega^2[M] + \mathrm{i}\Omega[C])\{z_0\} = \{V\}\mathrm{e}^{\mathrm{i}\beta}. \tag{53}$$

7. EQUATIONS OF MOTION, BY MODAL EXPANSION

In order to test the finite element formulation, the equations of motion for a simpler model will be obtained using the Rayleigh–Ritz method.

The new model consists of a flexible horizontal shaft, a rigid disk and two flexible bearings (Figure 6). The shaft is of uniform cross-section along its longitudinal axis and has two different stiffnesses along its two principal directions (Figure 2(a)). Its mass is supposed to be small compared with that of the disk, and consequently negligible. For bearings, only the direct stiffnesses, denoted as k_u and k_w, are considered. The unbalance is defined as shown in Figure 2(b). The only damping considered is the external one, acting on the disk (Figure 2(b)).

The shaft displacements in the x and z directions can be expressed as

$$u(y,t) = \left(1 - \frac{y}{L}\right)u_1(t) + \frac{y}{L}u_2(t) + f(y)q_x(t), \quad w(y,t) = \left(1 - \frac{y}{L}\right)w_1(t)$$

$$+ \frac{y}{L}w_2(t) + f(y)q_z(t), \tag{54}$$

where L is the shaft length, u_1, w_1, u_2 and w_2, are the displacements of the shaft ends, q_x and q_w are generalized independent co-ordinates, and $f(y)$ is the displacement function.

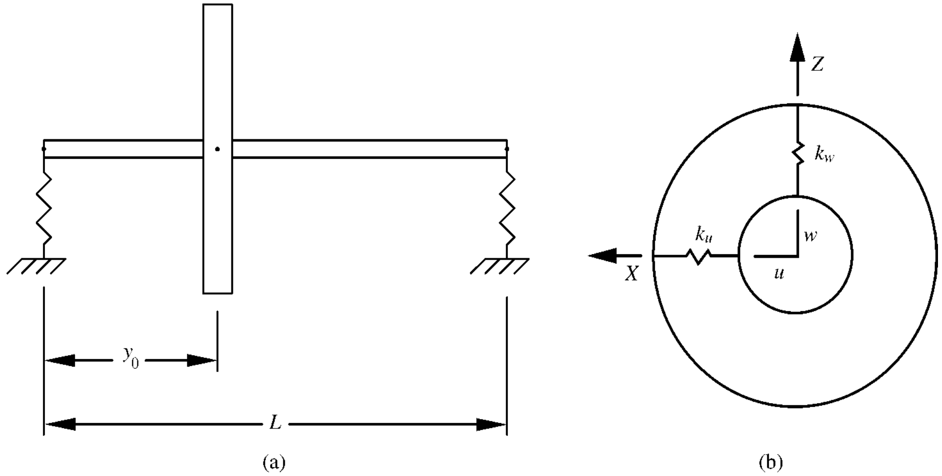

Figure 6. Simplified model of the rotor: (a) location of the disk, (b) bearing stiffness.

By substituting equations (54) into equation (11), we obtain the slopes of the shaft axis:

$$\theta(y,t)=\frac{w_2-w_1}{L}+\frac{\partial f}{\partial y}q_z \quad \text{and} \quad \psi(y,t)=-\frac{u_2-u_1}{L}-\frac{\partial f}{\partial y}q_x. \tag{55}$$

For $y=y_0$, the four equations (54) and (55) provide the displacements of the shaft ends (u_1, w_1, u_2, w_2) as functions of the displacements of the center of the disk $(u_0, w_0, \psi_0, \theta_0)$ and of the generalized co-ordinates (q_x, q_z):

$$\begin{aligned}
u_1 &= u_0+y_0\psi_0-(f_0-y_0g_0)q_x,\\
u_2 &= u_0-(L-y_0)\psi_0-[f_0+(L-y_0)g_0]q_x,\\
w_1 &= w_0-y_0\theta_0-(f_0-y_0g_0)q_z,\\
w_2 &= w_0+(L-y_0)\theta_0-[f_0+(L-y_0)g_0]q_z,
\end{aligned} \tag{56}$$

where $f_0=f(y_0)$ and $g_0=(\mathrm{d}f/\mathrm{d}y)(y_0)$.

The kinetic energy of the disk and the strain energy of the shaft are given by the expressions (14) and (26) respectively.

