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LETTERS TO THE EDITOR

VISCOUSLY DAMPED LINEAR SYSTEMS SUBJECTED TO DAMPING MODIFICATIONS

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1. INTRODUCTION

One of the most important problems in the analysis and design of a structural or mechanical system is to understand how the variations in system parameters affect the natural frequencies and mode shapes of the system. Very often, a structural designer faces the challenging problem of shifting a troublesome natural frequency away from the disturbing frequency or modifying a particular mode shape. Such a class of problems is labelled as sensitivity analysis.

Some of the recent studies in this field can be summarized as follows. Lin and Kim [1] proposed a method to derive structural design sensitivities, including frequency response function sensitivities and eigenvalue and eigenvector sensitivities from limited vibration test data. It is claimed that since the design sensitivities are calculated directly from the measured data, they are expected to be more realistic. Sivan and Ram [2] considered the backward problem: i.e., the problem of determining the structural modification needed to prescribe some natural frequencies and mode shapes in the presence of model uncertainty. Mottershead and Lallement [3] investigated the relationship between the sensitivity of antiresonances (zeros) and sensitivities of the natural frequencies and mode shapes of structural systems. They used unit-rank modifications for the mutual cancellation of a pole and a zero to form a vibration mode in structures. Lee *et al.* [4] developed a procedure for determining the sensitivities of eigenvalues and eigenvectors of damped vibratory systems with distinct eigenvalues. An algorithm was proposed where the sensitivity with respect to any parameter can be used.

In this paper, variations of the damping mechanism are considered and the eigencharacteristics of the system are obtained. A viscously damped n-degree-of-freedom linear discrete mechanical system is taken into account and a damper is added to the system. An explicit analytical expression for the eigenvectors is given after the formulation of the characteristic equation of the modified system. The receptance matrix of the modified system is also obtained in a straightforward manner. The sensitivities of the eigenvalues, eigenvectors and the receptance matrix with respect to the damping coefficient of the attached damper are calculated. A numerical example demonstrating the reliability of the expressions obtained is given for a two-degree-of-freedom system.

Some novel features of the present approach are as follows. While a numerical algorithm involving the solution of a system of equations is necessary in a conventional approach, a direct analytical formulation of the sensitivities with respect to damping constant is given. The sensitivity of the receptance matrix with respect to damping constant can be obtained

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in a straightforward manner. Modified values of the eigenvalues, eigenvectors and receptance matrix can also be calculated immediately.

2. THEORY

As is known, the motion of a linear discrete mechanical system with n d.o.f. is governed in the physical space by the matrix differential equation of order two

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{D}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0}, \tag{1}$$

where $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{K}$ are the $n \times n$ mass, damping and stiffness matrices, respectively and $\mathbf{q}$ is the $n \times 1$ vector of the generalized co-ordinates.

Suppose that a new damper is added to the mechanical system, for some reason, such that the modified damping matrix of the system can be written as

$$\tilde{\mathbf{D}} = \mathbf{D} + \mathbf{d}\mathbf{d}^{\mathrm{T}}, \tag{2}$$

where the vector $\mathbf{d}$ includes both the damping constant and the orientation information in the physical space [5].

The aim is to obtain the eigencharacteristics and the receptance matrix of the modified system and then to calculate their sensitivities with respect to a damping related parameter.

2.1. DETERMINATION OF THE EIGENCHARACTERISTICS AND THE RECEPTANCE MATRIX OF THE MODIFIED SYSTEM

If a solution of equation (1) with the modified damping matrix $\tilde{\mathbf{D}}$ from equation (2) is assumed in the form of

$$\mathbf{q}(t) = \mathbf{z}\mathrm{e}^{\lambda t}, \tag{3}$$

where λ and $\mathbf{z}$ represent an eigenvalue and the corresponding eigenvector, respectively, the eigenvalue problem

$$[\mathbf{M}\lambda^2 + \tilde{\mathbf{D}}\lambda + \mathbf{K}]\mathbf{z} = \mathbf{0} \tag{4}$$

is obtained. This means that the eigenvalues λ are obtained from the characteristic equation

$$p(\lambda) := \det[\mathbf{M}\lambda^2 + \tilde{\mathbf{D}}\lambda + \mathbf{K}] = 0. \tag{5}$$

