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ANALYTICAL SOLUTIONS OF THE SYSTEM OF TWO COUPLED PURE CUBIC NON-LINEAR OSCILLATORS EQUATIONS

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1. INTRODUCTION

The one-degree-of-freedom system whose motion is described with a differential equation with pure cubic non-linearity is discussed in references [1–6]. It is shown that there exists a closed form analytic solution of the equation. The motion is described with a Jacobi elliptic function [7]. For two-degrees-of-freedom systems which are described with a system of two coupled strong non-linear differential equations the closed form solutions are found only for some special cases (see references [8–11]). Analytic solutions based on Jacobi elliptic functions are obtained.

In this paper a special case of a two-degree-of-freedom system is considered. The non-linearities of the system are strong and of a cubic type and the differential equations are

$$\ddot{x} + x(x^2 + 3y^2) = \varepsilon f_1(x, y, \dot{x}, \dot{y}),$$
$$\ddot{y} + y(y^2 + 3x^2) = \varepsilon f_2(x, y, \dot{x}, \dot{y}), \qquad (1)$$

where x, y are generalized co-ordinates of the system, $(\dot{\ }) = \mathrm{d}/\mathrm{d}t$, $(\ddot{\ }) = \mathrm{d}^2/\mathrm{d}t^2$, εf_i are small non-linearities, $i = 1, 2$.

The aim of the paper is to obtain closed form analytic solutions for the system of two strongly coupled non-linear differential equations (1) for $\varepsilon = 0$. Using the generalized solutions the approximate solutions of equations (1) are denoted. The method of Krylov–Bogolubov [12] is extended for solving the system of differential equations whose generalized solutions are elliptic functions. The case when the cubic term is of a total type is specially considered. The approximate analytic solutions are compared with exact numeric ones.

2. GENERALIZED SOLUTIONS

Let us consider the case when $\varepsilon = 0$. System (1) transforms to

$$\ddot{x} + x^3 + 3xy^2 = 0, \qquad \ddot{y} + y^3 + 3x^2y = 0. \qquad (2)$$

These are two coupled differential equations with strong cubic non-linearities. There exists a Hamiltonian which represents the total energy function

$$H = \tfrac{1}{2}(x_1^2 + y_1^2) + \tfrac{1}{4}(x^4 + y^4) + \tfrac{3}{2}x^2y^2, \qquad (3)$$

where $x_1 = \mathrm{d}x/\mathrm{d}t$, $y_1 = \mathrm{d}y/\mathrm{d}t$.

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Using the direct Lyapunov method of stability [13] it can be concluded that the motion of system (2) is stable, as function (3) is positive definite and its time derivative is zero.

As shown below, the motion of the system is periodic and has an exact analytical solution of the form

$$x = A_1 \,\mathrm{cn}(\omega_1 t + \theta_1, k_1^2) + A_2 \,\mathrm{sd}(\omega_2 t + \theta_2, k_2^2) = A_1 \,\mathrm{cn}_1 + A_2 \,\mathrm{sd}_2,$$

$$y = A_1 \,\mathrm{cn}(\omega_1 t + \theta_1, k_1^2) - A_2 \,\mathrm{sd}(\omega_2 t + \theta_2, k_2^2) = A_1 \,\mathrm{cn}_1 - A_2 \,\mathrm{sd}_2, \tag{4}$$

where A_1, A_2 are the amplitudes of vibrations, θ_1, θ_2 are initial phases of vibrations, cn and sd are Jacobi elliptic functions [14] with the frequency parameters ω_1 and ω_2 and modulus k_1 and k_2.

Differentiating twice with respect to time the solutions (4) it follows that

$$\ddot{x} = -A_1 \omega_1^2 \mathrm{cn}_1 [1 - 2k_1^2 + 2k_1^2 \mathrm{cn}_1^2] + A_2 \omega_2^2 \mathrm{sd}_2 [2k_2^4 \mathrm{sd}_2^2 - 1],$$

$$\ddot{y} = -A_1 \omega_1^2 \mathrm{cn}_1 [1 - 2k_1^2 + 2k_1^2 \mathrm{cn}_1^2] - A_2 \omega_2^2 \mathrm{sd}_2 [2k_2^4 \mathrm{sd}_2^2 - 1]. \tag{5}$$

