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A SIMPLIFIED METHOD FOR NATURAL FREQUENCY ANALYSIS OF A MULTIPLE CRACKED BEAM

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A new method for natural frequency analysis of beam with an arbitrary number of cracks is developed on the bases of the transfer matrix method and rotational spring model of crack. The resulted frequency equation of a multiple cracked beam is general with respect to the boundary conditions including the more realistic (elastic) end supports and can be constructed analytically by using symbolic codes. The procedure proposed is advanced by elimination of numerical computation of the high order determinant so that the computer time for calculating natural frequencies in consequence is significantly reduced. Numerical computation has been carried out to investigate the effect of each crack, the number of cracks and boundary conditions on the natural frequencies of a beam.

1. INTRODUCTION

In the last two decades a lot of research effort has been devoted to developing an effective approach for detecting crack in structures. Such an approach must be developed based on the tools of the system identification, that includes mainly a modelling of the structure under consideration and its response measured *in situ*.

In the modelling of structures with cracks, it is necessary to underline the powerful finite element method (FEM) [1, 2]. This approach has no concurrence in application to large structures, but for specification of crack location in an element such as beam the analytical model of the element is more useful. In the analytical model of beams, crack is treated as a local change of stiffness (or flexibility) at a section of crack location. To model crack in this conception, Dimagoronas suggested the use of an equivalent rotational spring connecting both the sides of a beam at the crack position. Then, in reference [3] the proposed crack model has been validated by a general theory of cracked beams, that allows to calculate the stiffness of the equivalent spring as a function of crack depth. This model of crack is termed as transverse crack model. Since 1978, Adams *et al.* [4] have investigated the case of crack that was modelled by an axial spring (axial crack model), but stiffness of the spring has not been calculated. For application of a crack model to structural damage detection problem, there is a need for a relation between the crack parameters and some characteristics of the structure. This relation is often taken in the form of frequency equation of cracked beam.

Using the transverse model Rizos *et al.* [5] have constructed the equation for cantilever beam. Narkis [6] has given the equations for simply supported beam in both the cases of transverse and axial crack models. Boltezar *et al.* [7] did the same task for free–free beams with a transverse crack. Masoud *et al.* [8] considered the case of axially loaded fixed–fixed cracked beam. Nandwana *et al.* [9] and Tsai and Wang [10] developed the theory for stepped cantilevers and for Timoshenko beams. Furthermore, Liang [11] has shown that there exists a general form of the frequency equation for both the cantilever and simply supported beams. Morassi [12] has constructed a sensitivity equation of frequencies for beams with elastic supports at the beam ends, but he investigated only the case of symmetry of the supports. In reference [13], the frequency equation of cracked beam has been established in a general form for all of the classical boundary conditions, that is likely the equation given by Liang.

All the listed studies were concerned with the single crack; the dynamic behaviour of a double-cracked beam and a rotor with two cracks were investigated by Ruotolo *et al.* [14] and Sekhar [15]. A beam with an arbitrary number n of cracks was studied by Shifrin and Ruotolo [16], who proposed a new method for evaluating natural frequencies of such a beam, that requires to calculate determinant of $(n+2)$ order instead of $(4n+4)$-matrix determinant search as usually needed.

In this paper, a more simplified method for evaluating the natural frequencies of beams with n transverse cracks is investigated. This method is based on the use of rotational spring model of crack and the transfer matrix method, that leads to determinant calculation of a 4×4 matrix. It is possible to considerably reduce the computer time needed to evaluate natural frequencies in comparison even with the method developed in reference [16], if the number of cracks is more than two. Furthermore, the frequency equation established in this paper is more general with respect to the boundary condition including the elastic one. The obtained equation can be useful in theoretical investigation, for instance, in sensitivity analysis of frequencies to crack, boundary conditions or to the structural constants and, of course, it will be the main equation used for crack identification.

2. VIBRATION MODEL OF BEAM WITH n CRACKS

Let us consider a beam of length L, cross-section area $A = b\times h$, moment of inertia I and Young's modulus E with n cracks at positions $x_1,\dots,x_n$ $(x_0 = 0 \prec x_1 \prec x_2 \prec \cdots \prec x_{n-1} \prec x_n \prec L = x_{n+1})$, as shown in Figure 1. The crack at x_j are modelled by rotational spring of stiffness [3]

$$K_j = \frac{1}{\alpha_j}, \quad \text{where } \alpha_j = \frac{6\pi(1-\nu^2)h}{EI}\, I_c\left(\frac{a_j}{h}\right), \tag{1}$$

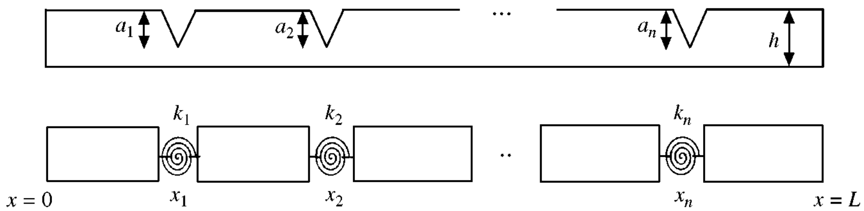

Figure 1. Model of multiple cracked beam.

where v is the Poisson coefficient, h the beam height, a_j the crack depth and the function $I_c(z)$ has the form

$$I_c(z) = 0{\cdot}6272z^2 - 1{\cdot}04533z^3 + 4{\cdot}5948z^4 - 9{\cdot}973z^5 + 20{\cdot}2948z^6$$
$$- 33{\cdot}0351z^7 + 47{\cdot}1063z^8 - 40{\cdot}7556z^9 + 19{\cdot}6z^{10}.$$

Free vibration of the beam is described by the equation

$$\varphi^{(\mathrm{IV})}(x) - \lambda^4\varphi(x) = 0 \quad \text{where } \lambda^4 = \omega^2\frac{\rho A}{EI} \tag{2}$$

with the condition at the crack position x_j

$$\varphi(x_j-0) = \varphi(x_j+0), \quad \varphi''(x_j+0) = \varphi''(x_j+0), \quad \varphi'''(x_j-0) = \varphi'''(x_j+0),$$
$$\varphi'(x_j-0) + \beta_j\varphi''(x_j-0) = \varphi'(x_j+0), \quad \text{where } \beta_j = EI\alpha_j. \tag{3}$$

Furthermore, the function $\varphi(x, \omega)$ must additionally satisfy boundary conditions, which can be expressed in the form

$$\begin{bmatrix} B^0_{11}B^0_{12}B^0_{13}B^0_{14} \\ B^0_{21}B^0_{22}B^0_{23}B^0_{24} \end{bmatrix}\begin{pmatrix} \varphi(+0) \\ \varphi'(+0) \\ EI\varphi'''(+0) \\ -EI\varphi''(+0)\end{pmatrix} = 0 = \begin{bmatrix} B^L_{11}B^L_{12}B^L_{13}B^L_{14} \\ B^L_{21}B^L_{22}B^L_{23}B^L_{24} \end{bmatrix}\begin{pmatrix} \varphi(L-0) \\ \varphi'(L-0) \\ -EI\varphi'''(L-0) \\ EI\varphi''(L-0)\end{pmatrix}, \tag{4}$$

where B^0_{ij}, B^L_{ij} are functions of the so-called boundary parameters $\bar{b} = (b_1, \dots, b_m)^{\mathrm{T}}$. The matrices in equation (3) further will be denoted by $\underline{\mathbf{B}}_0$ and $\underline{\mathbf{B}}_L$, that have the same dimensions 2×4.

