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# VIBRATION OF BEAMS WITH GENERALLY RESTRAINED BOUNDARY CONDITIONS USING FOURIER SERIES

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This work presents a method to find accurate vibration frequencies of beams with generally restrained boundary conditions using Fourier series. The suggested method is very convenient to find an accurate frequency parameter for beams with not only classical boundary conditions but also non-classical boundary conditions restrained by rotational and translational springs. Numerical results for various degenerate cases are compared with existing results for natural frequency obtained by conventional analysis. The results indicate that the present method can be used very effectively in the design of beams with various supporting conditions.



## 1. INTRODUCTION

Considerable research has been carried out with regard to the problem of free vibration of beams with elastical restraints. Chun [1] considered the free vibration of a Bernoulli–Euler beam hinged at one end by a rotational spring with constant spring stiffness and with the other end free. Maurizi [2] solved the problem of free vibration of a uniform beam hinged at one end by a rotational spring and subjected to the restraining action of a translational spring at the other end by using exact expression of trigonometric and hyperbolic function. Laura [3] obtained an approximate solution to the transverse vibrations of continuous beams subject to an axial force and carrying concentrated masses on elastic restraints by means of the classical Ritz method. Rao [4] derived exact frequency and normal mode shape expressions for generally restrained Bernoulli–Euler beams with unsymmetrical translations and rotations at either end. Abbas [5] has studied the problem of elastically restrained Timoshenko beams and presented some results for a degenerate case of Bernoulli–Euler beams. Rao [6] studied the free vibration and stability behavior of a simply supported uniform beam with non-linear elastic end restraints against rotation by using the standard finite-element formulation. Register [7] derived a general expression for the modal frequencies and investigated the eigenvalue for a beam with symmetric spring boundary conditions. Recently, Kang [8] studied the effects of the boundary stiffness and damping on modal parameters by using rotational and translational springs with complex stiffness. Grief [9, 10] presented a component mode method based on Fourier series for vibration of

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structures by using Lagrange's equations and Lagrange multipliers. Chung [11] presented a solution method for calculating the natural frequencies and modes of beams with any of the classical boundary conditions and with unlimited intermediate supports by using Fourier series in conjunction with Lagrange multipliers. Also, the applicability of Fourier series to the dynamic analysis of beams with arbitrary boundary conditions was studied by Wang [12].

In this study, the frequency equations for Bernoulli–Euler beams with general restraints are derived in matrix form by using Fourier series. The matrix form frequency equations make it very easy to solve the beams problems with generally restrained boundary conditions as well as classical boundary conditions by assigning appropriate restraint constants. For the purpose of illustration, several degenerate cases are considered. The numerical results are also presented in three-dimensional plots to show the effects of restraints on the natural frequencies.

## 2. GENERAL THEORETICAL FORMULATIONS

Consider a uniform Bernoulli–Euler beam with elastically restrained ends. The restraints are provided by either a translational or a rotational spring, or both at its ends. The equation of motion for free flexural vibrations of a uniform elastic beam ignoring shear deformation and rotary inertia effects is

$$EI \frac{\partial^4 w(x, t)}{\partial x^4} + \rho A \frac{\partial^2 w(x, t)}{\partial t^2} = 0, \tag{1}$$

where $w(x, t)$ is the lateral displacement at distance $x$ along the length of the beam and time $t$, $EI$ the flexural rigidity of the beam, $\rho$ the mass density and $A$ the cross-sectional area of the beam.

For any mode of vibration, the lateral displacement $w(x, t)$ may be written in the form

$$w(x, t) = \psi(x) \cos \omega t, \tag{2}$$

where $\psi(x)$ is the modal displacement function and $\omega$ the natural frequency. The function $\psi(x)$ may be written either as a Fourier sine series or as a cosine series. In the present study, we consider a Fourier sine series as a mode function. The function is defined in two separate regions, one for boundary points and the other for the intermediate region between the boundary points as follows:

$$\psi(x) = \begin{cases} \psi_0, & x = 0, \\ \psi_L, & x = L, \\ \sum_{m=1}^{\infty} A_m \sin \dfrac{m\pi x}{L}, & 0 < x < L. \end{cases} \tag{3}$$

Since direct differentiation of a Fourier sine series leads to a cosine series without the constant term, it is not considered to be a complete set of functions. To obtain correct series expressions for derivatives of a Fourier series, Stoke's transformation must be employed. Stoke's transformation consists of defining each derivative with an independent series and of integrating by parts the newly defined series to obtain the relationship between the Fourier coefficients.

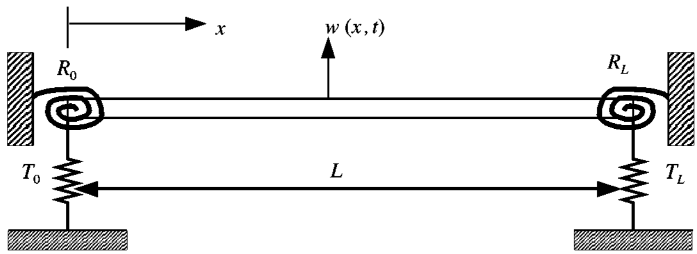


Figure 1. An elastic Bernoulli–Euler beam with rotational and translational restraints.

