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LETTERS TO THE EDITOR

SOLUTIONS IN THE FOURIER SERIES FORM, GIBBS PHENOMENA AND PADÉ APPROXIMANTS

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In a series of papers a so-called oscillator equation with an antisymmetric constant force [1–4] is considered of the form

$$\ddot{x} + \operatorname{sign}(x) = 0, \tag{1}$$

with the initial conditions

$$x(0) = 0, \qquad \dot{x}(0) = A, \tag{2}$$

where

$$\operatorname{sign}(x) = \begin{cases} +1 & \text{for } x > 0, \\ -1 & \text{for } z < 0. \end{cases}$$

Lipscomb and Mickens [2] obtained the solution for $x(t)$ over one period

$$x(t) = \begin{cases} -\dfrac{t}{2}(t - 2A) & \text{for } 0 \leqslant t \leqslant 2A, \\ \dfrac{t^2}{2} - 3At + 4A^2 & \text{for } 2A \leqslant t \leqslant 4A. \end{cases} \tag{3}$$

For values of t outside this interval, $x(t)$ can be determined from the periodicity condition in the following form:

$$x(t + nT) = x(t), \quad T = 4A. \tag{4}$$

Here T is a period, n is an integer.

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Solution (3) and (4) may be represented by the Fourier series as

$$x(t) = \frac{16A^2}{\pi^3} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^3} \sin\left\{(2m+1)\frac{\pi t}{2A}\right\}. \tag{5}$$

Solution (5) was also obtained by Govindan Potti *et al.* [3] in a direct way. Pilipchuk [4] obtained a closed-form analytical solution by making use of the saw-tooth transformation of the time proposed in reference [5]. Pilipchuk makes the following remark concerning the Fourier series [4]: ... "It should be noted that the Fourier series form gives, in principal, an approximate solution since it is impossible to account for the infinite number of terms. As long as one can keep "any number of terms", the above remark is not so important for the smooth time histories. However, it becomes very important when dealing with either a discontinuous function $x(t)$ or its discontinuous derivatives. It is known that the trigonometric series appear to be "bad working" around the discontinuities due to the Gibbs phenomenon. In terms of acceleration, the series performs an oscillating error near those points of time t at which the acceleration $\ddot{x}(t)$ has step-wise discontinuities switching its value from -1 to 1 or back as it is dictated by equation (1)".

The mentioned remark is true if one applies a simple summation of the Fourier series (as it is known that it leads to a so-called ill-posed problem). However, one can utilize the regularization properties of the Padé approximants [6–11].

As an example consider the function sign (x) in the interval $x \in [-\pi, \pi]$. This function has the following Fourier representation:

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \frac{1}{9}\sin 9x + \cdots\right). \tag{6}$$

The behaviour of $f(x)$ in a neighbourhood of the point $x = 0$ is well known. Namely, a so-called Gibbs phenomenon is observed.

A choice of the sign(x) function is motivated by an observation that it is one of the paradigm type functions exhibiting the so-called Gibbs phenomenon. If the Padé approximants can be satisfactorily used in this case, then also, a similar approach can be applied for other functions.

In order to obtain a limiting geometrical picture of the function $S_n(x)$ (being a part of the series (6)) for $n \to \infty$, one needs to extend the extent of a vertical line $x = 0$ linking the points $f(-0)$ and $f(0)$ by about 18% upwards and downwards. The diagonal Padé approximant $P(N, N, x)$ of the series (6) is given in reference [6], and has the following form:

$$P(N, N, x) = \frac{\sum_{j=0}^{[(N-1)/2]} q_{2j+1} \sin(2j+1)x}{1 + \sum_{j=1}^{[N/2]} s_{2j} \cos(2jx)}, \tag{7}$$

where

$$q_{2j+1} = \frac{4}{\pi}(2j+1)\left[\frac{1}{(2j+1)^2} + \sum_{i=1}^{[N/2]} \frac{s_{2i}}{(2j+1)^2 - (2i)^2}\right],$$

$$s_{2i} = 2(-1)^i \frac{(N!)^4 (2N+2i)!(2N-2i)!}{(N-1)!(N+1)!(N-2i)!(N+2i)![(2N)!]^2}.$$

TABLE 1

Numerical results

N	$\bar{x}(N)$	$P(N, N, \bar{x})$
2	0·68	1·0736
3	0·41	1·0419
4	0·28	1·0301
5	0·31	1·0242
6	0·16	1·0208
7	0·13	1·0185
8	0·11	1·0166
9	0·09	1·0152
10	0·08	1·0138

The numerical results are presented in Table 1, where $\bar{x}$ satisfies the relation $\max P(N, N, x) = P(N, N, \bar{x})$. From Table 1 it is seen, that in the case of the application of the trigonometric diagonal approximants the Gibbs effect does not achieve 1%.