3. THE TRANSFER MATRIX METHOD

For simplicity in writing, one introduces the following notations:

$$K_1(x) = \tfrac{1}{2}(\cosh x + \cos x), \quad K_3(x) = \tfrac{1}{2}(\cosh x - \cos x), \quad K_2(x) = \tfrac{1}{2}(\sinh x + \sin x),$$
$$K_4(x) = \tfrac{1}{2}(\sinh x - \sin x),$$
$$\bar{\mathbf{Z}}_j^- = \{Z^-_{j1}, Z^-_{j2}, Z^-_{j3}, Z^-_{j4}\}^{\mathrm{T}} = (\varphi(x_j-0);\ \varphi'(x_j-0);\ -EI\varphi'''(x_j-0);\ EI\varphi''(x_j-0)^{\mathrm{T}},$$
$$\bar{\mathbf{Z}}_j^+ = \{Z^+_{j1}, Z^+_{j2}, Z^+_{j3}, Z^+_{j4}\}^{\mathrm{T}} = (\varphi(x_j+0);\ \varphi'(x_j+0);\ EI\varphi'''(x_j+0);\ -EI\varphi''(x_j+0)^{\mathrm{T}}.$$

3.1. TRANSFER MATRIX FOR BEAM ELEMENT

Equation (2) is considered now in a beam segment (x_{j-1}, x_j). Its general solution can be represented as

$$\varphi(x, \omega) = \sum_{j=1}^{4} C_j K_j(\lambda\bar{x}), \tag{5}$$

where $\bar{x} = x - x_{j-1}$. The constants C_j, $j = 1, 2, 3, 4$ may be determined by substituting $\bar{x} = 0$ or $x = x_{j-1} + 0$ into function (5) and using the state vector $\bar{\mathbf{Z}}^+_{j-1}$ as follows:

$$C_1 = Z^+_{j-1,1}, \quad C_2 = Z^+_{j-1,2}/\lambda, \quad C_3 = -Z^+_{j-1,4}/EI\lambda^2, \quad C_4 = Z^+_{j-1,3}/EI\lambda^3. \tag{6}$$

Thus, substituting coefficients (6) again into function (5) yields

$$\varphi(x) = K_1(\lambda\bar{x})Z^+_{j-1,1} + \frac{K_2(\lambda\bar{x})}{\lambda} Z^+_{j-1,2} + \frac{K_4(\lambda\bar{x})}{EI\lambda^3} Z^+_{j-1,3} - \frac{K_3(\lambda\bar{x})}{EI\lambda^2} Z^+_{j-1,4}. \tag{7}$$

Using function (7), the relationships between the state vector $\bar{\mathbf{Z}}^-_j$ at the right end of the segment and the state vector at the left end $\bar{\mathbf{Z}}^+_{j-1}$ can be also obtained in the form

$$Z^-_{ij} = K_1(\lambda\ell_j)Z^+_{j-1,1} + \frac{K_2(\lambda\ell_j)}{\lambda} Z^+_{j-1,2} + \frac{K_4(\lambda\ell_j)}{EI\lambda^3} Z^+_{j-1,3} - \frac{K_3(\lambda\ell_j)}{EI\lambda^2} Z^+_{j-1,4},$$

$$Z^-_{j2} = \lambda K_4(\lambda\ell_j)Z^+_{j-1,1} + K_1(\lambda\ell_j)Z^+_{j-1,2} + \frac{K_3(\lambda\ell_j)}{EI\lambda^2} Z^+_{j-1,3} - \frac{K_2(\lambda\ell_j)}{EI\lambda} Z^+_{j-1,4},$$

$$Z^-_{j3} = -\lambda^3 EIK_2(\lambda\ell_j)Z^+_{j-1,1} - \lambda^2 EIK_3(\lambda\ell_j)Z^+_{j-1,2} - K_1(\lambda\ell_j)Z^+_{j-1,3} + \lambda K_4(\lambda\ell_j)Z^+_{j-1,4},$$

$$Z^-_{j4} = \lambda^2 EIK_3(\lambda\ell_j)Z^+_{j-1,1} + \lambda EIK_4(\lambda\ell_j)Z^+_{j-1,2} + \frac{K_2(\lambda\ell_j)}{\lambda} Z^+_{j-1,3} - K_1(\lambda\ell_j)Z^+_{j-1,4},$$

here $\ell_j = x_j - x_{j-1}$. The last equation can be rewritten in the matrix form

$$\bar{\mathbf{Z}}^-_j = \underline{\mathbf{T}}_j \bar{\mathbf{Z}}^+_{j-1} \tag{8}$$

with $j = 1, 2, \ldots, n+1$ and matrix $\underline{\mathbf{T}}_j = \underline{\mathbf{T}}(\lambda, \ell_j)$, where matrix function

$$\underline{\mathbf{T}}_j(\lambda, \ell) = \begin{bmatrix} K_1(\lambda\ell) & \lambda^{-1}K_2(\lambda\ell) & K_4(\lambda\ell)/EI\lambda^3 & -K_3(\lambda\ell)/EI\lambda^2 \\ \lambda K_4(\lambda\ell) & K_1(\lambda\ell) & K_3(\lambda\ell)/EI\lambda^2 & -K_2(\lambda\ell)/EI\lambda \\ -\lambda^3 EIK_2(\lambda\ell) & -\lambda^2 EIK_3(\lambda\ell) & -K_1(\lambda\ell) & \lambda K_4(\lambda\ell) \\ \lambda^2 EIK_3(\lambda\ell) & \lambda EIK_4(\lambda\ell) & \lambda^{-1}K_2(\lambda\ell) & -K_1(\lambda\ell) \end{bmatrix}$$

is the transfer matrix of the beam segment.

3.2. TRANSFER MATRIX FOR JOINT (CRACK)

Conditions (3), in use of the notation for the state vectors $\bar{\mathbf{Z}}^-_j$, $\bar{\mathbf{Z}}^+_j$, can be rewritten as

$$\bar{\mathbf{Z}}^+_j = \underline{\mathbf{J}}_j \bar{\mathbf{Z}}^-_j, \tag{9}$$

where $j = 1, 2, \ldots, n$. The matrix $\underline{\mathbf{J}}_j = \underline{\mathbf{J}}(\alpha_j)$ with the matrix function

$$\underline{\mathbf{J}}(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

is called here the joint transfer matrix for crack.