By introducing equations (54) in equation (26), the strain energy may be expressed as

$$U=\tfrac{1}{2}k_m(q_x^2+q_z^2)+\tfrac{1}{2}k_d[(q_x^2-q_z^2)\cos(2\Omega t)-2q_xq_z\sin(2\Omega t)], \tag{57}$$

where k_m and k_d are the mean and the deviatoric stiffnesses of the shaft, given by

$$k_m=EI_m\int_0^L f''(y)^2\,\mathrm{d}y \quad \text{and} \quad k_d=EI_d\int_0^L f''(y)^2\,\mathrm{d}y.$$

Assuming the displacement function to be the first mode shape in the bending of a beam with constant cross-section, simply supported at both ends, i.e., $f(y)=\sin(\pi y/L)$, the stiffnesses become

$$k_m=\frac{\pi^4}{2}\frac{EI_m}{L^3} \quad \text{and} \quad k_d=\frac{\pi^4}{2}\frac{EI_d}{L^3}.$$

The expression for the virtual work is

$$\delta L=-\,k_{u1}u_1\delta u_1-k_{w1}w_1\delta w_1-k_{u2}u_2\delta u_2-k_{w2}w_2\delta w_2-c\dot{u}_0\delta u_0-c\dot{w}_0\delta w_0-mg\delta w_0, \tag{58}$$

where u_1, w_1, u_2 and w_2, are the displacements of the shaft ends, u_0 and w_0 are the displacements of the center of the disk, k_{u1}, k_{w1}, k_{u2} and k_{w2} are the stiffness coefficients of the bearings, c is the damping coefficient, m is the mass of the disk, and g is the gravitational acceleration. Substituting equations (56) into equation (58), the virtual work becomes a function of the disk displacements u_0, w_0, ψ_0, θ_0 and generalized co-ordinates q_x, q_z.

Considering the expressions for the kinetic energy, strain energy and virtual work, the application of Lagrange's equations gives

$$\begin{bmatrix}[M] & [0_{42}]\\ [0_{24}] & [0_{22}]\end{bmatrix}\begin{Bmatrix}\{\ddot{\delta}_0\}\\ \{\ddot{\delta}_q\}\end{Bmatrix}+\begin{bmatrix}[C] & [0_{42}]\\ [0_{24}] & [0_{22}]\end{bmatrix}\begin{Bmatrix}\{\dot{\delta}_0\}\\ \{\dot{\delta}_q\}\end{Bmatrix}+\begin{bmatrix}[K_0] & [K_{0q}]\\ [K_{q0}] & [K_q]\end{bmatrix}\begin{Bmatrix}\{\delta_0\}\\ \{\delta_q\}\end{Bmatrix}=\begin{Bmatrix}\{Q\}\\ \{0_2\}\end{Bmatrix}, \tag{59}$$

where matrices $[M]$, $[C]$ and vector $\{Q\}$ are identical to $[M_D]$, $[C_D]$ and $\{Q_D\}$, respectively, from equation (16):

$$\{\delta_0\} = \{u_0 \;\; w_0 \;\; \psi_0 \;\; \theta_0\}^{\mathrm{T}}, \qquad \{\delta_q\} = \{q_x \;\; q_z\}^{\mathrm{T}},$$

$$[K_0] = \begin{bmatrix} k_{u1} + k_{u2} & 0 & k_{u1}y_0 - k_{u2}(L - y_0) & 0 \\ & k_{w1} + k_{w2} & 0 & -k_{w1}y_0 + k_{w2}(L - y_0) \\ & & k_{u1}y_0^2 + k_{u2}(L - y_0)^2 & 0 \\ (sym) & & & k_{w1}y_0^2 + k_{w2}(L - y_0)^2 \end{bmatrix},$$

$$[K_q(t)] = \begin{bmatrix} \begin{array}{c} k_{u1}(f_0 - y_0 g_0)^2 \\ + k_{u2}[f_0 + (L - y_0)g_0]^2 \end{array} & 0 \\ 0 & \begin{array}{c} k_{w1}(f_0 - y_0 g_0)^2 \\ + k_{w2}[f_0 + (L - y_0)g_0]^2 \end{array} \end{bmatrix}$$

$$+ k_m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k_d \begin{bmatrix} \cos 2\Omega t & -\sin 2\Omega t \\ -\sin 2\Omega t & -\cos 2\Omega t \end{bmatrix},$$

$$[K_{0q}] = [K_{q0}]^{\mathrm{T}} = \begin{bmatrix} \begin{array}{c} -k_{u1}(f_0 - y_0 g_0) \\ -k_{u2}[f_0 + (L - y_0)g_0] \end{array} & 0 \\ 0 & \begin{array}{c} -k_{w1}(f_0 - y_0 g_0) \\ -k_{w2}[f_0 + (L - y_0)g_0] \end{array} \\ \begin{array}{c} -k_{u1}y_0(f_0 - y_0 g_0) \\ + k_{u2}(L - y_0)[f_0 + (L - y_0)g_0] \end{array} & 0 \\ 0 & \begin{array}{c} k_{w1}y_0(f_0 - y_0 g_0) \\ - k_{w2}(L - y_0)[f_0 + (L - y_0)g_0] \end{array} \end{bmatrix}$$

The last two equations of system (59) give

$$\{\delta_q\} = -[K_q(t)]^{-1}[K_{q0}]\{\delta_0\},$$

and therefore, the first four equations provide the system, and

$$[M]\{\ddot{\delta}_0\} + [C]\{\dot{\delta}_0\} + [K(t)]\{\delta_0\} = \{Q(t)\}, \tag{60}$$

where

$$[K(t)] = [K_0] - [K_{0q}][K_q(t)]^{-1}[K_{q0}].$$

System (60) gives a first order differential system of type (45), with eight equations and eight unknown functions.