By using the well-known formula

$$\det(\mathbf{A} + \alpha\mathbf{b}\mathbf{b}^{\mathrm{T}}) = (\det\mathbf{A})(1 + \alpha\mathbf{b}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{b}) \tag{6}$$

for the determinant of the sum of a regular square matrix and a dyadic [6], $p(\lambda)$ can be reformulated as

$$p(\lambda) = p_0(\lambda)(1 + \lambda\mathbf{d}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{d}), \tag{7}$$

where

$$\mathbf{A} = \mathbf{M}\lambda^2 + \mathbf{D}\lambda + \mathbf{K}, \qquad p_0(\lambda) = \det \mathbf{A} \tag{8}$$

are introduced. $p_0(\lambda)$ is nothing else but the characteristic polynomial of the original system. Hence, the characteristic equation of the modified system is

$$1 + \lambda \mathbf{d}^{\mathrm{T}}(\mathbf{M}\lambda^2 + \mathbf{D}\lambda + \mathbf{K})^{-1}\mathbf{d} = 0. \tag{9}$$

Eigenvectors of the modified system $\mathbf{z}$ are to be obtained from equation (4) The kth eigenvector $\mathbf{z}_k$ can be shown to be

$$\mathbf{z}_k = (\mathbf{M}\lambda_k^2 + \mathbf{D}\lambda_k + \mathbf{K})^{-1}\mathbf{d}. \tag{10}$$

Proof. The correctness of this statement can be shown by substituting the expression (10) into the equation (4):

$$\begin{aligned}[(\mathbf{M}\lambda_k^2 + \mathbf{D}\lambda_k + \mathbf{K}) + \lambda_k \mathbf{d}\mathbf{d}^{\mathrm{T}}](\mathbf{M}\lambda_k^2 + \mathbf{D}\lambda_k + \mathbf{K})^{-1}\mathbf{d} &= \mathbf{d} + \lambda_k \mathbf{d}[\mathbf{d}^{\mathrm{T}}(\mathbf{M}\lambda_k^2 + \mathbf{D}\lambda_k + \mathbf{K})^{-1}\mathbf{d}] \\ &= \mathbf{d}[1 + \lambda_k \mathbf{d}^{\mathrm{T}}(\mathbf{M}\lambda_k^2 + \mathbf{D}\lambda_k + \mathbf{K})^{-1}\mathbf{d}] \\ &= 0.\end{aligned}$$

The last line is equal to zero, as the kth eigenvalue satisfies the characteristic equation given by equation (9).

If the mechanical system in equation (1) is harmonically excited, then the motion of the system is governed by the inhomogeneous differential equation

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{D}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \bar{\mathbf{F}}\mathrm{e}^{\mathrm{i}\omega t}, \tag{11}$$

where $\bar{\mathbf{F}}$ is the forcing vector and ω denotes the forcing frequency.

Substitution of

$$\mathbf{q}(t) = \bar{\mathbf{q}}\mathrm{e}^{\mathrm{i}\omega t} \tag{12}$$

into equation (11) yields the relation

$$\bar{\mathbf{q}} = \mathbf{H}(\omega)\bar{\mathbf{F}} \tag{13}$$

between the constant parts of the input and response vectors. The complex-valued matrix

$$\mathbf{H}(\omega) = (-\,\omega^2\mathbf{M} + \mathrm{i}\omega\mathbf{D} + \mathbf{K})^{-1} \tag{14}$$

is referred to as the complex frequency response matrix or the receptance matrix.

One can denote the receptance matrix of the modified system as $\tilde{\mathbf{H}}(\omega)$:

$$\tilde{\mathbf{H}}(\omega) = (-\,\omega^2\mathbf{M} + \mathrm{i}\omega\tilde{\mathbf{D}} + \mathbf{K})^{-1}. \tag{15}$$

The aim now is to express $\tilde{\mathbf{H}}(\omega)$ in terms of the receptance matrix of the original system $\mathbf{H}(\omega)$ and $\mathbf{d}$. To this end, a matrix inversion formula can be used from matrix theory which gives the inverse of the sum of a non-singular square matrix and a dyadic [7]:

$$(\bar{\mathbf{A}} + \mathbf{u}\mathbf{v}^{\mathrm{T}})^{-1} = \bar{\mathbf{A}}^{-1} - \bar{\mathbf{A}}^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}}\bar{\mathbf{A}}^{-1}/(1 + \mathbf{v}^{\mathrm{T}}\bar{\mathbf{A}}^{-1}\mathbf{u}). \tag{16}$$