3.3. TRANSFER MATRIX FOR BEAM WITH n INTERNAL JOINTS (CRACKS)

From equations (8), and (9) one obtains $\bar{\mathbf{Z}}_j^+ = \underline{\mathbf{J}}_j\underline{\mathbf{T}}_j\bar{\mathbf{Z}}_{j-1}^+ = \underline{\mathbf{Q}}_j\bar{\mathbf{Z}}_{j-1}^+$ for $j = 1, \ldots, n$ so that, in consequence, $\bar{\mathbf{Z}}_n^+ = \underline{\mathbf{Q}}_n\underline{\mathbf{Q}}_{n-1}\ldots\underline{\mathbf{Q}}_1\bar{\mathbf{Z}}_0^+$. Multiplying both the sides of the last equation by the matrix $\underline{\mathbf{T}}_{n+1}$, then applying formula (8), one obtains the equation

$$\bar{\mathbf{Z}}_{n+1}^- = \underline{\mathbf{T}}_{n+1}\underline{\mathbf{Q}}_n\underline{\mathbf{Q}}_{n-1}\ldots\underline{\mathbf{Q}}_1\bar{\mathbf{Z}}_0^+ = \underline{\mathbf{Q}}\bar{\mathbf{Z}}_0^+. \tag{10}$$

The matrix

$$\underline{\mathbf{Q}} = \underline{\mathbf{T}}_{n+1}\underline{\mathbf{Q}}_n\underline{\mathbf{Q}}_{n-1}\cdots\underline{\mathbf{Q}}_1 = \underline{\mathbf{T}}_{n+1}\underline{\mathbf{J}}_n\underline{\mathbf{T}}_n\underline{\mathbf{J}}_{n-1}\cdots\underline{\mathbf{J}}_2\underline{\mathbf{T}}_2\underline{\mathbf{J}}_1\underline{\mathbf{T}}_1 \tag{11}$$

is the transfer matrix for the beam with n internal cracks. It is a matrix function of the frequency ω, crack positions $\bar{\mathbf{x}}_c = (x_1, \ldots, x_n)^{\mathrm{T}}$ and the so-called crack magnitude $\bar{\boldsymbol{\alpha}}_c = (\alpha_1, \ldots, \alpha_n)^{\mathrm{T}}$.

4. FREQUENCY EQUATION FOR BEAM WITH n CRACKS

Using the notation introduced above for the state vectors $\bar{\mathbf{Z}}_j, \bar{\mathbf{Z}}_j^+$, the boundary condition (4) can be rewritten in the form

$$\underline{\mathbf{B}}_0\bar{\mathbf{Z}}_0^+ = 0; \underline{\mathbf{B}}_L\bar{\mathbf{Z}}_{n+1}^- = 0. \tag{12}$$

Applying second condition (12) to equation (10) yields $\underline{\mathbf{B}}_L\bar{\mathbf{Z}}_{n+1}^- = \underline{\mathbf{B}}_L\underline{\mathbf{Q}}\bar{\mathbf{Z}}_0^+ = 0$, that, in combination with the first equation in equation (12), leads to a system of four linear equations

$$\underline{\mathbf{A}}\bar{\mathbf{Z}}_0^+ = 0, \tag{13}$$

where $\underline{\mathbf{A}}$ is a 4×4 matrix of the form

$$\underline{\mathbf{A}} = \begin{bmatrix} \underline{\mathbf{B}}_0 \\ \underline{\mathbf{B}}_L\underline{\mathbf{Q}} \end{bmatrix} = \begin{bmatrix} B_{11}^0 & B_{12}^0 & B_{13}^0 & B_{14}^0 \\ B_{21}^0 & B_{22}^0 & B_{23}^0 & B_{24}^0 \\ \sum_j^4 B_{1j}^L Q_{j1} & \sum_j^4 B_{1j}^L Q_{j2} & \sum_j^4 B_{1j}^L Q_{j3} & \sum_j^4 B_{1j}^L Q_{j4} \\ \sum_j^4 B_{2j}^L Q_{j1} & \sum_j^4 B_{2j}^L Q_{j2} & \sum_j^4 B_{2j}^L Q_{j3} & \sum_j^4 B_{2j}^L Q_{j4} \end{bmatrix}, \tag{14}$$

Q_{ij} are elements of the transfer matrix $\underline{\mathbf{Q}}$. For the existence of non-zero solution of system (14), determinant of the matrix $\underline{\mathbf{A}}$ must be made to vanish. Since the matrix $\underline{\mathbf{A}}$ is a function of the frequency ω represented through the so-called frequency parameter $\lambda = \sqrt[4]{\omega^2\rho A/EI}$, the crack characteristics $\bar{\mathbf{x}}_c, \bar{\boldsymbol{\alpha}}_c$ and the boundary parameters $\bar{\mathbf{b}} = (b_1, \ldots, b_m)^{\mathrm{T}}$, the frequency equation with respect to the frequency parameter λ takes the form

$$f(\lambda, \bar{\mathbf{x}}_c, \bar{\boldsymbol{\alpha}}_c, \bar{\mathbf{b}}) = \det\underline{\mathbf{A}} = 0. \tag{15}$$

Thus, natural frequency analysis of a beam with an arbitrary number n of internal cracks leads to solving equation (15) with respect to λ for given parameters $\bar{\mathbf{x}}_c, \bar{\boldsymbol{\alpha}}_c, \bar{\mathbf{b}}$. In order to construct the function $f(\lambda, \bar{\mathbf{x}}_c, \bar{\boldsymbol{\alpha}}_c, \bar{\mathbf{b}})$, one has to calculate, first, the transfer 4×4 matrix $\underline{\mathbf{Q}}$, then, determinant of the matrix $\underline{\mathbf{A}}$, which has also dimensions 4×4. These tasks might be done with any symbolic code, for example, the MAPLE. Thus, the function $f(\lambda, \bar{\mathbf{x}}_c, \bar{\boldsymbol{\alpha}}_c, \bar{\mathbf{b}})$ can be determined analytically. In comparison with the procedure proposed in reference [16], which requires to search determinant of an $(n+2)$ dimension matrix, here one has to compute only a 4×4 matrix determinant. This fact demonstrates the advantage and effectiveness of the method proposed herein. In most cases of boundary conditions the determinant can be obtained easily by hand. In fact, for the classical boundary condition such as simple supports, fixed ends or cantilever beam the frequency equations (15) are, respectively, the following:

$$f_S(\lambda) = Q_{12}Q_{43} - Q_{13}Q_{42} = 0, \tag{16}$$

$$f_F(\lambda) = Q_{13}Q_{24} - Q_{23}Q_{14} = 0, \tag{17}$$

$$f_C(\lambda) = Q_{33}Q_{44} - Q_{43}Q_{34} = 0. \tag{18}$$

In the more general case of the elastic end supports, the frequency equation (15) takes the form

$$\begin{aligned} f_e(\lambda) = {} & D_0 + D_{11}\alpha_L + D_{12}\beta_L + D_{13}\alpha_0 + D_{14}\beta_0 + D_{21}\alpha_L\beta_L + D_{22}\alpha_0\beta_0 \\ & + D_{23}\alpha_0\alpha_L + D_{24}\alpha_0\beta_L + D_{25}\alpha_L\beta_0 + D_{26}\beta_0\beta_L + D_{31}\alpha_0\alpha_L\beta_L \\ & + D_{32}\alpha_0\alpha_L\beta_0 + D_{33}\alpha_0\beta_0\beta_L + D_{34}\beta_0\alpha_L\beta_L + D_4\alpha_0\beta_0\alpha_L\beta_L = 0, \end{aligned} \tag{19}$$

where

$$\begin{aligned} &D_0 = Q_{13}Q_{24} - Q_{14}Q_{23}, \quad D_{24} = Q_{23}Q_{32} - Q_{22}Q_{33}, \quad D_{11} = Q_{13}Q_{44} - Q_{14}Q_{43}, \\ &D_{25} = Q_{14}Q_{41} - Q_{11}Q_{44}, \quad D_{12} = Q_{24}Q_{33} - Q_{23}Q_{34}, \quad D_{26} = Q_{21}Q_{34} - Q_{24}Q_{31}, \\ &D_{13} = Q_{12}Q_{23} - Q_{13}Q_{22}, \quad D_{31} = Q_{32}Q_{43} - Q_{42}Q_{33}, \quad D_{14} = Q_{14}Q_{21} - Q_{11}Q_{24}, \\ &D_{32} = Q_{11}Q_{42} - Q_{12}Q_{41}, \quad D_{21} = Q_{44}Q_{33} - Q_{43}Q_{34}, \quad D_{33} = Q_{22}Q_{31} - Q_{21}Q_{32}, \\ &D_{22} = Q_{11}Q_{22} - Q_{21}Q_{12}, \quad D_{34} = Q_{41}Q_{34} - Q_{44}Q_{31}, \quad D_{23} = Q_{12}Q_{43} - Q_{13}Q_{42}, \\ &\qquad\qquad D_4 = Q_{31}Q_{42} - Q_{41}Q_{32}, \end{aligned}$$

$\alpha_0 = EI/k_{0R}$, $\beta_0 = EI/k_{0A}$, $\alpha_L = EI/k_{LR}$, $\beta_L = EI/k_{LA}$ and $k_{0R}, k_{0A}, k_{LR}, k_{LA}$ are stiffnesses of the rotational and transverse springs at the left and right ends of the beam respectively. Equation (19) is general for a lot of boundary conditions and covers, for instance, equations (16)–(18) as particular cases.

5. NUMERICAL RESULTS AND DISCUSSION

To illustrate the present method of analysis, numerical results given below are obtained for the beam considered in reference [16], that has the following properties: $L = 0{\cdot}8$ m, $b = 0{\cdot}02$ m, $h = 0{\cdot}02$ m, $E = 2{\cdot}1 \times 10^{11}$ N/m^2 and $\rho = 7800$ kg/m^3.

5.1. COMPARISON WITH PREVIOUS RESULTS

In reference [16], the authors have presented a comparison of their result with the one given in reference [14] for the case of double-cracked cantilever (Figures 2–4 in reference [16]).

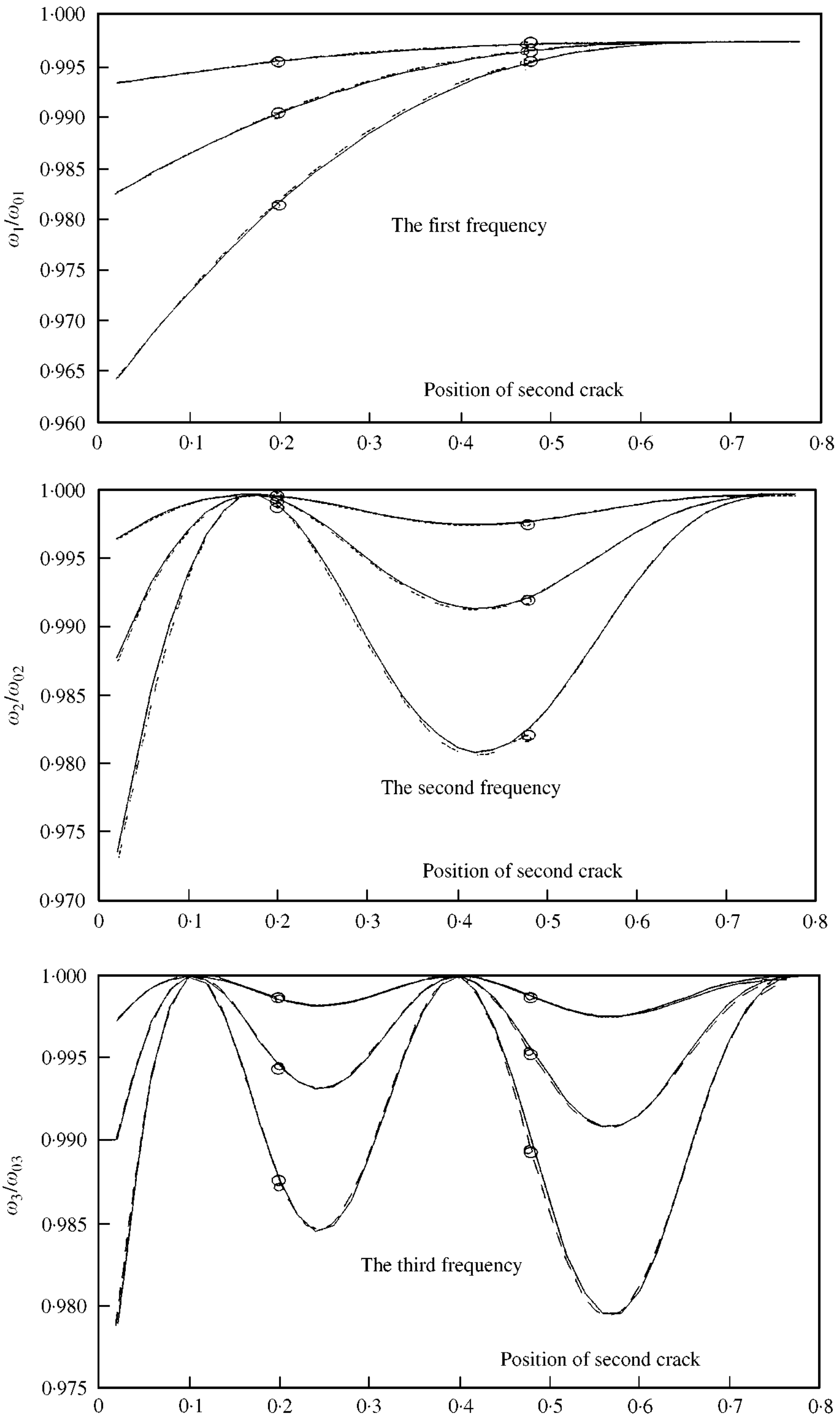

Figure 2. Comparison with the previous results: effect of second crack on the first three natural frequencies of cantilever beam for various crack depths (1–10%; 2–20%; 3–30%). ------, reference [16]; ○, reference [14].