8. NUMERICAL RESULTS

Two models are used for testing the finite element method and the transfer matrix method as applied to asymmetrical rotors. Model 1 is as shown in Figure 6, but the mass of the shaft is not neglected, and the bearings are as shown in Figure 3. The fixed characteristics of model 1 are given in Table 1. The variable parameter is the factor of shaft asymmetry, defined as the rate between the deviatory and the mean area moments of the shaft cross-section, I_d/I_m. Model 2 is similar to model 1, only the bearing damping is replaced by an external damping of coefficient $c = 50$ N s/m, and the stiffness cross-coefficients of the bearings are neglected.

Using the finite element formulation, the shaft is simulated by two elements, delimited by disk station and shaft ends. It was proven that when using four elements (dividing each of the previous elements by two), the effect on calculated critical speeds and steady state response is insignificant.

The number of intervals per period was set at $N = 90$ for stability analysis and $N = 360$ for steady state response evaluation. These values were established as the minimum acceptable by testing the convergence of numerical calculations. The response amplitude is observed at the disk-to-shaft attachment.

If the shaft is symmetric, the motion of the rotor can be observed in both the fixed and the rotating frames of reference. In the first case, the steady state response is obtained as the solution of a system of differential equations with constant coefficients (see section 6.3), while in the second case we are dealing with the numerical integration of a periodic system (see section 6.1). Figure 7 gives the results obtained by the two methods, for model 1. The rotational speed was varied from 75 to 300 rad/s, in increments of 5 rad/s. As can be observed, there are two peaks, corresponding to principal critical speeds excited by the mass unbalance. The relative error has an average value of 1·3% and a maximum value of 3·4% (at 75 rad/s). The accuracy of the algorithm used to find the steady state solution of a periodic system is verified by this example.

The effect of shaft asymmetry on the steady state response is studied next, using model 1, and the results are plotted in Figure 8(a), for a rotational speed increment of 5 rad/s, and in Figure 8(b), for a rotational speed increment of 1 rad/s. As can be observed from Figure 8(a), when the factor of shaft asymmetry passes from 0 to 0·2, each one of the two peaks from Figure 7 splits into two other peaks. Additionally, there are two new peaks, corresponding to secondary critical speeds excited by the rotor weight (at 105 and 115 rad/s). To understand this behavior, we need to analyze the stability of the motion.

TABLE 1

Details of model 1

Element	Details
Disk	$m = 2$ kg, $J_x = J_z = 0{\cdot}005$ kg m^2, $J_y = 0{\cdot}01$ kg m^2 $m_u d = 0{\cdot}004$ kg m, $\beta = 45^\circ$
Shaft	$y_0 = 0{\cdot}4$ m, $L = 1$ m, $E = 2\times 10^{11}$ N m^2, $\rho = 7750$ kg/m^3 $I_m = 4\times 10^{-8}$ m^4, $A = 0{\cdot}693\times 10^{-3}$ m^2
Bearing—$y = 0$	$[K_p] = \begin{bmatrix} 3{\cdot}5 & -1 \\ -1 & 5{\cdot}5 \end{bmatrix} \times 10^5$ N/m, $[C_P] = \begin{bmatrix} 26 & -8 \\ -8 & 20 \end{bmatrix}$ N s/m
Bearing—$y = L$	$[K_p] = \begin{bmatrix} 4{\cdot}5 & -1 \\ -1 & 3 \end{bmatrix} \times 10^5$ N/m, $[C_P] = \begin{bmatrix} 24 & -4 \\ -4 & 30 \end{bmatrix}$ N s/m

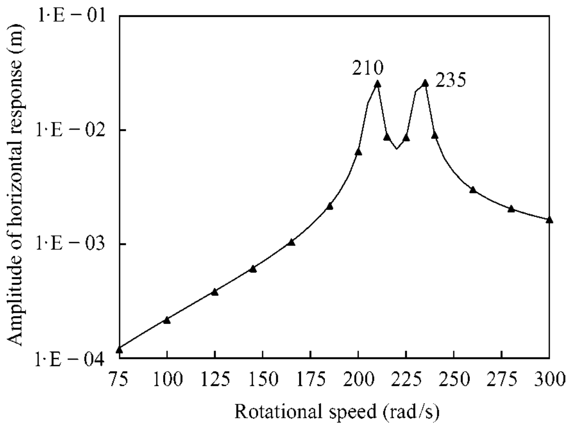

Figure 7. Steady state response for model 1, with symmetric cross-section of shaft. Comparison between the responses obtained using the autonomous system of motion in fixed frame of reference and the periodic system of motion in rotating frame of reference: ——, fixed frame; ▲, rotating frame.

