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ANALYTICAL AND NUMERICAL STUDY OF A NON-STANDARD FINITE DIFFERENCE SCHEME FOR THE UNPLUGGED VAN DER POL EQUATION

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The van der Pol equation [1] corresponds to a non-linear oscillatory system that has both input and output sources of energy. This equation is given by the expression

$$\ddot{x} + x = \mu(1 - x^2)\dot{x}, \tag{1}$$

where μ is a non-negative parameter. For the case where no input in energy exists, equation (1) reduces to

$$\ddot{x} + x = -\mu x^2 \dot{x}. \tag{2}$$

This equation can be appropriately called the "unplugged" van der Pol equation and all of its solutions are expected to oscillate with decreasing amplitude to zero. The approach to zero of its solutions can be shown by using an energy argument [2]. Writing equation (2) in system form gives

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -x - \mu x^2 y. \tag{3a, b}$$

Multiplying the first equation by $2x$, the second equation by $2y$, and adding yields the result

$$\frac{\mathrm{d}r^2}{\mathrm{d}t} = -2\mu x^2 y^2, \qquad r^2 = x^2 + y^2. \tag{4}$$

Since the right-hand side of equation (4) is non-positive for all values of x and y, it follows that [2]

$$\operatorname*{Lim}_{t \to \infty} r^2 = 0 \Rightarrow x(t) \to 0, \qquad y(t) \to 0. \tag{5}$$

It is of interest to note that elementary methods of numerical integration, applied to the unplugged van der Pol equation (2), do not give results that are consistent with equation (5). In fact, the use of the forward Euler method leads to a scheme for which the fixed-point $(\bar{x}, \bar{y}) = (0, 0)$ is always unstable [3]. Thus, the dynamics of the discrete equations are inconsistent with those of the original differential equation (2).

The main purpose of this paper is to show that a dynamically consistent finite-difference scheme can be constructed for the unplugged van der Pol equation using the non-standard procedures investigated by Mickens [4, 5]. Further, this scheme requires a restriction on the step-size that depends on the initial conditions and the damping parameter μ.

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To begin with, equation (2) can be rewritten in an alternative system form of two coupled first order differential equations; they are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y - \left(\frac{\mu}{3}\right)x^3, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -x. \tag{6a, b}$$

It is this form that will be used in the work to follow. Applying the methods of Mickens [4, 5] to equations (6), the following finite-difference scheme is obtained:

$$\frac{x_{k+1} - x_k}{\phi} = y_k - \left(\frac{\mu}{3}\right)x_k^3, \qquad \frac{y_{k+1} - y_k}{\phi} = -x_{k+1}, \tag{7a, b}$$

where $t_k = hk$, with $h = \Delta t$; x_k is an approximation to $x(t_k)$; and ϕ is the denominator function [6] which is taken to be

$$\phi = 2\sin(h/2). \tag{8}$$

Note that to proceed with the numerical calculation, x_{k+1} is calculated from equation (7a) in terms of x_k and y_k; then y_{k+1} is determined using equation (7b) and the previously calculated x_{k+1} from equation (7a). The discrete model for the second order differential equation (2) can be obtained by eliminating y_k and y_{k+1} in equation (7); this leads to the expression

$$\frac{x_{k+1} - 2x_k + x_{k-1}}{\phi^2} + x_k = -\mu\left(\frac{x_k^2 + x_k x_{k-1} + x_{k-1}^2}{3}\right)\left(\frac{x_k - x_{k-1}}{\phi}\right). \tag{9}$$

Observe that the second-order derivative is replaced by a central difference representation, while the first order derivative is a backward Euler scheme. Of importance is the fact that the x^2 term is symmetric in the discrete functions x_k and x_{k-1}, i.e.,

$$x^2 \to \frac{x_k^2 + x_k x_{k-1} + x_{k-1}^2}{3}. \tag{10}$$

For small μ, i.e., $0 < \mu \ll 1$, the non-linear, second order difference equation (9) can be solved by means of a discrete form [7] of the method of slowly varying amplitude and phase [1]. The result is that [8]

$$x_k \to 0, \qquad k \to \infty. \tag{11}$$

In more detail, x_k oscillates with a damped amplitude. Thus, the finite-difference scheme, given by equation (9), has the same dynamical behavior as the original differential equation (2) for small μ.

For arbitrary but positive values of μ, the dynamical behavior of the solutions to equations (7) and (9) can be obtained by using these equations as devices for the numerical integration of equation (2). The following initial conditions are used in the work to come:

$$x(0) = x_0, \qquad y(0) = y_0 = 0. \tag{12}$$

To use equation (9), both x_0 and x_1 must be known. Since x_0 is given, along with y_0, x_1 can be determined by use of a Taylor series expansion,

$$x_1 = x(h) = x(0) + \dot{x}(0)h + O(h^2). \tag{13}$$

Using the initial conditions and the differential equation (2), it follows that

$$x(0) = x_0, \qquad \dot{x}(0) = -(\mu/3)\,x_0^3 \tag{14}$$

and

$$x_1 = x_0 \left[1 - (\mu x_0^2/3)h + O(h^2)\right]. \tag{15}$$

The physics of this case dictates that

$$|x_1| < |x_0|. \tag{16}$$

Further, an examination of equation (15) shows that if μ and/or x_0 are large enough, then the condition of equation (16) can be violated. In other words, there exists a critical step-size, h^*, and a constant, C, such that

$$h^* = \left(\frac{3}{\mu x_0^2}\right) C, \tag{17}$$

where it is expected that $C = O(1)$.