Substituting equations (4) and (5) into equation (1) and separating the terms with the functions cn_1, cn_1^3, sd_2, sd_2^3 a system of four algebraic equations is obtained:

$$\mathrm{cn}_1: \quad -\omega_1^2(1 - 2k_1^2) = 0,$$

$$\mathrm{cn}_1^3: \quad \omega_1^2 k_1^2 - 2A_1^2 = 0,$$

$$\mathrm{sd}_2: \quad \omega_2^2(2k_2^2 - 1) = 0,$$

$$\mathrm{sd}_2^3: \quad -\omega_2^2 k_2^2 (1 - k_2^2) + 2A_2^2 = 0. \tag{6}$$

From equations (6) the parameters of the elliptic functions are

$$\omega_1^2 = 4A_1^2, \qquad k_1^2 = 1/2,$$

$$\omega_2^2 = 8A_2^2, \qquad k_2^2 = 1/2. \tag{7}$$

It can be seen that the modulus k_1 and k_2 have equal values and the modulus is constant. The frequencies of the vibrations are different in the general case, and depend on the amplitudes of vibrations.

Substituting equation (7) into equation (4) the general solutions are

$$x = A_1 \mathrm{cn}(2A_1 t + \theta_1, 1/2) + A_2 \mathrm{sd}(2\sqrt{2} A_2 t + \theta_2, 1/2),$$

$$y = A_1 \mathrm{cn}(2A_1 t + \theta_1, 1/2) - A_2 \mathrm{sd}(2\sqrt{2} A_2 t + \theta_2, 1/2), \tag{8}$$

where A_1, A_2, θ_1 and θ_2 depend on the initial conditions. Assume that the initial conditions are in the general form

$$x(0) = x_0, \, y(0) = y_0, \qquad \dot{x}(0) = \dot{x}_0, \, \dot{y}(0) = \dot{y}_0. \tag{9}$$

Substituting the initial conditions (9) into equation (8) the following system of equations is obtained:

$$x_0 = A_1 \operatorname{cn}(\theta_1, 1/2) + A_2 \operatorname{sd}(\theta_2, 1/2),$$

$$y_0 = A_1 \operatorname{cn}(\theta_1, 1/2) - A_2 \operatorname{sd}(\theta_2, 1/2),$$

$$\dot{x}_0 = -A_1\omega_1 \operatorname{sn}(\theta_1, 1/2)\operatorname{dn}(\theta_1, 1/2) + A_2\omega_2 \operatorname{cd}(\theta_2, 1/2)\operatorname{nd}(\theta_2, 1/2),$$

$$\dot{y}_0 = -A_1\omega_1 \operatorname{sn}(\theta_1, 1/2)\operatorname{dn}(\theta_1, 1/2) - A_2\omega_2 \operatorname{cd}(\theta_2, 1/2)\operatorname{nd}(\theta_2, 1/2). \tag{10}$$

Solving system (10) gives

$$A_1 = \tfrac{1}{2}[(x_0 + y_0)^4 + 2(\dot{x}_0 + \dot{y}_0)^2]^{1/4},$$

$$A_2 = \frac{1}{2\sqrt{2}}[(x_0 - y_0)^4 + 2(\dot{x}_0 - \dot{y}_0)^2]^{1/4}, \tag{11}$$

and the phase angles of vibrations are obtained by solving the equations

$$\operatorname{sc}(\theta_1, 1/2)\operatorname{dn}(\theta_1, 1/2) = -\frac{1}{2A_1}\frac{\dot{x}_0 + \dot{y}_0}{x_0 + y_0},$$

$$\operatorname{cs}(\theta_2, 1/2)\operatorname{nd}(\theta_2, 1/2) = \frac{1}{2\sqrt{2}A_2}\frac{\dot{x}_0 - \dot{y}_0}{x_0 - y_0}, \tag{12}$$

where sc, cs, nd and dn are the Jacobi elliptic functions [15].