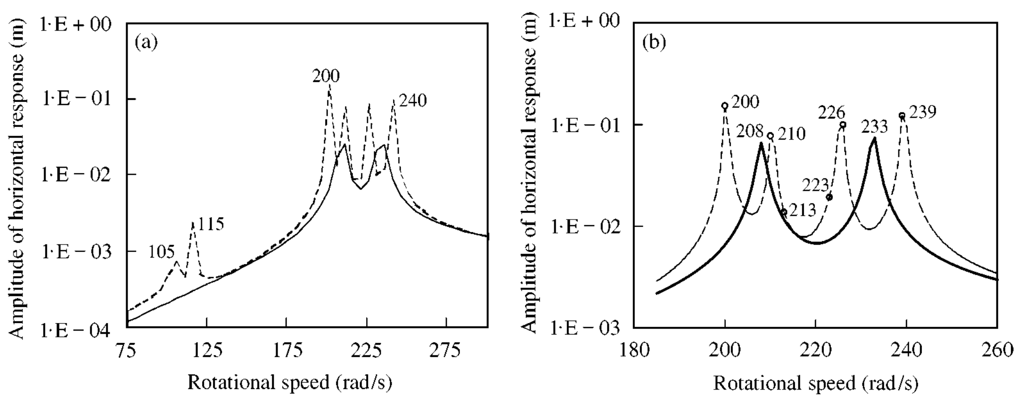

Figure 8. Steady state response for model 1. (a) Effect of shaft asymmetry: ——, symmetrical; - - - - -, factor of shaft asymmetry 0·2. (b) Detail of the region of main critical speeds.

Figure 9 gives the regions of instability obtained for model 1, for factors of shaft asymmetry varying from 0 to 0·3, with increments of 0·05. The results were obtained by varying the rotational speed between 75 and 300 rad/s, in increments of 1 rad/s. For the symmetric shaft, no instability interval has been identified. For the asymmetric shaft, there are three regions of instability, of widths increasing with the shaft asymmetry until joining into a single band. For a factor of shaft asymmetry of 0·2, the 6 critical speeds delimiting the three regions of instability are 200, 210, 213, 223, 226 and 239 rad/s. Consequently, the split of the two peaks observed in Figures 8(a) and (b) indicates two regions of instability. The evaluation of the steady state response could not identify the middle region of instability. One may note that the periodic solution is meaningless inside the instability regions.

The response amplitudes of models 1 and 2 are compared next, for a factor of shaft asymmetry of 0.2. For model 2, Figure 10 shows two peaks corresponding to principal critical speeds excited by the mass unbalance and one peak for a secondary critical speed excited by the rotor weight. A stability analysis shows a single instability region, bounded by the speeds of the two principal peaks. The different behavior for model 1 is due basically to the bearing cross-stiffness. Replacing the bearing damping with the external damping

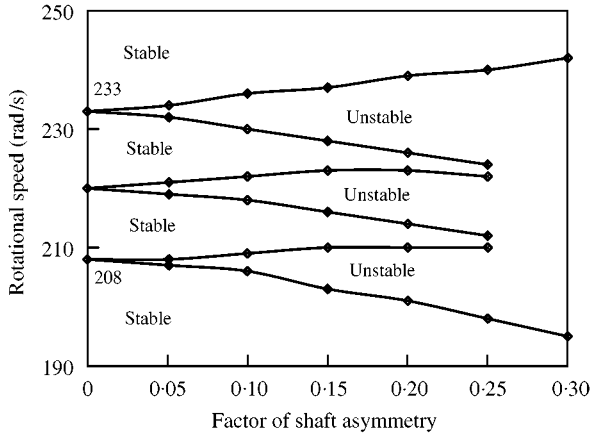

Figure 9. Instability regions for model 1; effect of shaft asymmetry.

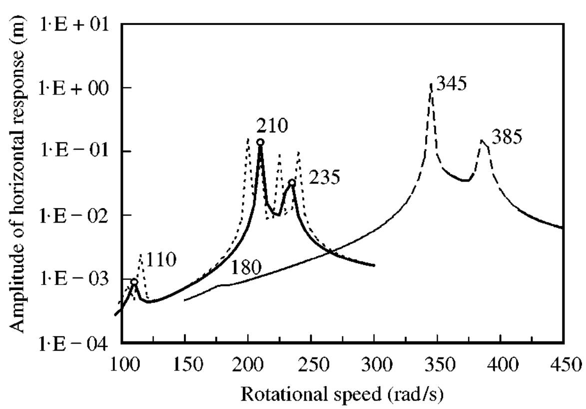

Figure 10. Steady state response, for a factor of shaft asymmetry of 0·2. Comparison between models 1 and 2: ·······, model 1; ——, model 2; - - - - -, model 2, massless shaft.

$c = 50$ N s/m has a smaller effect, while the direct damping of the two bearings equals the external damping ($c = c_{u1} + c_{u2} = c_{w1} + c_{w2}$).