In use of the method developed in the present paper for the same beam, numerical computation has been carried out and results were compared also with those given in reference [16]. Graphics in Figure 2, where continuous lines correspond to results of this work, show quite close results to those obtained in references [14, 16].

5.2. EFFECT OF CRACK POSITION AND DEPTH

Figure 3 gives ratios of the first three natural frequencies for the beam with single crack to the corresponding frequencies of the uncracked beam in three cases of boundary condition simply supported, fixed ends and cantilever beam respectively. These ratios shown by graphics in the figure are obtained as functions of the crack position for given crack depths 0·002, 0·004 and 0·006 m. Figure 4 shows the frequency ratios versus position of the fourth crack with the given three cracks at 0·04, 0·08 and 0·12 m of the same depth 6 mm (30% of the height) and under different boundary conditions. A fact that might be derived from graphics given in the figures is that there exist certain positions in the beam, at which cracks do not affect certain natural frequencies. Such positions are called here critical points for a given frequency. For each frequency the corresponding critical points are different and they have been listed in Table 1. The critical points do not depend upon the number of cracks that occurred in the beam and their existence is useful to detect crack position if unchange of a certain frequency is recognized.

5.3. EFFECT OF NUMBER OF CRACKS

The frequencies ratios versus the number of cracks in the cases of classical boundary condition have been computed and shown in Figures 5 and 6. The different lines numbered

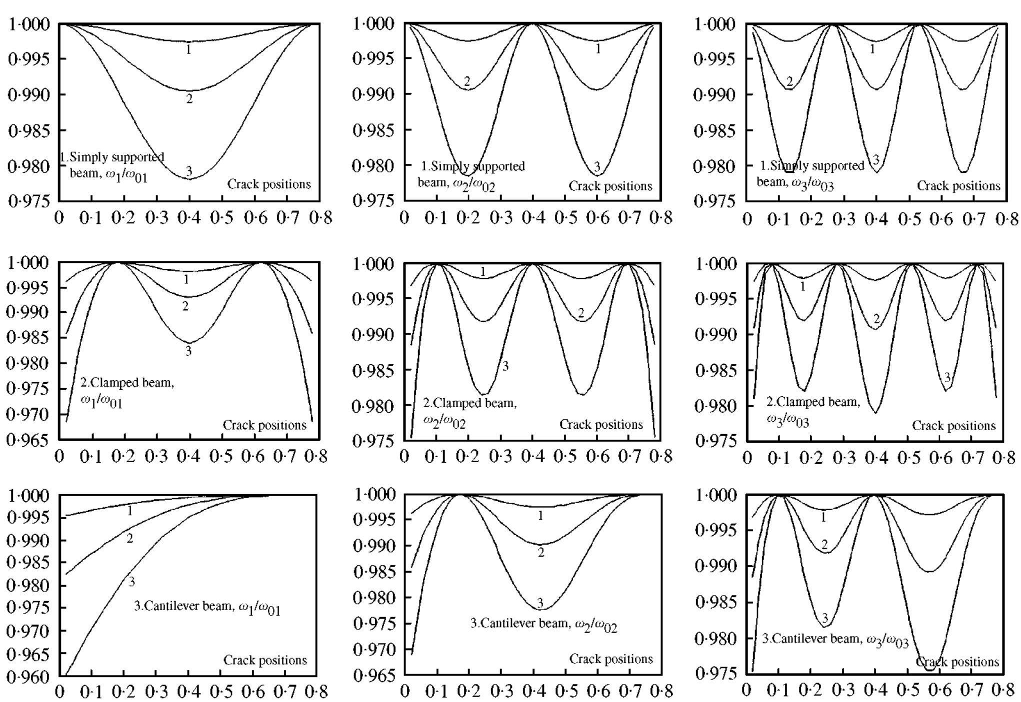

Figure 3. Effect of single crack on the first three natural frequencies of a beam for different boundary conditions and for various crack depths (1–10%; 2–20%; 3–30%).

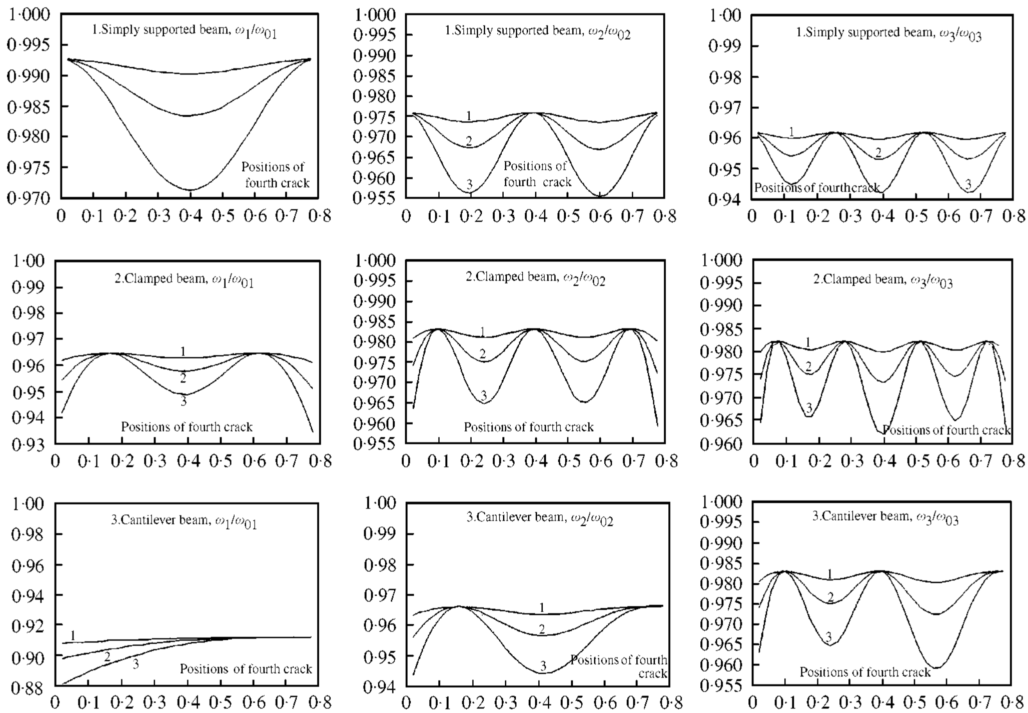

Figure 4. Effect of the fourth crack on the first three natural frequencies of a beam for different boundary conditions and for various crack depths (1–10%; 2–20%; 3–30%).