(1) Now assume that the motion of system starts with zero velocity, i.e.,

$$\dot{x}_0 = 0, \qquad \dot{y}_0 = 0. \tag{13}$$

The parameters of the system are

$$A_1 = \frac{x_0 + y_0}{2}, \qquad \theta_1 = 0,$$

$$A_2 = \frac{x_0 - y_0}{2\sqrt{2}}, \qquad \theta_2 = \mathrm{K}(1/2), \tag{14}$$

where $\mathrm{K}(1/2) = 1{\cdot}85407$ is the total elliptic integral of the first kind [15]. The solutions of equations (2) for the initial conditions (13) are

$$x = \frac{x_0 + y_0}{2}\operatorname{cn}[(x_0 + y_0)t, 1/2] + \frac{x_0 - y_0}{2\sqrt{2}}\operatorname{sd}[(x_0 - y_0)t + \mathrm{K}(1/2), 1/2],$$

$$y = \frac{x_0 + y_0}{2}\operatorname{cn}[(x_0 + y_0)t, 1/2] - \frac{x_0 - y_0}{2\sqrt{2}}\operatorname{sd}[(x_0 - y_0)t + \mathrm{K}(1/2), 1/2]. \tag{15}$$

For the initial conditions $x_0 = 1{\cdot}5$, $\dot{x}_0 = 0$, $y_0 = 0{\cdot}5$, $\dot{y}_0 = 0$ solutions (15) are plotted in Figure 1. The motions of both the oscillators are periodic.

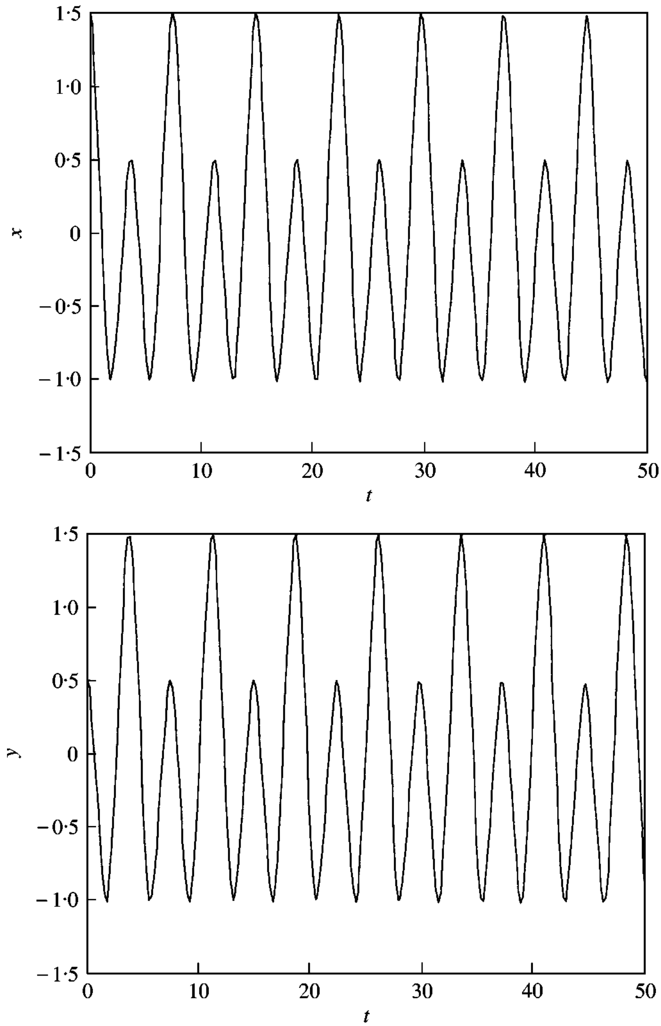


Figure 1. The x–t and y–t diagrams for the initial conditions $x_0 = 1{\cdot}5$, $x_0 = 0$, $y_0 = 0{\cdot}5$, $y_0 = 0$.

(2) If the initial conditions are

$$x(0) = x_0, \qquad y(0) = 0, \qquad \dot{x}(0) = 0, \qquad \dot{y}(0) = 0, \tag{16}$$

the solutions are

$$x = x_0 \,\mathrm{cn}\,(x_0 t, 1/2), \qquad y = 0, \tag{17}$$

since

$$\mathrm{cn}\,(x_0 t, 1/2) = \frac{1}{\sqrt{2}}\,\mathrm{sd}\,(x_0 t + \mathrm{K}(1/2), 1/2). \tag{18}$$

Then, only one oscillator moves and the other is motionless. In Figure 2, solutions (17) for the initial conditions $x(0) = 1{\cdot}5$, $y(0) = 0$, $\dot{x}(0) = 0$, $\dot{y}(0) = 0$ are plotted. The movement of the oscillator is periodic.