Figure 10 shows also that if we ignore the mass of the shaft, the instability region of model 2 passes from 210–235 rad/s to 345–385 rad/s. This effect is very big because the mass of the shaft is 2·7 times larger than the mass of the disk (see Table 1).

The response amplitudes obtained by finite element formulation are compared next with those obtained by the Rayleigh–Ritz method. Two factors of shaft asymmetry are used for model 2. As can be observed from Figure 11, the results obtained by the two methods are very close, which validates the finite element formulation developed for asymmetric rotor-bearing systems.

9. CONCLUSIONS

In this paper, a finite element procedure for rotor-bearing systems is generalized to include the effects of the shaft asymmetry. In order to deal with the particular form of the

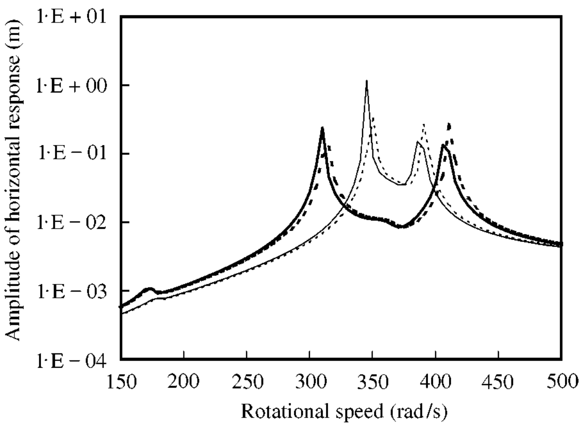

Figure 11. Steady state response for model 2. Comparison between the responses obtained using the finite element method (FEM) and the Rayleigh–Ritz method (RRM), for two factors of shaft asymmetry (fsa): ——, FEM, fsa 0·2; ······, RRM, fsa 0·2; – – – – –, RRM, fsa 0·4.

equations of motion (ordinary differential equations with periodic coefficients), the finite element method is applied in conjunction with a time-transfer matrix method, based on Floquet's theory. The time-transfer matrix method was tested by observing the motion of a rotor with symmetric shaft in both fixed and rotating frames of reference. The finite element procedure was compared, for a rotor-bearing system with a massless shaft and undamped bearings, with a modal expansion method.

Numerical examples have shown that the finite element method in conjunction with the time-transfer matrix method is a convenient way to predict the behavior of asymmetric rotors.

REFERENCES

1. A. H. Nayfeh 1981 *Introduction to Perturbation Techniques*. New York: John Wiley.
2. L. Meirovitch 1970 *Methods of Analytical Dynamics*. New York: McGraw-Hill Book Co.
3. S. T. Noah and G. R. Hopkins 1982 *Journal of Applied Mechanics* **49**, 217–223. A generalized Hill's method for the stability analysis of parametrically excited dynamic systems.
4. P. Friedmann, C. E. Hammond and T. H. Woo 1977 *International Journal for Numerical Methods in Engineering* **11,** 1117–1136. Efficient numerical treatment of periodic systems with applications to stability problems.
5. P. Friedmann 1990 *Computers and Structures* **35**, 329–347. Numerical methods for the treatment of periodic systems with applications to structural dynamics and helicopter rotor dynamics.
6. S. C. Sinha, R. Pandiyan and J. S. Bibb 1996 *Journal of Vibration and Acoustics* **118**, 209–219. Liapunov–Floquet transformation: computation and applications to periodic systems.
7. A. A. El-Marhomy 1991 *Journal of Applied Mechanics* **58**, 1056–1063. Dynamic stability of elastic rotor-bearing systems via Liapunov's direct method.
8. D. Ardayfio and D. A. Frohrib 1976 *Journal of Engineering for Industry* **98**, 1161–1165. Instabilities of an asymmetric rotor with asymmetric shaft mounted on symmetric elastic supports.
9. P. J. Brosens and S. H. Crandall 1961 *Journal of Applied Mechanics* **28**, 355–362. Whirling of unsymmetrical rotors.
10. T. Kotera and S. Yano 1980 *Bulletin of the Japan Society of Mechanical Engineers* **23**, 1194–1199. Instability of motion of a disc supported by an asymmetric shaft in asymmetric bearings (influence of external damping).