TABLE 1

The critical points for the first three natural frequencies in the cases of classical boundary condition

Boundary condition type	First frequency	Second frequency	Third frequency
Simpy supported ends	—	(1) 0·40000	(1) 0·26672 (2) 0·53328
Fixed ends	(1) 0·17928 (2) 0·62072	(1) 0·10608 (2) 0·40000 (3) 0·69392	(1) 0·07632 (2) 0·28544 (3) 0·51456 (4) 0·72368
Cantilever	—	(1) 0·17312	(1) 0·10608 (2) 0·39688

from 1 to 6 in the figures correspond to the crack depths of 5, 10, 15, 20, 25 and 30%. Of course, behaviour of the frequencies ratios strongly depends also on the distribution of the cracks in the beam. Figure 5 shows the result in the case when cracks are distributed inside the left quarter of the beam and Figure 6 shows the one for the cracks distributed all along the beam. It is clear that increase of the number of cracks, in general, reduces the frequencies.

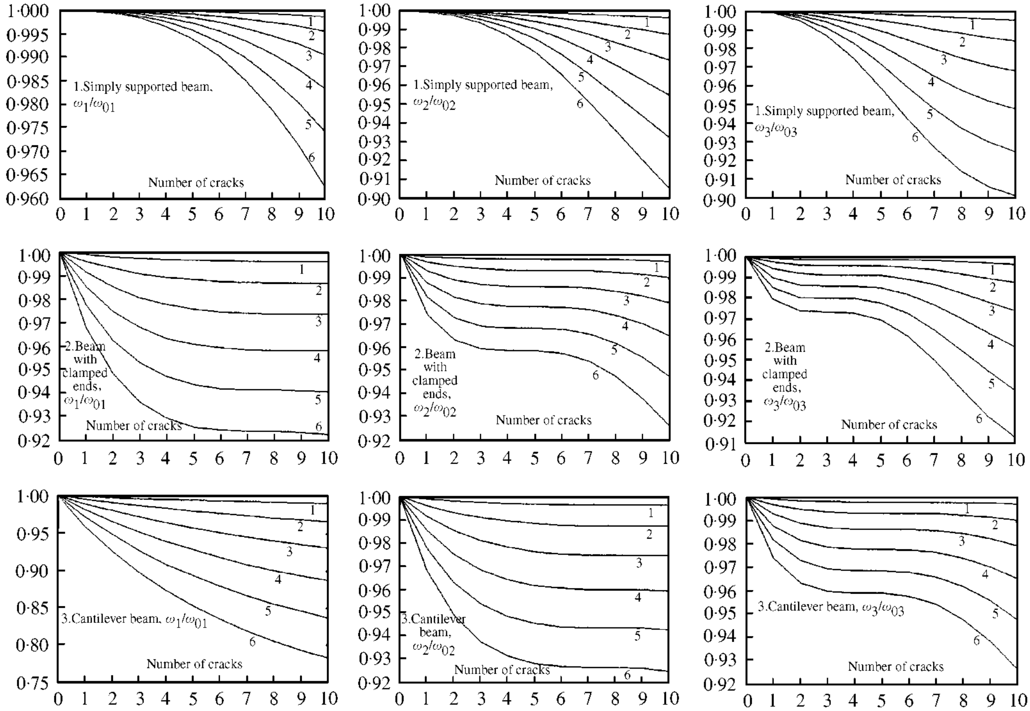

Figure 5. Effect of the number of cracks on the natural frequencies for different boundary conditions and for various crack depths (1–5%; 2–10%; 3–15%; 4–20%; 5–25%; 6–30%).

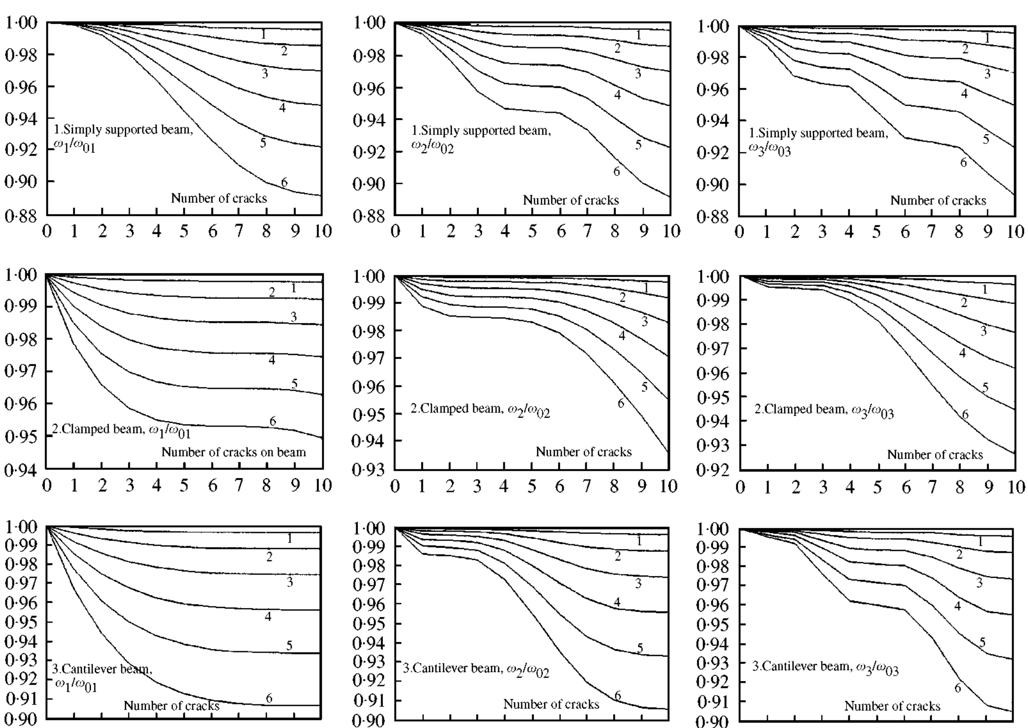

Figure 6. Effect of the number of cracks on the natural frequencies for different boundary conditions and various crack depths (1–5%; 2–10%; 3–15%; 4–20%; 5–25%; 6–30%).