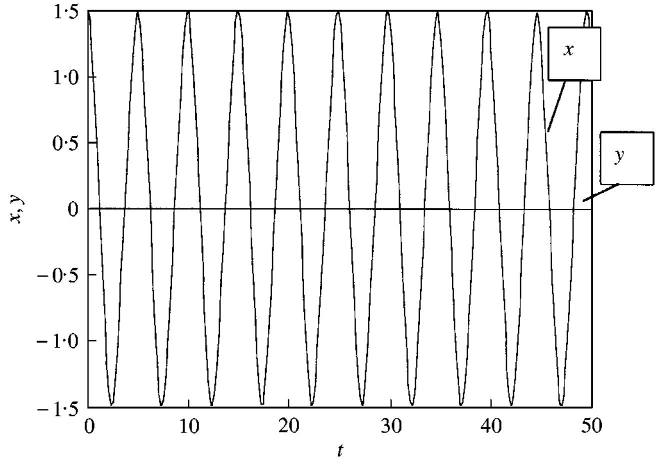


Figure 2. The x–t and y–t diagrams for the initial conditions $x_0 = 1{\cdot}5$, $x_0 = 0$, $y_0 = 0$, $y_0 = 0$.

3. APPROXIMATE SOLUTIONS

Now consider the case when the motions are described with equations (1). According to the Krylov–Bogolubov method [12] we assume that the trial solutions are in the form of general solutions (4) and they are

$$x = A_1(t)\,\mathrm{cn}\,[\psi_1(t), 1/2] + A_2(t)\,\mathrm{sd}\,[\psi_2(t), 1/2] = A_1\mathrm{cn}_1 + A_2\mathrm{sd}_2,$$

$$y = A_1(t)\,\mathrm{cn}\,[\psi_1(t), 1/2] - A_2(t)\,\mathrm{sd}\,[\psi_2(t), 1/2] = A_1\mathrm{cn}_1 - A_2\mathrm{sd}_2, \qquad (19)$$

where

$$\psi_1(t) = \int_t \omega_1(t)\,\mathrm{d}t + \theta_1(t),$$

$$\psi_2(t) = \int_t \omega_2(t)\,\mathrm{d}t + \theta_2(t). \qquad (20)$$

The first time derivatives have the same form as the first time derivatives of equation (4) and they are

$$\dot{x} = -A_1\omega_1\mathrm{sn}_1\mathrm{dn}_1 + A_2\omega_2\mathrm{cd}_2\mathrm{nd}_2,$$

$$\dot{y} = -A_1\omega_1\mathrm{sn}_1\mathrm{dn}_1 - A_2\omega_2\mathrm{cd}_2\mathrm{nd}_2, \qquad (21)$$

where

$$\dot{A}_1\mathrm{cn}_1 - A_1\dot{\theta}_1\mathrm{sn}_1\mathrm{dn}_1 + \dot{A}_2\dot{\theta}_2\mathrm{sd}_2 + A_2\mathrm{cd}_2\mathrm{nd}_2 = 0,$$

$$\dot{A}_1\mathrm{cn}_1 - A_1\dot{\theta}_1\mathrm{sn}_1\mathrm{dn}_1 - \dot{A}_2\dot{\theta}_2\mathrm{sd}_2 - A_2\mathrm{cd}_2\mathrm{nd}_2 = 0. \qquad (22)$$

The second time derivatives of the solutions are

$$\ddot{x} = -\dot{A}_1\omega_1 \mathrm{sn}_1 \mathrm{dn}_1 - A_1\dot{\omega}_1 \mathrm{sn}_1 \mathrm{dn}_1 - A_1\omega_1(\omega_1 + \dot{\theta}_1)\mathrm{cn}_1(1 - 2k_1^2 + 2k_1^2 cn_1^2)$$