It should be noted that reference [7] has been devoted to the problem of the Padé approximants convergence and the Gibbs phenomena. Among others, the convergence of the Padé approximants to the sign(x) function and the essential decrease of the Gibbs phenomena in this case has been proved mathematically.

In order to illustrate the results the function $x(t) = \text{sign}(x)$ has been computed using series (6) and formula (7) (see also reference [6]) for $N = 10$ using the Mathematica package. The obtained solutions are shown in Figure 1. In addition, the function $x(t)$ has been calculated using series (3) and the diagonal Padé approximant using series (5) together with formulas (1)–(3) and the comments given in reference [6]. The obtained results are presented in Figure 2.

In conclusion one can say that in order to obtain the coefficients of the Padé approximants one needs to solve a system of linear algebraic equations. This requirement

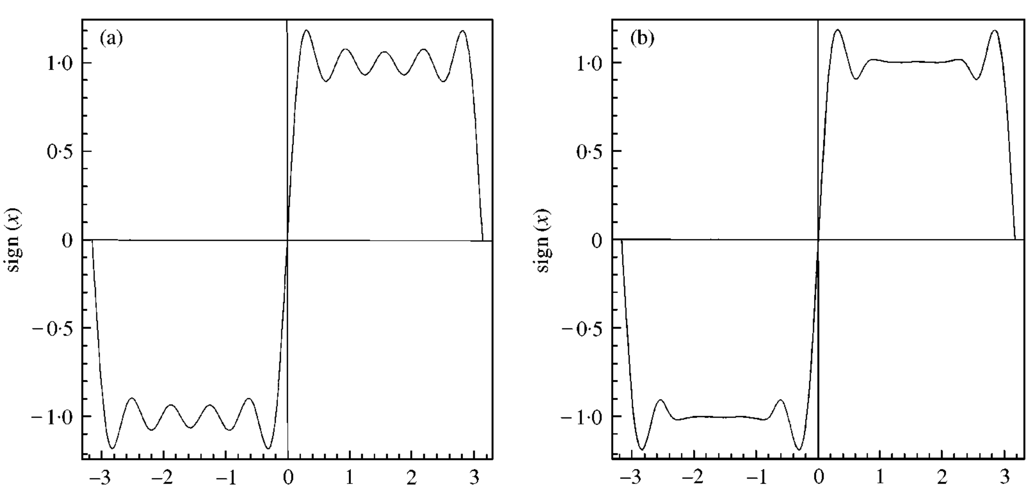

Figure 1. Approximations to $x(t) = \text{sign}(x)$ using (a) formula (6) and (b) formula (7).

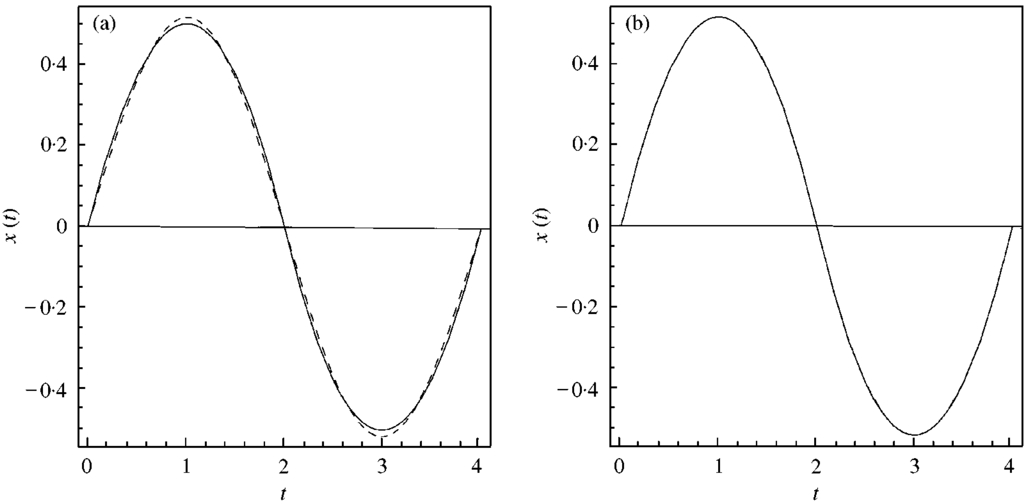

Figure 2. The approximation of the function $x(t) = \text{sign}(x)$ using series (3) (solid curve) and the diagonal Padé approximant using series (5) (dashed curve) for (a) one and (b) three terms of the series (5).

can lead, as in the previous case while summing the Fourier series, to an ill-posed problem. However, the Padé approximants possess the autocorrection (or self-correction) properties [10, 11], which omit this drawback.

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