11. H. F. BLACK and A. J. MCTERNAN 1968 *Journal of Mechanical Engineering Science* **10**, 252–261. Vibration of a rotating asymmetric shaft supported in asymmetric bearings.
12. T. IWATSUBO, R. KAWAI and T. MIYAJI 1980 *Bulletin of the Japan Society of Mechanical Engineers* **23**, 934–937. On the stability of a rotating asymmetric shaft supported by asymmetric bearings.
13. P. M. GUILHEN, P. BERTHIER, G. FERRARIS and M. LALANNE 1988 *Journal of Vibration, Acoustics, Stress and Reliability in Design* **110**, 288–294. Instability and unbalance response of dissymmetric rotor-bearing systems.
14. H. D. NELSON and J. M. MCVAUGH 1976 *Journal of Engineering for Industry* **98**, 593–600. The dynamics of rotor-bearing systems using finite elements.
15. M. LALANNE and G. FERRARIS 1990 *Rotordynamics Prediction in Engineering*. New York: John Wiley.
16. T. INAGAKI, H. KANKI and K. SHIRAKI 1980 *Journal of Mechanical Design* **102**, 147–157. Response analysis of a general asymmetric rotor-bearing system.
17. G. GENTA 1988 *Journal of Sound and Vibration* **124**, 27–53. Whirling of unsymmetrical rotors: a finite element approach based on complex coordinates.
18. Y. KANG, Y.-P. SHIH and A.-C. LEE 1992 *Journal of Vibration and Acoustics* **114**, 194–208. Investigation on the steady-state responses of asymmetric rotors.

APPENDIX A: MATRICES

A.1. TRANSFORMATION MATRICES

$$[T_2] = \begin{bmatrix} \cos\Omega t & \sin\Omega t \\ -\sin\Omega t & \cos\Omega t \end{bmatrix}, \qquad [T_4] = \begin{bmatrix} [T_2] & \\ & [T_2]^{\mathrm{T}} \end{bmatrix}, \qquad [T_8] = \begin{bmatrix} [T_4] & \\ & [T_4] \end{bmatrix}. \tag{A.1–3}$$

$$[H_2] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad [H_4] = \begin{bmatrix} [H_2] & \\ & [H_2]^{\mathrm{T}} \end{bmatrix}, \qquad [H_8] = \begin{bmatrix} [H_4] & \\ & [H_4] \end{bmatrix}. \tag{A.4–6}$$

A.2. DISK MATRICES

$$[M_D] = \begin{bmatrix} m & & & \\ & m & & \\ & & J_x & \\ & & & J_x \end{bmatrix}, \qquad [C_D] = \begin{bmatrix} c & & & \\ & c & & \\ & & 0 & J_y\Omega \\ & & -J_y\Omega & 0 \end{bmatrix}, \tag{A.7, 8}$$

$$\{Q_D\} = m_u \mathrm{d}\Omega^2 \begin{Bmatrix} \sin(\Omega t + \beta) \\ \cos(\Omega \mathrm{t} + \beta) \\ 0 \\ 0 \end{Bmatrix} - mg \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix}, \tag{A.9}$$

$$[\bar{C}_D] = \begin{bmatrix} c & 2\Omega m & & \\ -2\Omega m & c & & \\ & & 0 & \Omega(J_y - 2J_x) \\ & & -\Omega(J_y - 2J_x) & 0 \end{bmatrix}, \tag{A.10}$$

$$[\bar{K}_D] = \begin{bmatrix} -\Omega^2 m & \Omega c & & \\ -\Omega c & -\Omega^2 m & & \\ & & \Omega^2(J_y - J_x) & \\ & & & \Omega^2(J_y - J_x) \end{bmatrix}, \tag{A.11}$$

$$\{\bar{Q}_D\} = m_u \mathrm{d}\Omega^2 \begin{Bmatrix} \sin(\beta) \\ \cos(\beta) \\ 0 \\ 0 \end{Bmatrix} + mg \begin{Bmatrix} \sin(\Omega t) \\ \cos(\Omega t) \\ 0 \\ 0 \end{Bmatrix}. \tag{A.12}$$

A.3. SHAFT ELEMENT MATRICES

$$[M_t] = \frac{\rho A L}{420} \begin{bmatrix} 156 & 0 & -22L & 0 & 54 & 0 & 13L & 0 \\ & 156 & 0 & 22L & 0 & 54 & 0 & -13L \\ & & 4L^2 & 0 & -13L & 0 & -3L^2 & 0 \\ & & & 4L^2 & 0 & 13L & 0 & -3L^2 \\ & & & & 156 & 0 & 22L & 0 \\ & & (sym) & & & 156 & 0 & -22L \\ & & & & & & 4L^2 & 0 \\ & & & & & & & 4L^2 \end{bmatrix}, \tag{A.13}$$

$$[M_r] = \frac{\rho I_m}{30L} \begin{bmatrix} 36 & 0 & -3L & 0 & -36 & 0 & -3L & 0 \\ & 36 & 0 & 3L & 0 & -36 & 0 & 3L \\ & & 4L^2 & 0 & 3L & 0 & -L^2 & 0 \\ & & & 4L^2 & 0 & -3L & 0 & -L^2 \\ & & & & 36 & 0 & 3L & 0 \\ & & (sym) & & & 36 & 0 & -3L \\ & & & & & & 4L^2 & 0 \\ & & & & & & & 4L^2 \end{bmatrix}, \tag{A.14}$$