$$+ \dot{A}_2\omega_2 \mathrm{cd}_2 \mathrm{nd}_2 + A_2\dot{\omega}_2 \mathrm{cd}_2 \mathrm{nd}_2 + A_2\omega_2(\omega_2 + \dot{\theta}_2)\mathrm{sd}_2[(2k_2^2 - 1) - 2k_2^2(1 - k_2^2)\mathrm{sd}_2^2],$$

$$\ddot{y} = -\dot{A}_1\omega_1 \mathrm{sn}_1 \mathrm{dn}_1 - A_1\dot{\omega}_1 \mathrm{sn}_1 \mathrm{dn}_1 - A_1\omega_1(\omega_1 + \dot{\theta}_1)\mathrm{cn}_1(1 - 2k_1^2 + 2k_1^2 cn_1^2)$$

$$- \dot{A}_2\omega_2 \mathrm{cd}_2 \mathrm{nd}_2 - A_2\dot{\omega}_2 \mathrm{cd}_2 \mathrm{nd}_2 - A_2\omega_2(\omega_2 + \dot{\theta}_2)\mathrm{sd}_2[(2k_2^2 - 1) \tag{23}$$

$$- 2k_2^2(1 - k_2^2)\mathrm{sd}_2^2], \tag{24}$$

Substituting equations (19) and (23) into equation (1) and using relations (22) results in a system of two first order differential equations:

$$-4A_1\dot{A}_1 \mathrm{sn}_1 \mathrm{dn}_1 - 2A_1^2\dot{\theta}_1 \mathrm{cn}_1^3 + 4\sqrt{2}A_2\dot{A}_2 \mathrm{cd}_2 \mathrm{nd}_2 - \sqrt{2}A_2^2\dot{\theta}_2 \mathrm{sd}_2^3 = \varepsilon f_1,$$

$$-4A_1\dot{A}_1 \mathrm{sn}_1 \mathrm{dn}_1 - 2A_1^2\dot{\theta}_1 \mathrm{cn}_1^3 - 4\sqrt{2}A_2\dot{A}_2 \mathrm{cd}_2 \mathrm{nd}_2 + \sqrt{2}A_2^2\dot{\theta}_2 \mathrm{sd}_2^3 = \varepsilon f_2. \tag{25}$$

Transforming equations (22) and (24) gives the following system of equations:

$$A_1^2\dot{\theta}_1 = -\frac{\varepsilon}{4}(f_1 + f_2)\,\mathrm{cn}_1,$$

$$A_1\dot{A}_1 = -\frac{\varepsilon}{4}(f_1 + f_2)\,\mathrm{sn}_1\,\mathrm{dn}_1,$$

$$A_2^2\dot{\theta}_2 = -\frac{\varepsilon}{8\sqrt{2}}(f_1 - f_2)\,\mathrm{sd}_2,$$

$$A_2\dot{A}_2 = \frac{\varepsilon}{8\sqrt{2}}(f_1 - f_2)\,\mathrm{cd}_2\,\mathrm{nd}_2. \tag{26}$$

This system of equations gives the four unknown functions $A_1(t)$, $A_2(t)$, $\theta_1(t)$ and $\theta_2(t)$. To find the solutions of these equations is not an easy task. We will introduce the averaging procedure for both the angles ψ_1 and ψ_2. The averaged first order differential equations are

$$\dot{\theta}_1 = -\frac{\varepsilon}{4A_1^2}\frac{1}{4\mathrm{K}_1}\frac{1}{4\mathrm{K}_2}\int_0^{4\mathrm{K}_1}\int_0^{4\mathrm{K}_2}(f_1 + f_2)\,\mathrm{cn}_1\,\mathrm{d}\psi_1\,\mathrm{d}\psi_2,$$

$$\dot{A}_1 = -\frac{\varepsilon}{4A_1}\frac{1}{4\mathrm{K}_1}\frac{1}{4\mathrm{K}_2}\int_0^{4\mathrm{K}_1}\int_0^{4\mathrm{K}_2}(f_1 + f_2)\,\mathrm{sn}_1\,\mathrm{dn}_1\,\mathrm{d}\psi_1\,\mathrm{d}\psi_2, \tag{27}$$

$$\dot{\theta}_2 = -\frac{\varepsilon}{8\sqrt{2}A_2^2}\frac{1}{4\mathrm{K}_1}\frac{1}{4\mathrm{K}_2}\int_0^{4\mathrm{K}_1}\int_0^{4\mathrm{K}_2}(f_1 - f_2)\,\mathrm{sd}_2\,\mathrm{d}\psi_1\,\mathrm{d}\psi_2,$$