By identifying $-\omega^2\mathbf{M} + \mathrm{i}\omega\mathbf{D} + \mathbf{K} = \bar{\mathbf{A}}$, $\mathrm{i}\omega\mathbf{d} = \mathbf{u}$, and $\mathbf{d} = \mathbf{v}$, the following equation can be written for the receptance matrix of the modified system, upon recognizing that $\bar{\mathbf{A}}^{-1} = \mathbf{H}(\omega)$:

$$\tilde{\mathbf{H}}(\omega) = \mathbf{H}(\omega) - (\mathrm{i}\omega\mathbf{H}(\omega)\mathbf{d}\mathbf{d}^{\mathrm{T}}\mathbf{H}(\omega))/(1 + \mathrm{i}\omega\mathbf{d}^{\mathrm{T}}\mathbf{H}(\omega)\mathbf{d}). \tag{17}$$

Since the eigencharacteristics and the receptance matrix of the modified system have been obtained, in the next section, the corresponding sensitivities will be determined.

2.2. CALCULATION OF THE SENSITIVITIES OF THE EIGENCHARACTERISTICS AND THE RECEPTANCE MATRIX

Let it be assumed that α denotes some damping-related parameter upon which the vector $\mathbf{d}$ depends. If equation (9) is differentiated partially with respect to α,

$$\lambda_k' := \frac{\partial \lambda_k}{\partial \alpha} = \frac{2\lambda_k \mathbf{d}'^{\mathrm{T}}\boldsymbol{\Lambda}_k\mathbf{d}}{(1/\lambda_k) + \lambda_k\mathbf{d}^{\mathrm{T}}\boldsymbol{\Lambda}_k(2\mathbf{M}\lambda_k + \mathbf{D})\boldsymbol{\Lambda}_k\mathbf{d}} \tag{18}$$

is obtained where the prime denotes the partial derivative with respect to α, and where the matrix

$$\boldsymbol{\Lambda}_k = (\mathbf{M}\lambda_k^2 + \mathbf{D}\lambda_k + \mathbf{K})^{-1} \tag{19}$$

is introduced. Formula (18) gives the sensitivity of the eigenvalue λ_k with respect to α.

The sensitivity of the eigenvector $\mathbf{z}_k$ with respect to the parameter α can be obtained by differentiating the expression (10) with respect to α partially to get

$$\mathbf{z}_k' := \partial\mathbf{z}_k/\partial\alpha = -\lambda_k'\boldsymbol{\Lambda}_k(2\mathbf{M}\lambda_k + \mathbf{D})\boldsymbol{\Lambda}_k\mathbf{d} + \boldsymbol{\Lambda}_k\mathbf{d}'. \tag{20}$$

It is reasonable to assume that the increment of the original damping matrix $\mathbf{D}$ is of the form

$$\mathbf{d}\mathbf{d}^{\mathrm{T}} = c\mathbf{d}_1\mathbf{d}_1^{\mathrm{T}}, \tag{21}$$

where c represents the physical damping constant of the added viscous damper. This in turn means $\mathbf{d} = \sqrt{c}\mathbf{d}_1$.

The sensitivities with respect to the damping constant c are easily obtained from equations (18) and (20) as

$$\lambda_k' = \frac{\lambda_k\mathbf{d}_1^{\mathrm{T}}\boldsymbol{\Lambda}_k\mathbf{d}_1}{(1/\lambda_k) + c\lambda_k\mathbf{d}_1^{\mathrm{T}}\boldsymbol{\Lambda}_k(2\mathbf{M}\lambda_k + \mathbf{D})\boldsymbol{\Lambda}_k\mathbf{d}_1}, \tag{22}$$

$$\mathbf{z}_k' = -\lambda_k'\sqrt{c}\boldsymbol{\Lambda}_k(2\mathbf{M}\lambda_k + \mathbf{D})\boldsymbol{\Lambda}_k\mathbf{d}_1 + (1/2\sqrt{c})\boldsymbol{\Lambda}_k\mathbf{d}_1. \tag{23}$$