The value of C can be determined by the following procedure. Select particular values for h and x_0, and use equations (7) to obtain numerical values for x_k and y_k, at a given small value for the parameter μ. Increase the value of μ until overflow occurs. One can also fix h and μ, and increase the value of x_0 until overflow occurs. Using both procedures, it was found that h^* is given by the relation

$$h^* = 2\left(\frac{3}{\mu x_0^2}\right), \tag{18}$$

which corresponds to $C = 2$. (In general, to two significant decimal places, $C = 1{\cdot}99$ for all sets of (h, x_0, μ) studied.)

In actual numerical work, a value for the step-size close to

$$h = h^*/20, \tag{19}$$

would be used to make sure that both the initial rapid transient effects are resolved as well as the oscillatory behavior of the motion. Figure 1 presents a typical set of plots for values of (h, x_0, μ) that satisfy the condition $h < h^*$, where h^* is given by equation (18).

The following is a summary of the major results presented in this paper:

(a) A non-standard finite-difference scheme was constructed for the unplugged van der Pol equation. This discrete model resolves the difficulties [3] arising from the use of the forward Euler scheme for the derivatives in equations (6), i.e.,

$$\frac{x_{k+1} - x_k}{h} = y_k - (\mu/3)\, x_k^3, \qquad \frac{y_{k+1} - y_k}{h} = -\, x_k. \tag{20a, b}$$

Note that in addition to having the denominator function $\phi = 2\sin(h/2)$, in place of h in equations (20), the non-standard scheme, in equations (7), also replaces the x, on the right-hand side of equation (6b), by its value at the advanced time t_{k+1}. This is a very critical change; see Mickens [4] for a discussion of this issue.

(2) A major discovery was that of a critical step-size, h^*, whose value depended on both the damping parameter μ and the initial value x_0. For $h > h^*$, the numerical solution overflowed. To the best of our knowledge, this is the first time that such a restriction has been explicitly stated. It may be postulated that this situation is a generic feature for oscillatory problems.

(3) Future work on this topic will involve the finding of other examples of this phenomenon. A good set of equations to study comprises the Lotka–Volterra equations [9]. Previous work has shown that it is quite difficult to construct discrete finite-difference models that are dynamically consistent with the original differential equations [10].

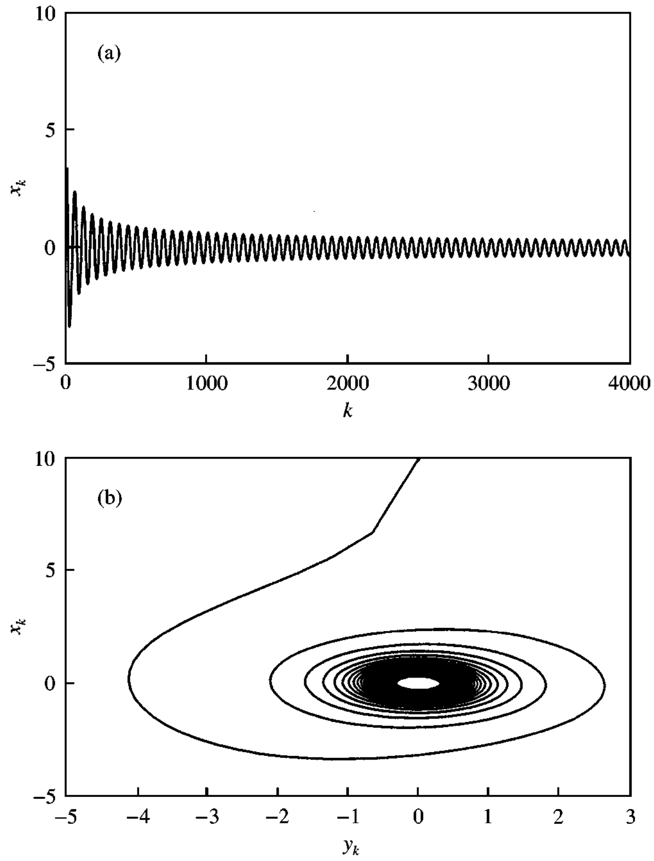

Figure 1. Plots of (a) x_k versus k and (b) x_k versus y_k for the parameter values $x_0 = 10$, $y_0 = 0$, $\mu = 0{\cdot}1$, $h = 0{\cdot}1$.

Finally, it should be indicated that all of the results found for the non-standard discretization of the system equations (6) generalize also to the system representation of equation (2) given by equations (3). For this case, the discrete system equations are

$$\frac{x_{k+1} - x_k}{\phi} = y_k, \qquad \frac{y_{k+1} - y_k}{\phi} = -\, x_{k+1} - \mu x_{k+1}^2 y_k, \tag{21a, b}$$

where $\phi = 2\sin(h/2)$ and the x-functions in equation (3b) are evaluated at the advanced time step, t_{k+1} [4]. The elimination of y_k results in the following second order difference equation for x_k:

$$\frac{x_{k+1} - 2x_k + x_{k-1}}{\phi^2} + x_k = -\, \mu x_k^2 \left(\frac{x_k - x_{k-1}}{\phi}\right). \tag{22}$$

The significant feature here is the use of a backward Euler representation for the first order derivative.

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