$$\dot{A}_2 = \frac{\varepsilon}{8\sqrt{2}A_1}\frac{1}{4\mathrm{K}_1}\frac{1}{4\mathrm{K}_2}\int_0^{4\mathrm{K}_1}\int_0^{4\mathrm{K}_2}(f_1 - f_2)\,\mathrm{cd}_2\,\mathrm{nd}_2\,\mathrm{d}\psi_1\,\mathrm{d}\psi_2,$$

where $\mathrm{K}_1 = \mathrm{K}_2 = \mathrm{K}(1/2)$ is the total elliptic integral of the first kind [7].

4. AN EXAMPLE

Consider the case when the small non-linearities are

$$f_1 = 3x^2y + y^3, \qquad f_2 = 3xy^2 + x^3. \tag{28}$$

For $\varepsilon = 1$ the non-linearities are of the total cubic type and are considered in reference [11]. Using the assumed solutions (19) functions (27) are

$$\begin{aligned} f_1 &= 4A_1^3\,\mathrm{cn}_1^3 - 4A_2^3\,\mathrm{sd}_2^3,\\ f_2 &= 4A_1^3\,\mathrm{cn}_1^3 + 4A_2^3\,\mathrm{sd}_2^3. \end{aligned} \tag{29}$$

Substituting equation (28) into equation (26) and integrating the averaged equations gives

$$\begin{aligned} A_1 &= A_1(0) = const.,\\ \theta_1 &= -2A_1(0)\,\varepsilon\left[\frac{1}{4\mathrm{K}_1}\int_0^{4\mathrm{K}_1}\mathrm{cn}_1^4\,\mathrm{d}\psi_1\right]t + \theta_1(0) = -\frac{2}{3}A_1(0)\,\varepsilon t + \theta_1(0),\\ A_2 &= A_2(0) = const.,\\ \theta_2 &= \frac{\varepsilon A_2(0)}{\sqrt{2}}\left[\frac{1}{4\mathrm{K}_2}\int_0^{4\mathrm{K}_2}\mathrm{sd}_2^4\,\mathrm{d}\psi_2\right]t + \theta_2(0) = \frac{4\varepsilon A_2(0)}{3\sqrt{2}}\,t + \theta_2(0), \end{aligned} \tag{30}$$

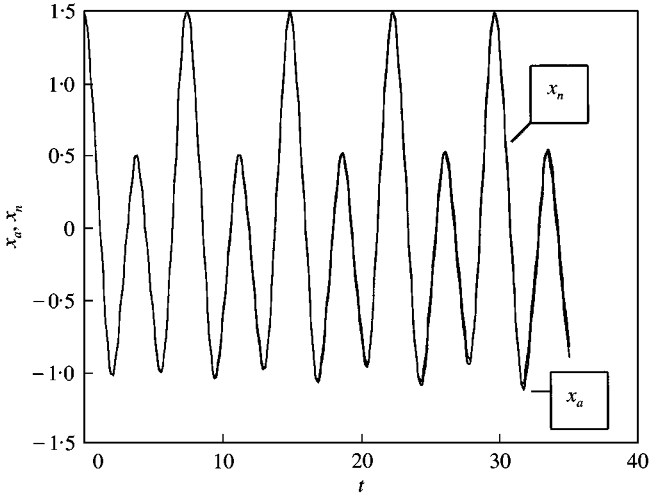
where $A_1(0)$, $A_2(0)$, $\theta_1(0)$ and $\theta_2(0)$ are the initial amplitudes and phases of the system.