$$[M_{d,c}] = [M_d] = \frac{\rho I_d}{30L} \begin{bmatrix} 36 & 0 & -3L & 0 & -36 & 0 & -3L & 0 \\ & -36 & 0 & -3L & 0 & 36 & 0 & -3L \\ & & 4L^2 & 0 & 3L & 0 & -L^2 & 0 \\ & & & -4L^2 & 0 & 3L & 0 & L^2 \\ & & & & 36 & 0 & 3L & 0 \\ & & (sym) & & & -36 & 0 & 3L \\ & & & & & & 4L^2 & 0 \\ & & & & & & & -4L^2 \end{bmatrix}, \tag{A.15}$$

$$[M_{d,s}] = \frac{\rho I_d}{30L}\begin{bmatrix} 0 & -36 & 0 & -3L & 0 & 36 & 0 & -3L \\ & 0 & 3L & 0 & 36 & 0 & 3L & 0 \\ & & 0 & 4L^2 & 0 & -3L & 0 & -L^2 \\ & & & 0 & 3L & 0 & -L^2 & 0 \\ & & & & 0 & -36 & 0 & 3L \\ & & (sym) & & & 0 & -3L & 0 \\ & & & & & & 0 & 4L^2 \\ & & & & & & & 0 \end{bmatrix}, \tag{A.16}$$

$$[G] = \frac{\rho I_p}{30L}\begin{bmatrix} 0 & -36 & 0 & -3L & 0 & 36 & 0 & -3L \\ & 0 & -3L & 0 & -36 & 0 & -3L & 0 \\ & & 0 & 4L^2 & 0 & -3L & 0 & -L^2 \\ & & & 0 & -3L & 0 & L^2 & 0 \\ & & & & 0 & -36 & 0 & 3L \\ & & (skew\text{–}sym) & & & 0 & 3L & 0 \\ & & & & & & 0 & 4L^2 \\ & & & & & & & 0 \end{bmatrix}, \tag{A.17}$$

$$[K_m] = \frac{EI_m}{L^3}\begin{bmatrix} 12 & 0 & -6L & 0 & -12 & 0 & -6L & 0 \\ & 12 & 0 & 6L & 0 & -12 & 0 & 6L \\ & & 4L^2 & 0 & 6L & 0 & 2L^2 & 0 \\ & & & 4L^2 & 0 & -6L & 0 & 2L^2 \\ & & & & 12 & 0 & 6L & 0 \\ & & (sym) & & & 12 & 0 & -6L \\ & & & & & & 4L^2 & 0 \\ & & & & & & & 4L^2 \end{bmatrix}, \tag{A.18}$$

$$[K_{d,c}] = [K_d] = \frac{EI_d}{L^3}\begin{bmatrix} 12 & 0 & -6L & 0 & -12 & 0 & -6L & 0 \\ & -12 & 0 & -6L & 0 & 12 & 0 & -6L \\ & & 4L^2 & 0 & 6L & 0 & 2L^2 & 0 \\ & & & -4L^2 & 0 & 6L & 0 & -2L^2 \\ & & & & 12 & 0 & 6L & 0 \\ & & (sym) & & & -12 & 0 & 6L \\ & & & & & & 4L^2 & 0 \\ & & & & & & & -4L^2 \end{bmatrix}, \tag{A.19}$$

$$[K_{d,s}] = \frac{EI_d}{L^3}\begin{bmatrix} 0 & -12 & 0 & -6L & 0 & 12 & 0 & -6L \\ & 0 & 6L & 0 & 12 & 0 & 6L & 0 \\ & & 0 & 4L^2 & 0 & -6L & 0 & 2L^2 \\ & & & 0 & 6L & 0 & 2L^2 & 0 \\ & & & & 0 & -12 & 0 & 6L \\ & & (sym) & & & 0 & -6L & 0 \\ & & & & & & 0 & 4L^2 \\ & & & & & & & 0 \end{bmatrix}, \tag{A.20}$$

$$\{Q_A\} = \tfrac{1}{2}\rho g A L \begin{Bmatrix} 0 \\ 1 \\ 0 \\ \frac{1}{6}L \\ 0 \\ 1 \\ 0 \\ -\frac{1}{6}L \end{Bmatrix}, \qquad \{\bar{Q}_A\} = -\tfrac{1}{2}\rho g A L \begin{Bmatrix} -\sin\Omega t \\ \cos\Omega t \\ \frac{1}{6}L\sin\Omega t \\ \frac{1}{6}L\cos\Omega t \\ -\sin\Omega t \\ \cos\Omega t \\ -\frac{1}{6}L\sin\Omega t \\ -\frac{1}{6}L\cos\Omega t \end{Bmatrix}. \tag{A.21, 22}$$