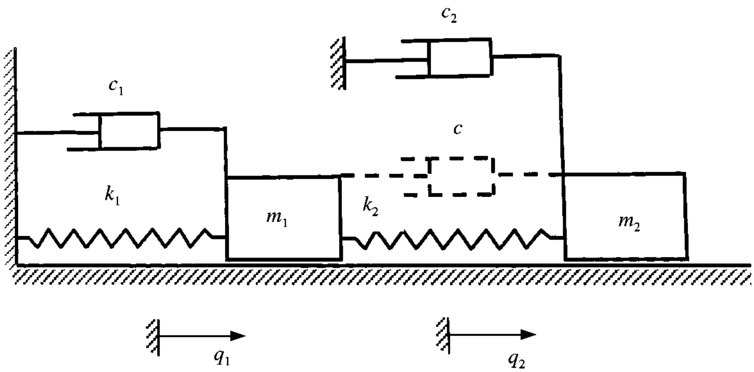

Figure 1. Sample system with two degrees of freedom.

The sensitivity of the receptance matrix with respect to the damping constant c can be obtained in a straightforward manner by differentiating equation (17) with respect to c to obtain

$$\tilde{\mathbf{H}}'(\omega) = -\frac{\mathrm{i}\omega\mathbf{H}(\omega)\mathbf{d}_1\mathbf{d}_1^{\mathrm{T}}\mathbf{H}(\omega)}{(1+\mathrm{i}\omega c\mathbf{d}_1^{\mathrm{T}}\mathbf{H}(\omega)\mathbf{d}_1)^2}. \tag{24}$$

Since the sensitivities of the eigencharacteristics and of the receptance matrix have been obtained, now the approximate formulae for the modified values of the eigenvalues, eigenvectors and receptance matrix can be given, if the damping constant of the added viscous damper is changed by an amount Δc around its nominal value c:

$$\lambda_{k_{\mathrm{mod}}} \approx \lambda_k(c) + \lambda_k'(c)\Delta c, \tag{25}$$

$$\mathbf{z}_{k_{\mathrm{mod}}} \approx \mathbf{z}_k(c) + \mathbf{z}_k'(c)\Delta c, \tag{26}$$

$$\tilde{\mathbf{H}}(\omega)_{\mathrm{mod}} \approx \tilde{\mathbf{H}}(\omega) + \tilde{\mathbf{H}}'(\omega)\Delta c. \tag{27}$$

3. NUMERICAL EVALUATIONS

This section is devoted to the testing of the reliability of the expressions obtained. The simple system in Figure 1 is taken as an illustrative example. It consists of a vibrational system with two d.o.f. in which every mass is acted upon by an inertial viscous damper. The physical parameters are chosen as $m_1 = 2$ kg, $m_2 = 1$ kg; $k_1 = 2$ N/m, $k_2 = 1$ N/m; $c_1 = 0{\cdot}35$ N/m/s, $c_2 = 0{\cdot}15$ N/m/s. It is assumed that a relative viscous damper of constant $c = 1$ N/m/s is to be added between the two masses as depicted in dashed lines.

The mass and stiffness matrices of the system are simply

$$\mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{K} = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix},$$

Table 1

Eigenvalues of the system in Figure 1

	Direct solution of the eigenvalue problem	From equation (9)
$\lambda_{1,2}$	$-0{\cdot}15090485 \pm 0{\cdot}74232271\,i$	$-0{\cdot}15090484 \pm 0{\cdot}74232272\,i$
$\lambda_{3,4}$	$-0{\cdot}76159515 \pm 1{\cdot}07828287\,i$	$-0{\cdot}76159515 \pm 1{\cdot}07828286\,i$

Table 2

Eigenvectors of the system in Figure 1

	Direct solution of the eigenvalue problem	From equation (10)
$\tilde{\mathbf{y}}_{1,2}$	$\begin{bmatrix} 1 \\ 1{\cdot}48465548 \mp 0{\cdot}64543792\,i \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1{\cdot}48465546 \mp 0{\cdot}64543791\,i \end{bmatrix}$
$\tilde{\mathbf{y}}_{3,4}$	$\begin{bmatrix} 1 \\ -1{\cdot}45965548 \mp 1{\cdot}07068561\,i \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1{\cdot}45965548 \mp 1{\cdot}07068560\,i \end{bmatrix}$

Table 3

Eigenvalues of the system in Figure 1, *if the damping coefficient is changed slightly by an amount Δc around its nominal value $c = 1$*