The approximate solutions for system (27) are

$$\begin{aligned} x &= A_1(0)\,\mathrm{cn}\left[2A_1(0)t\left(1 - \frac{\varepsilon}{3}\right) + \theta_1(0), 1/2\right]\\ &\quad + A_2(0)\,\mathrm{sd}\left[2\sqrt{2}A_2(0)t\left(1 + \frac{\varepsilon}{3}\right) + \theta_2(0), 1/2\right],\\ y &= A_1(0)\,\mathrm{cn}\left[2A_1(0)t\left(1 - \frac{\varepsilon}{3}\right) + \theta_1(0), 1/2\right]\\ &\quad - A_2(0)\,\mathrm{sd}\left[2\sqrt{2}A_2(0)t\left(1 + \frac{\varepsilon}{3}\right) + \theta_2(0), 1/2\right]. \end{aligned} \tag{31}$$

Consider next the initial conditions $x_0 = 1{\cdot}5$, $\dot{x}_0 = 0$, $y_0 = 0{\cdot}5$, $\dot{y}_0 = 0$, i.e., $A_1(0) = 1$, $\theta_1(0) = 0$, $A_2(0) = \sqrt{2}/4$, $\theta_2(0) = \mathrm{K}(1/2) = 1{\cdot}85407$. For the case when the small parameter is $\varepsilon = 0{\cdot}01$ the analytical solutions of the equations of motion are

$$\begin{aligned} x_a &= \mathrm{cn}(1{\cdot}9934t, 1/2) + 0{\cdot}3525\,\mathrm{sd}(1{\cdot}0033t + 1{\cdot}85407, 1/2),\\ y_a &= \mathrm{cn}(1{\cdot}9934t, 1/2) - 0{\cdot}3525\,\mathrm{sd}(1{\cdot}0033t + 1{\cdot}85407, 1/2). \end{aligned} \tag{32}$$



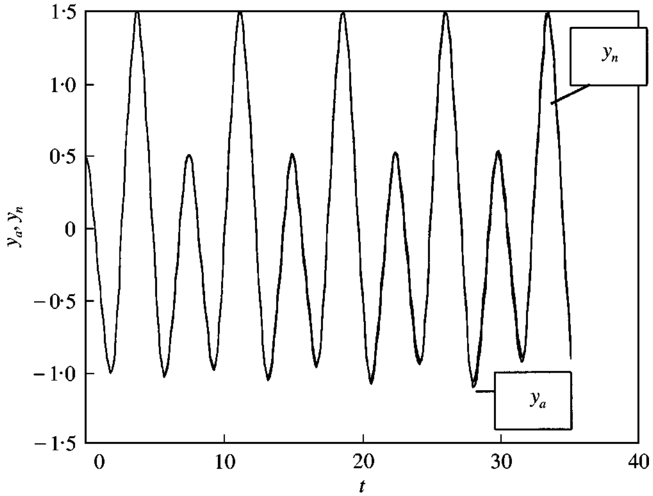


Figure 3. The analytically, x_a–t, y_a–t, and numerically, x_n–t, y_n–t, obtained diagrams for the small parameter $\varepsilon = 0{\cdot}01$.

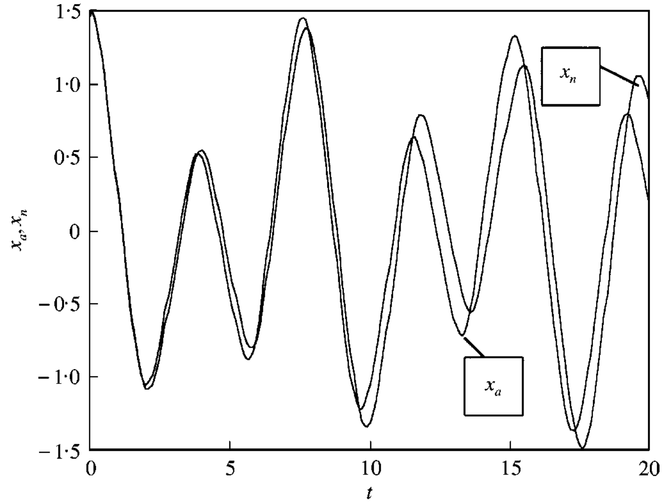
These solutions (x_a, y_a) are compared with those obtained numerically (x_n, y_n) applying the Runge–Kutta method. From Figure 3, it can be seen that the motions are periodical. The differences between the approximate analytic and exact numeric results are negligible even for large values of time.