A.4. BEARING MATRICES

$$[K_P] = \begin{bmatrix} k_{uu} & k_{uw} \\ k_{wu} & k_{ww} \end{bmatrix}, \qquad [C_P] = \begin{bmatrix} C_{uu} & C_{uw} \\ C_{wu} & C_{ww} \end{bmatrix}, \tag{A.23, 24}$$

$$[\bar{K}_P] = \left(\frac{k_{uu}+k_{ww}}{2} - \Omega\frac{c_{uw}-c_{wu}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left(\frac{k_{uw}+k_{wu}}{2} - \Omega\frac{c_{uu}-c_{ww}}{2}\right)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
$$+\left(\frac{k_{uu}-k_{ww}}{2} + \Omega\frac{c_{uw}-c_{wu}}{2}\right)\begin{bmatrix} \cos(2\Omega t) & \sin(2\Omega t) \\ \sin(2\Omega t) & -\cos(2\Omega t) \end{bmatrix}$$
$$+\left(\frac{k_{uw}+k_{wu}}{2} + \Omega\frac{c_{uu}-c_{ww}}{2}\right)\begin{bmatrix} -\sin(2\Omega t) & \cos(2\Omega t) \\ \cos(2\Omega t) & \sin(2\Omega t) \end{bmatrix}, \tag{A.25}$$

$$[\bar{C}_P] = \frac{c_{uu}+c_{ww}}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{c_{uw}-c_{wu}}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \frac{c_{uu}-c_{ww}}{2}\begin{bmatrix} \cos(2\Omega t) & \sin(2\Omega t) \\ \sin(2\Omega t) & -\cos(2\Omega t) \end{bmatrix}$$
$$+\frac{c_{uw}+c_{wu}}{2}\begin{bmatrix} -\sin(2\Omega t) & \cos(2\Omega t) \\ \cos(2\Omega t) & \sin(2\Omega t) \end{bmatrix}. \tag{A.26}$$

APPENDIX B: NOMENCLATURE

A	cross-section area of the shaft
$[A(t)]$	periodic matrix of dimension $m \times m$
$[B(y)]$	matrix defined by $[B] = (\partial^2/\partial y^2)[N]$

c	external damping coefficient
c_{uu}, c_{uw}, c_{wu}, c_{ww}	damping coefficients of the bearings, Figure 3
$[C]$	damping matrix (subscripts D, A, P: disk, shaft, bearing)
d	radius defining the unbalance position, Figure 2
$[D(y)]$	matrix given by $[D] = [(\partial/\partial y)N]$
E	Young's modulus
$f(y)$	displacement function
$\{f\}$	force vector of dimension m
g	gravitational acceleration
$[G]$	gyroscopic matrix
$\lfloor H_j \rfloor_{,j=2,4,8}$	constant transformation matrices, Appendix A
$[I_m]$	$m \times m$ unity matrix
I	second moments of area for shaft (subscripts x, z, m, d, p: along x'- and z'-axis, mean, deviatoric, polar)
J	moment of inertia of the disk (subscripts x, y: transverse, polar)
k_{uu}, k_{uw}, k_{wu}, k_{ww}	stiffness coefficients of the bearings, Figure 3
k_m, k_d	mean and deviatory stiffness of the shaft
$[K]$	stiffness matrix (subscripts D, A, P: disk, shaft, bearing)
L	length of a shaft element, also shaft length
δL	virtual work of external forces (subscripts D, A, P: disk, shaft, bearing)
m	mass of the disk
m_u	mass of the unbalance
$[M]$	damping matrix (subscripts D, A, P: disk, shaft, bearing)
N	number of intervals in a period
N_d, N_n	number of disks, number of nodes of the shaft partition
$[N(y)]$	matrix of shape functions for shaft elements
q	generalized displacements (subscripts x, z: along X- and Z-axis)
$\{Q\}$	forces vector (subscripts D, A, P, L, u, w: disk, shaft, bearing, liaison, unbalance, weight)
t	time
T	period
T	kinetic energy (subscripts D, A: disk, shaft)
$\lfloor T_j \rfloor_{,j=2,4,8}$	periodic transformation matrices, Appendix A
u, w	lateral deflections of the shaft in the in fixed frame
U	strain energy
$\{x\}$	vector of dimension m, containing displacements and velocities
(xyz)	rotating frame
$(x'y'z')$	principal axes of the shaft cross-section
(XYZ)	fixed reference frame
y	axial distance along shaft element or shaft
β	angle defining the unbalance position, Figure 2
$\{\delta\}$	nodal displacements vector (subscripts 0, e, p: disk, shaft element, bearing)
θ, ψ	angular deflections of the shaft
ρ	mass per unit volume for shaft
φ	angle of rotation of the rotor
φ, θ, ψ	Euler's angles
$[\Phi(t)]$, $[\Phi(t, s)]$	transfer matrix of system (1)
ω_x, ω_y and ω_z	angular velocities of the shaft cross-section
Ω	rotational speed of the rotor

Special symbols

$\dot{x} = \mathrm{d}x/\mathrm{d}t$	differentiation with respect to time
$x' = \mathrm{d}x/\mathrm{d}\varphi$	differentiation with respect to angle of rotation
$\bar{x}$, $\{\bar{x}\}$, $[\bar{X}]$	with reference to the rotating frame (for scalars, vectors or matrices)