Δc	From equation (9)	From equation (25)
0	$-0{\cdot}15090485 \pm 0{\cdot}74232271\,i$ $-0{\cdot}76159515 \pm 1{\cdot}07828287\,i$	$-0{\cdot}15090485 \pm 0{\cdot}74232271\,i$ $-0{\cdot}76159515 \pm 1{\cdot}07828287\,i$
0·001	$-0{\cdot}15093052 \pm 0{\cdot}74240530\,i$ $-0{\cdot}76231948 \pm 1{\cdot}07758724\,i$	$-0{\cdot}15093060 \pm 0{\cdot}74240530\,i$ $-0{\cdot}76231940 \pm 1{\cdot}07758770\,i$
0·002	$-0{\cdot}15095602 \pm 0{\cdot}74248789\,i$ $-0{\cdot}76304398 \pm 1{\cdot}07689069\,i$	$-0{\cdot}15095634 \pm 0{\cdot}74248787\,i$ $-0{\cdot}76304366 \pm 1{\cdot}07689254\,i$
0·003	$-0{\cdot}15098137 \pm 0{\cdot}74257048\,i$ $-0{\cdot}76376863 \pm 1{\cdot}07619322\,i$	$-0{\cdot}15098209 \pm 0{\cdot}74257047\,i$ $-0{\cdot}76376791 \pm 1{\cdot}07619737\,i$
0·004	$-0{\cdot}15100656 \pm 0{\cdot}74265308\,i$ $-0{\cdot}76449344 \pm 1{\cdot}07549482\,i$	$-0{\cdot}15100783 \pm 0{\cdot}74265306\,i$ $-0{\cdot}76449217 \pm 1{\cdot}07550221\,i$
0·005	$-0{\cdot}15103158 \pm 0{\cdot}74273567\,i$ $-0{\cdot}76521842 \pm 1{\cdot}07479550\,i$	$-0{\cdot}15103358 \pm 0{\cdot}74273565\,i$ $-0{\cdot}76521642 \pm 1{\cdot}07480705\,i$

and the damping matrix reads as

$$\tilde{\mathbf{D}} = \mathbf{D} + c\mathbf{d}_1\mathbf{d}_1^{\mathrm{T}} = \begin{bmatrix} 0{\cdot}35 & 0 \\ 0 & 0{\cdot}15 \end{bmatrix} + 1\begin{bmatrix} 1 \\ -1 \end{bmatrix}[1 \quad -1] = \begin{bmatrix} 1{\cdot}35 & -1 \\ -1 & 1{\cdot}15 \end{bmatrix},$$

where, as noted previously, according to equation (21) the vectors $\mathbf{d}$ and $\mathbf{d}_1$ are interrelated as $\mathbf{d} = \sqrt{c}\,\mathbf{d}_1$.

TABLE 4

Eigenvectors of the system in Figure 1, if the damping coefficient is changed slightly by an amount Δc around its nominal value $c = 1$