The analytic (x_a, y_a) and numeric solutions (x_n, y_n) which are determined for $\varepsilon = 0{\cdot}1$ are shown in Figure 4. The approximate analytic solutions are

$$x_a = \mathrm{cn}(1{\cdot}94t, 1/2) + 0{\cdot}3525\,\mathrm{sd}(1{\cdot}033t + 1{\cdot}85407, 1/2),$$

$$y_a = \mathrm{cn}(1{\cdot}94t, 1/2) - 0{\cdot}3525\,\mathrm{sd}(1{\cdot}033t + 1{\cdot}85407, 1/2). \tag{33}$$

It can be seen that the solutions are periodic. The approximate analytic solutions differ significantly from the exact numeric results. The difference is higher for larger values of time.



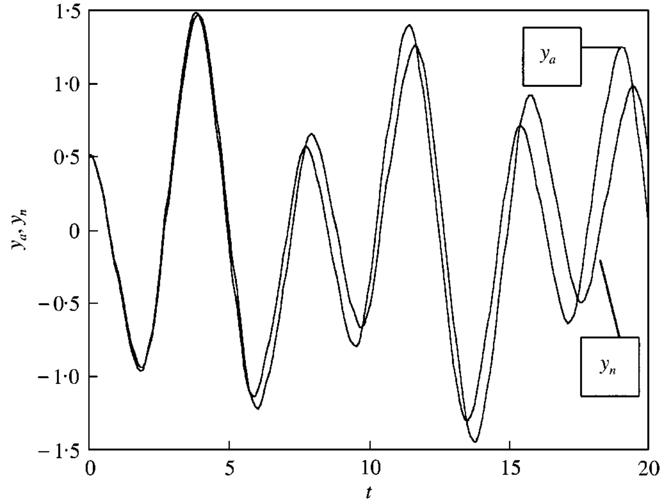


Figure 4. The analytically, x_a–t, y_a–t, and numerically, x_n–t, y_n–t, obtained diagrams for the small parameter $\varepsilon = 0{\cdot}1$.

REFERENCES

1. S. B. Yuste and J. D. Bejarano 1986 *Journal of Sound and Vibration* **110**, 347–350. Construction of approximate analytical solutions to a new class of non-linear oscillator equations.
2. S. B. Yuste and J. D. Bejarano 1989 *Journal of Control* **49**, 1127–1141. Extension and improvement to the Krylov–Bogolubov methods using elliptic functions.
3. L. Cveticanin 1999 *Journal of Sound and Vibration* **224**, 952–960. The conservation laws and the exact solutions for the singular non-linear oscillator.
4. S. H. Chen and Y. K. Cheung 1996 *Journal of Sound and Vibration* **192**, 453–464. An elliptic perturbation method for certain strongly non-linear oscillators.
5. M. Belhaq and F. Lakrad 2000 *Journal of Sound and Vibration* **234**, 547–553. The elliptic multiple scales method for a class of autonomous strongly non-linear oscillators.
6. Z. M. Ge and F. N. Ku 2000 *Journal of Sound and Vibration* **236**, 554–560. Subharmonic Melnikov functions for strongly odd non-linear oscillators with large perturbations.

7. P. F. Byrd and M. D. Friedman 1954 *Handbook of Elliptic Integrals for Engineers and Physicists*. Berlin: Springer-Verlag.
8. L. Cveticanin 1998 *Journal of Sound and Vibration* **212**, 173–178. Some particular solutions which describe the motion of the rotor.
9. L. Cveticanin 1998 *Journal of Sound and Vibration* **214**, 325–338. Analytical methods for solving strongly non-linear differential equations.
10. R. Manasevich, J. Mawhin and F. Zanolin 1999 *Nonlinear Analysis* **36**. Periodic solutions of some complex-valued Lienard and Rayleigh equations.
11. L. Cveticanin 2000 *Journal of Sound and Vibration* (accepted). The motion of a two mass system with non-linear connection.
12. S. B. Yuste and J. D. Bejarano 1990 *Journal of Sound and Vibration* **139**, 151–163. Improvement of a Krylov–Bogolubov method that uses Jacobi elliptic functions.
13. W. Hahn 1959 Theorie und Anwendung der direkten Method von Ljapunov. Berlin: Springer-Verlag.
14. M. Abramowitz and I. A. Stegun 1979 Handbook of Mathematical Functions. Moscow: Nauka.
15. I. S. Gradstein and I. M. Rjizhik 1971 Tablici integralov, summ, rjadov i proizvedenij. Moscow: Nauka.