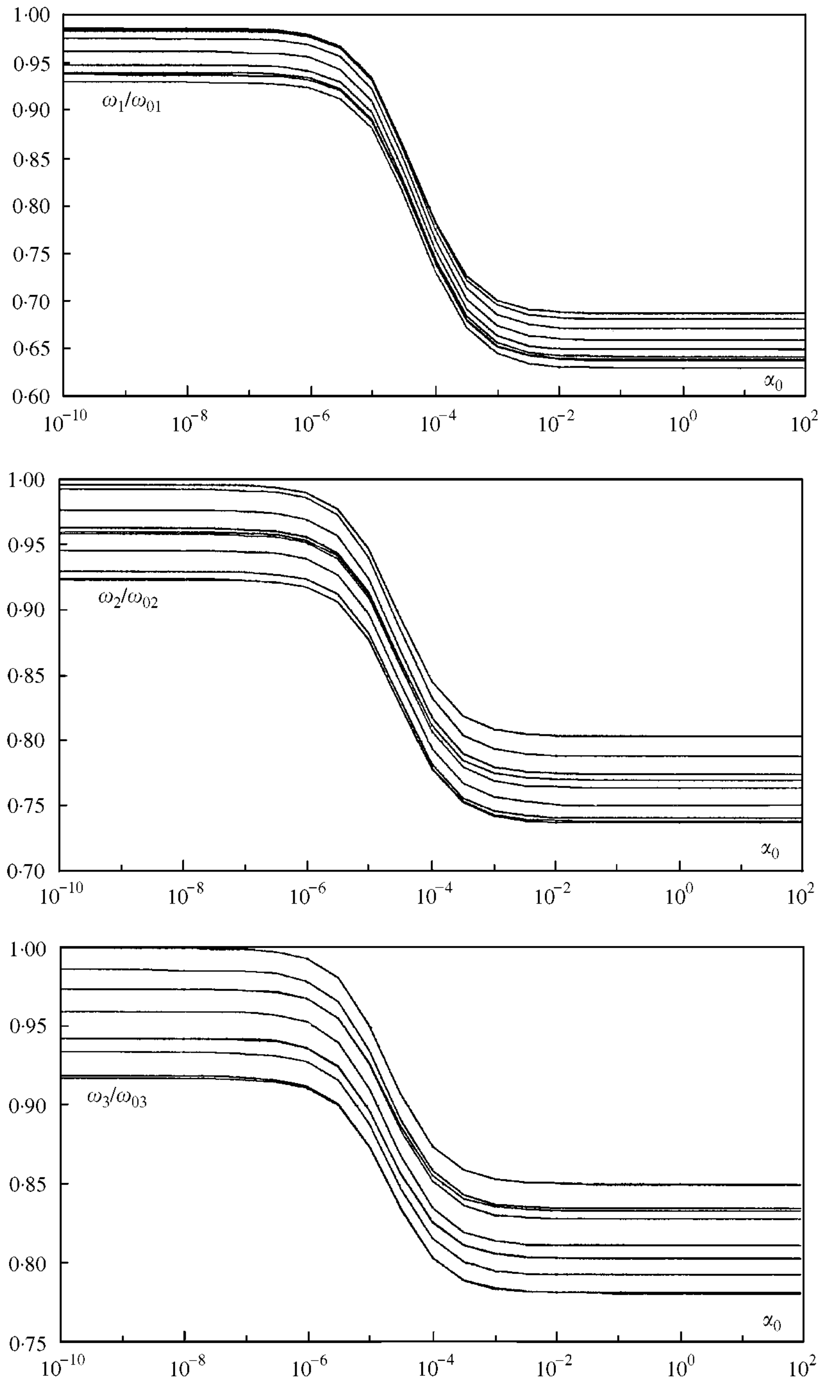

Figure 7. Effect of the boundary parameter α_0 on the first three natural frequencies for various numbers of cracks (from 0 to 9).

5.4. EFFECT OF ELASTIC BOUNDARY CONDITION

Effect of elastic boundary condition in three particular cases upon the natural frequencies of the cracked beam is considered by using equation (19). In the first case, only rotational

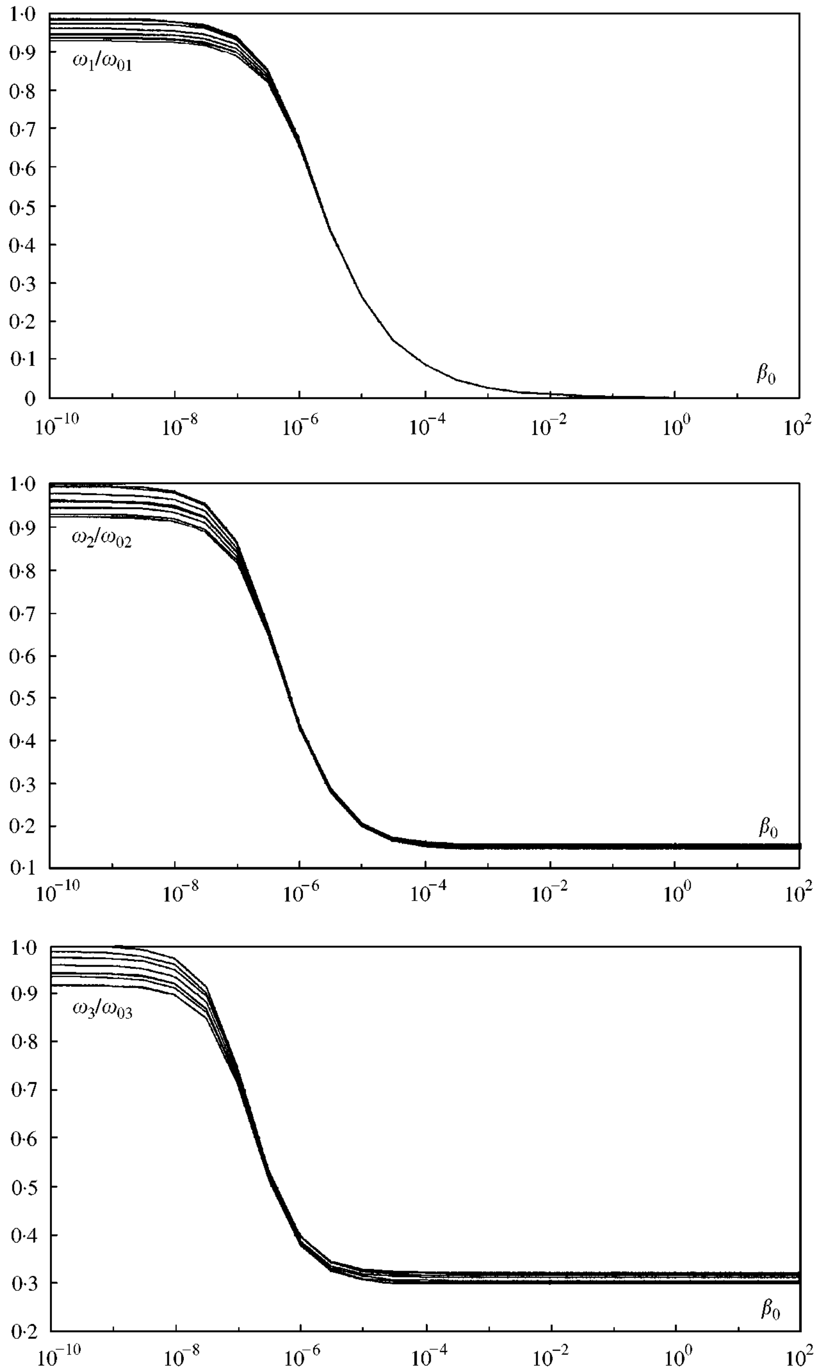

Figure 8. Effect of the boundary parameter β_0 on the first three natural frequencies for various numbers of cracks (from 0 to 9).