Δc	$\mathbf{z}_{1,2}$ from equation (10)	$\mathbf{z}_{1,2}$ from equation (26)
0	$\begin{bmatrix} 1{\cdot}08601933 \mp 1{\cdot}22293758\,\mathrm{i} \\ 0{\cdot}82303426 \mp 2{\cdot}51659904\,\mathrm{i} \end{bmatrix}$	$\begin{bmatrix} 1{\cdot}08601933 \mp 1{\cdot}22293758\,\mathrm{i} \\ 0{\cdot}82303426 \mp 2{\cdot}51659904\,\mathrm{i} \end{bmatrix}$
0·001	$\begin{bmatrix} 1{\cdot}08656220 \mp 1{\cdot}22354890\,\mathrm{i} \\ 0{\cdot}82344568 \mp 2{\cdot}51785703\,\mathrm{i} \end{bmatrix}$	$\begin{bmatrix} 1{\cdot}08698119 \mp 1{\cdot}22204841\,\mathrm{i} \\ 0{\cdot}82414323 \mp 2{\cdot}51491302\,\mathrm{i} \end{bmatrix}$
0·002	$\begin{bmatrix} 1{\cdot}08710481 \mp 1{\cdot}22415991\,\mathrm{i} \\ 0{\cdot}82385688 \mp 2{\cdot}51911438\,\mathrm{i} \end{bmatrix}$	$\begin{bmatrix} 1{\cdot}08794450 \mp 1{\cdot}22115924\,\mathrm{i} \\ 0{\cdot}82525220 \mp 2{\cdot}51322699\,\mathrm{i} \end{bmatrix}$
0·003	$\begin{bmatrix} 1{\cdot}08764714 \mp 1{\cdot}22477061\,\mathrm{i} \\ 0{\cdot}82426789 \mp 2{\cdot}52037111\,\mathrm{i} \end{bmatrix}$	$\begin{bmatrix} 1{\cdot}08890709 \mp 1{\cdot}22027007\,\mathrm{i} \\ 0{\cdot}82636116 \mp 2{\cdot}51154096\,\mathrm{i} \end{bmatrix}$
0·004	$\begin{bmatrix} 1{\cdot}08818920 \mp 1{\cdot}22538102\,\mathrm{i} \\ 0{\cdot}82467869 \mp 2{\cdot}52162722\,\mathrm{i} \end{bmatrix}$	$\begin{bmatrix} 1{\cdot}08986967 \mp 1{\cdot}21938090\,\mathrm{i} \\ 0{\cdot}82747013 \mp 2{\cdot}50985494\,\mathrm{i} \end{bmatrix}$
0·005	$\begin{bmatrix} 1{\cdot}08873099 \mp 1{\cdot}22599111\,\mathrm{i} \\ 0{\cdot}82508928 \mp 2{\cdot}52288269\,\mathrm{i} \end{bmatrix}$	$\begin{bmatrix} 1{\cdot}09083226 \mp 1{\cdot}21849173\,\mathrm{i} \\ 0{\cdot}82857910 \mp 2{\cdot}50816891\,\mathrm{i} \end{bmatrix}$

Δc	$\mathbf{z}_{3,4}$ from equation (10)	$\mathbf{z}_{3,4}$ from equation (26)
0	$\begin{bmatrix} 0{\cdot}24142735 \pm 0{\cdot}14646051\,\mathrm{i} \\ -0{\cdot}19558759 \mp 0{\cdot}47227467\,\mathrm{i} \end{bmatrix}$	$\begin{bmatrix} 0{\cdot}24142735 \pm 0{\cdot}14646051\,\mathrm{i} \\ -0{\cdot}19558759 \mp 0{\cdot}47227467\,\mathrm{i} \end{bmatrix}$
0·001	$\begin{bmatrix} 0{\cdot}24154803 \pm 0{\cdot}14653372\,\mathrm{i} \\ -0{\cdot}19568536 \mp 0{\cdot}47251075\,\mathrm{i} \end{bmatrix}$	$\begin{bmatrix} 0{\cdot}24130355 \pm 0{\cdot}14608066\,\mathrm{i} \\ -0{\cdot}19600777 \mp 0{\cdot}47208685\,\mathrm{i} \end{bmatrix}$
0·002	$\begin{bmatrix} 0{\cdot}24166865 \pm 0{\cdot}14660690\,\mathrm{i} \\ -0{\cdot}19578308 \mp 0{\cdot}47274671\,\mathrm{i} \end{bmatrix}$	$\begin{bmatrix} 0{\cdot}24117975 \pm 0{\cdot}14570082\,\mathrm{i} \\ -0{\cdot}19642795 \mp 0{\cdot}47189902\,\mathrm{i} \end{bmatrix}$
0·003	$\begin{bmatrix} 0{\cdot}24178922 \pm 0{\cdot}14668004\,\mathrm{i} \\ -0{\cdot}19588075 \mp 0{\cdot}47298255\,\mathrm{i} \end{bmatrix}$	$\begin{bmatrix} 0{\cdot}24178922 \pm 0{\cdot}14668004\,\mathrm{i} \\ -0{\cdot}19588075 \mp 0{\cdot}47298255\,\mathrm{i} \end{bmatrix}$
0·004	$\begin{bmatrix} 0{\cdot}24190972 \pm 0{\cdot}14675314\,\mathrm{i} \\ -0{\cdot}19597837 \mp 0{\cdot}47321828\,\mathrm{i} \end{bmatrix}$	$\begin{bmatrix} 0{\cdot}24093215 \pm 0{\cdot}14494112\,\mathrm{i} \\ -0{\cdot}19726831 \mp 0{\cdot}47152337\,\mathrm{i} \end{bmatrix}$
0·005	$\begin{bmatrix} 0{\cdot}24203016 \pm 0{\cdot}14682621\,\mathrm{i} \\ -0{\cdot}19607595 \mp 0{\cdot}47345389\,\mathrm{i} \end{bmatrix}$	$\begin{bmatrix} 0{\cdot}24080835 \pm 0{\cdot}14456128\,\mathrm{i} \\ -0{\cdot}19768849 \mp 0{\cdot}47133555\,\mathrm{i} \end{bmatrix}$

The eigenvalues of the system in Figure 1 are given in Table 1. The complex numbers in the first column are the eigenvalues obtained directly by solving the eigenvalue problem of the system. The complex numbers in the second column are obtained by solving equation (9) with MATLAB. The agreement of the complex numbers in both columns is excellent.

In order to gain insight into how accurately the eigenvectors can be obtained by the present method, the eigenvectors of the system in Figure 1 are given in Table 2 according to the state-space representation $\tilde{\mathbf{x}}_j^{\mathrm{T}} = [\tilde{\mathbf{y}}_j^{\mathrm{T}} \;\; \lambda_j \tilde{\mathbf{y}}_j^{\mathrm{T}}]$.

The eigenvectors in the first column are obtained directly by solving the eigenvalue problem of the system in Figure 1. The eigenvectors in the second column are determined by using equation (10) and then normalizing appropriately. The agreement here is also excellent.

It is in order to test also the correctness of the formula (17) for the receptance matrix of the system after the damper attachment. The receptance matrix which is directly written

Table 5

Receptance matrix of the system in Figure 1, if the damping coefficient is changed slightly by an amount Δc around its nominal value $c = 1$

Δc	Receptance matrix from equation (17)	Receptance matrix from equation (27)
0	$\begin{bmatrix} -0{\cdot}12881412 - 0{\cdot}06276617\,i & -0{\cdot}05087906 + 0{\cdot}06779085\,i \\ -0{\cdot}05087906 + 0{\cdot}06779085\,i & -0{\cdot}17625909 - 0{\cdot}12380955\,i \end{bmatrix}$	$\begin{bmatrix} -0{\cdot}12881412 - 0{\cdot}06276617\,i & -0{\cdot}05087906 + 0{\cdot}06779085\,i \\ -0{\cdot}05087906 + 0{\cdot}06779085\,i & -0{\cdot}17625909 - 0{\cdot}12380955\,i \end{bmatrix}$
0·001	$\begin{bmatrix} -0{\cdot}12877345 - 0{\cdot}06274422\,i & -0{\cdot}05094161 + 0{\cdot}06776036\,i \\ -0{\cdot}05094161 + 0{\cdot}06776036\,i & -0{\cdot}17616308 - 0{\cdot}12376755\,i \end{bmatrix}$	$\begin{bmatrix} -0{\cdot}12882204 - 0{\cdot}06275525i & -0{\cdot}05086781 + 0{\cdot}06777393\,i \\ -0{\cdot}05086781 + 0{\cdot}06777393\,i & -0{\cdot}17627497 - 0{\cdot}12378338\,i \end{bmatrix}$
0·002	$\begin{bmatrix} -0{\cdot}12873286 - 0{\cdot}06272227\,i & -0{\cdot}05100406 + 0{\cdot}06772986\,i \\ -0{\cdot}05100406 + 0{\cdot}06772986\,i & -0{\cdot}17606722 - 0{\cdot}12372554\,i \end{bmatrix}$	$\begin{bmatrix} -0{\cdot}12882997 - 0{\cdot}06274433\,i & -0{\cdot}05085656 + 0{\cdot}06775700\,i \\ -0{\cdot}05085656 + 0{\cdot}06775700\,i & -0{\cdot}17629084 - 0{\cdot}12375722\,i \end{bmatrix}$
0·003	$\begin{bmatrix} -0{\cdot}12869233 - 0{\cdot}06270032\,i & -0{\cdot}05106640 + 0{\cdot}06769934\,i \\ -0{\cdot}05106640 + 0{\cdot}06769934\,i & -0{\cdot}17597153 - 0{\cdot}12368350\,i \end{bmatrix}$	$\begin{bmatrix} -0{\cdot}12883790 - 0{\cdot}06273341\,i & -0{\cdot}05084531 + 0{\cdot}06774008\,i \\ -0{\cdot}05084531 + 0{\cdot}06774008\,i & -0{\cdot}17630672 - 0{\cdot}12373105\,i \end{bmatrix}$
0·004	$\begin{bmatrix} -0{\cdot}12865188 - 0{\cdot}06267836\,i & -0{\cdot}05112863 + 0{\cdot}06766882\,i \\ -0{\cdot}05112863 + 0{\cdot}06766882\,i & -0{\cdot}17587599 - 0{\cdot}12364143\,i \end{bmatrix}$	$\begin{bmatrix} -0{\cdot}12884583 - 0{\cdot}06272249\,i & -0{\cdot}05083406 + 0{\cdot}06772315\,i \\ -0{\cdot}05083406 + 0{\cdot}06772315\,i & -0{\cdot}17632259 - 0{\cdot}12370489\,i \end{bmatrix}$
0·005	$\begin{bmatrix} -0{\cdot}12861149 - 0{\cdot}06265640\,i & -0{\cdot}05119076 + 0{\cdot}06763828\,i \\ -0{\cdot}05119076 + 0{\cdot}06763828\,i & -0{\cdot}17578060 - 0{\cdot}12359935\,i \end{bmatrix}$	$\begin{bmatrix} -0{\cdot}12885375 - 0{\cdot}06271156\,i & -0{\cdot}05082281 + 0{\cdot}06770623\,i \\ -0{\cdot}05082281 + 0{\cdot}06770623\,i & -0{\cdot}17633847 - 0{\cdot}12367872\,i \end{bmatrix}$