constraint at the left end is investigated. In this case $0 \prec \alpha_0 \prec \infty$, β_0, α_L, β_L equal 0. In the second one, $0 \prec \beta_0 \prec \infty$, α_0, α_L, β_L equal 0, i.e., only effect of transverse spring at the left end is studied. In the last case considered herein, β_0, β_L equal 0 and $\alpha_0 = \alpha_L = \alpha$. In all these cases, the frequency ratios ω_j/ω_{0j}, $j = 1, 2, 3$ (where ω_{0j} are natural frequencies computed

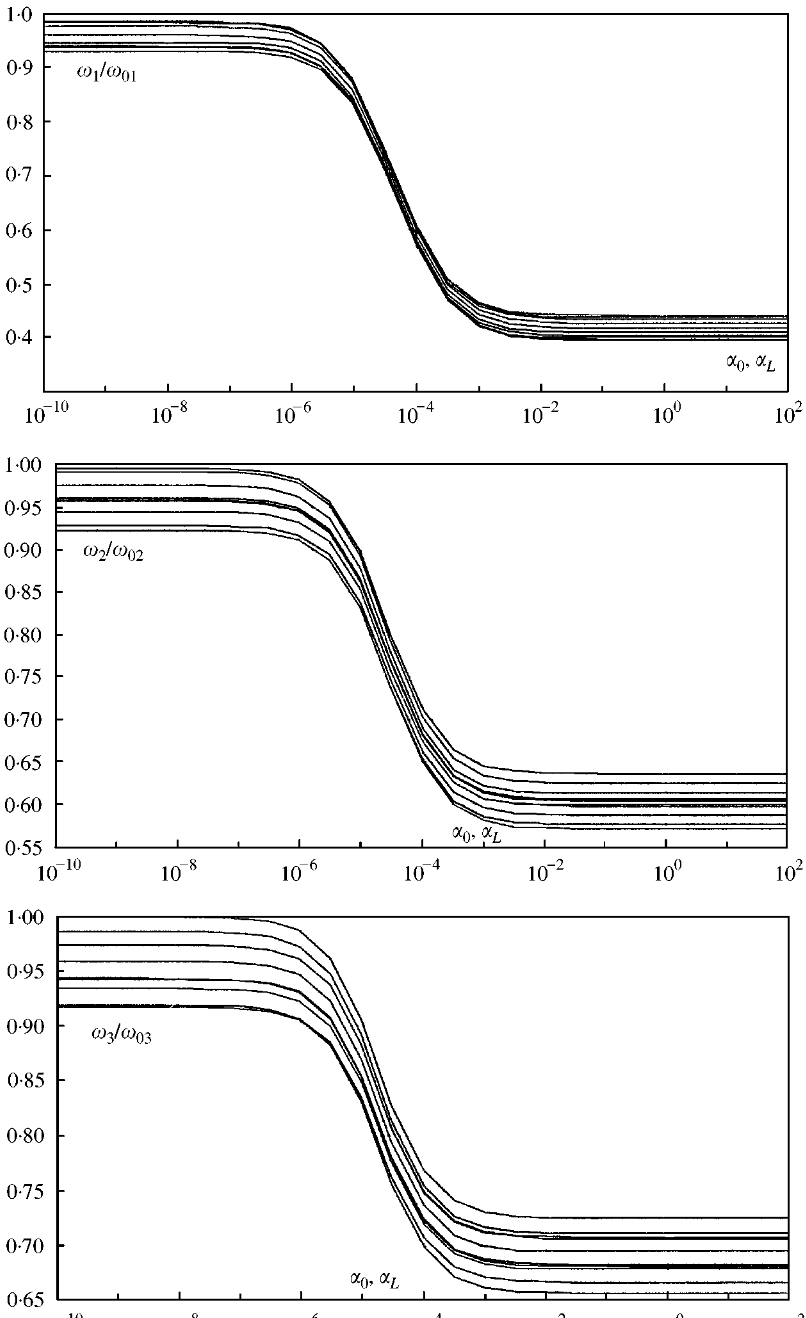

Figure 9. Effect of the boundary parameter α_0, α_L on the first three natural frequencies for various numbers of cracks (from 0 to 9).

for zero values of the boundary parameters) versus α or β for various numbers of cracks (from 0 to 9) have been computed and shown in Figures 7, 8 and 9 respectively. Figure 7 shows the frequency ratio versus parameter α_0, which is the relative flexibility of the rotational spring at the left beam end and Figure 9 shows change of the one in case of simultaneous variation of both α_0 and α_L (the rotational spring parameter at the right bound of the beam). From the figures it may be noted that effect of the boundary parameters in rotation constraint is significant only in the range from 10^{-6} to 10^{-2} of the parameters. Outside of the interval the number of cracks is more effective on the frequency change. Effect of the transverse constraint described by the parameter β at the ends of the beam on the natural frequencies is shown in Figure 8. In this case, the significant change range of the frequency ratios is from 10^{-8} to 10^{-4}. Moreover, increase in the boundary parameter β may reduce the first frequency to zero and for values of the parameter β_0 beginning from 10^{-7} the number of cracks does not affect the natural frequencies.

6. CONCLUSIONS

The transfer matrix method has been developed for natural frequency analysis of a multiple cracked beam based on the rotational spring model of crack. Using this method the frequency equation for a beam with an arbitrary number of cracks was obtained by determinant calculation of only 4×4 dimension matrix and for a more general (elastic) boundary condition. This considerably reduces computer time for evaluating the natural frequencies and hence is an advantage of the method developed in this paper. The obtained frequency equation has been used to investigate the effect of crack position and depth, number of cracks and elastic end constraints on the natural frequencies of a beam. One of the results obtained is that independent of the number of cracks there exists a set of positions in beam at which the presence of crack does not affect certain natural frequencies of the beam. These positions for a given frequency are called the critical points. Furthermore, the numerical computation shows also that increase in the number of cracks, in general, reduces all natural frequencies for any boundary condition at the ends of beam. Finally, the natural frequencies are sensitive to the elastic boundary conditions only for spring constants ranged in some limited interval. Outside the interval the number of cracks has a more significant effect on the natural frequencies.

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