from the equation (15) for $\omega = 2\,\text{rad/s}$ reads as

$$\tilde{\mathbf{H}}(\omega) = \begin{bmatrix} -0{\cdot}12881412 - 0{\cdot}06276617\mathrm{i} & -0{\cdot}05087906 + 0{\cdot}06779085\mathrm{i} \\ -0{\cdot}05087906 + 0{\cdot}06779085\mathrm{i} & -0{\cdot}17625909 - 0{\cdot}12380955\mathrm{i} \end{bmatrix}.$$

Formula (17) in connection with equation (21) yields exactly the same matrix, which is not repeated for the sake of brevity.

Now, one can test the reliability of the sensitivity-based formulae (25)–(27). To this end, the system in Figure 1 is denoted as the "nominal" system and it is assumed that the damping constant c changes by an amount Δc around its nominal value $c = 1$, due to some reason, which in turn causes a modification of the system.

The eigenvalues of the system, thus modified are calculated by solving the characteristic equation (9) numerically and then using the sensitivity-based formula (25), considering equation (22). The results are given in Table 3. The complex numbers in the first column are "exact" values obtained from equation (9), whereas those of the second column come from the approximate formula (25).

As can be seen from the table, the agreement of the numbers in both columns is excellent, especially for small values of Δc. This in turn means that formula (25) gives very accurate approximations for the eigenvalues of the modified system without having to resolve the characteristic equation (9) with the parameters of the modified system.

In a similar manner, the eigenvectors of the modified system are calculated first by equation (10) and then by the sensitivity-based formula (26), considering equation (23). The results are collected in Table 4. In Table 4(a) are shown those eigenvectors which correspond to the eigenvalues $\lambda_{1,2}$ having smaller real parts, whereas those in Table 4(b) correspond to $\lambda_{3,4}$ with greater real parts.

The vectors in the first column are "exact" vectors obtained from equation (10) and those in the second column come from equation (26), considering equation (23). One sees clearly that the agreement of the vectors in both columns is very good. This means that formula (26) gives accurate approximations for the eigenvectors of the modified system.

The receptance matrix of the modified system is calculated by evaluating formula (17), which denotes the "exact" expression of the receptance matrix, and then using the sensitivity-based formula (27), considering equation (24). The results are given in Table 5 for $\omega = 2$ rad/s. Inspection of the matrices in both columns indicates clearly that formula (27) yields accurate approximations for the receptance matrix of the modified system.

4. CONCLUSIONS

This study is concerned with a linear discrete mechanical system which is damped viscously. The attachment of an additional viscous damper will naturally change the damping matrix of the system. The main concern is the establishment of the characteristic equation, determination of the eigenvectors and the receptance matrix of the system thus modified. Further, the eigenvalue, eigenvector and receptance sensitivities with respect to the attached damping constant are derived.

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