The derivatives of $\psi(x)$ based on the usual definitions of Fourier series become

$$\frac{\mathrm{d}\psi(x)}{\mathrm{d}x} = \frac{\psi_L - \psi_0}{L} + \sum_{m=1}^{\infty}\left[\frac{2}{L}\{\psi_L(-1)^m - \psi_0\} + \alpha_m A_m\right]\cos\alpha_m x, \quad 0 \leqslant x \leqslant L, \tag{4}$$

$\alpha_m = m\pi/L$,

$$\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} = -\sum_{m=1}^{\infty}\alpha_m\left[\frac{2}{L}\{\psi_L(-1)^m - \psi_0\} + \alpha_m A_m\right]\sin\alpha_m x, \quad 0 < x < L,$$

$$\psi''(0) = \psi''_0, \quad \psi''(L) = \psi''_L, \tag{5}$$

$$\frac{\mathrm{d}^3\psi(x)}{\mathrm{d}x^3} = \frac{\psi''_L - \psi''_0}{L} + \sum_{m=1}^{\infty}\left[\frac{2}{L}\{\psi''_L(-1)^m - \psi''_0\} - \alpha_m^2\left(\frac{2}{L}\{\psi_L(-1)^m - \psi_0\} + \alpha_m A_m\right)\right]$$
$$\times\cos\alpha_m x, \quad 0 \leqslant x \leqslant L \tag{6}$$

and

$$\frac{\mathrm{d}^4\psi(x)}{\mathrm{d}x^4} = -\sum_{m=1}^{\infty}\alpha_m\left[\frac{2}{L}\{\psi''_L(-1)^m - \psi''_0\} - \alpha_m^2\left(\frac{2}{L}\psi_L(-1)^m - \psi_0\} + \alpha_m A_m\right)\right]\times\sin\alpha_m x,$$
$$0 < x < L. \tag{7}$$

The function can also be represented by Fourier cosine series in a similar manner.

To obtain a general expression for the flexure of beams, we consider free–free beam having both rotational and translational springs at its ends, as shown in Figure 1. Substitution of equations (2)–(7) into equation (1) results in the equation

$$\sum_{m=1}^{\infty}\left[-\omega^2\psi(x) - \frac{EI}{\rho A}\alpha_m\left\{\frac{2}{L}(\psi''_L(-1)^m - \psi''_0) - \alpha_m^2\left(\frac{2}{L}(\psi_L(-1)^m - \psi_0) + \alpha_m A_m\right)\right\}\sin\alpha_m x\right]$$
$$\times\cos\omega t = 0, \tag{8}$$

where $\omega$ is the natural circular frequency of the system. Then, we can rewrite above equation (8) as follows:

$$\sum_{m=1}^{\infty}\left[\left(\frac{EI\alpha_m^4}{\rho A} - \omega^2\right)A_m + \frac{2EI}{\rho AL}\{\alpha_m(\psi''_0 - (-1)^m\psi''_L) + \alpha_m^3(\psi_L(-1)^m - \psi_0)\}\right]\sin\alpha_m\cos\omega t = 0. \tag{9}$$

Therefore, the coefficient $A_m$ can be written in terms of $\psi''_0$, $\psi''_L$, $\psi_0$, and $\psi_L$ as follows:

$$A_m = \sum_{m=1}^{\infty} \frac{2}{\alpha_m^3 L} \frac{\omega_n^2}{\omega^2 - \omega_n^2} \{(\psi''_0 - (-1)^m \psi''_L) - \alpha_m^2 (\psi_0 - (-1)^m \psi_L)\}, \quad \omega_n^2 = \frac{EI}{\rho A} \alpha_m^4. \quad (10)$$

Hence, the displacement function for the free vibration of a beam having no geometrical constraints at both ends becomes

$$w(x, t) = \sum_{m=1}^{\infty} \frac{2}{\alpha_m^3 L} \frac{\omega_n^2}{\omega^2 - \omega_n^2} \{(\psi''_0 - (-1)^m \psi''_L) - \alpha_m^2 (\psi_0 - (-1)^m \psi_L)\} \sin \alpha_m x \cos \omega t. \quad (11)$$

The elastically restrained boundary conditions of the beam shown in Figure 1 are as follows:

$$T_0 w_0 = -EI \frac{\partial^3 w}{\partial x^3}, \qquad R_0 \frac{\partial w}{\partial x} = EI \frac{\partial^2 w}{\partial x^2} \quad \text{at } x = 0, \quad (12, 13)$$

$$T_L w_L = EI \frac{\partial^3 w}{\partial x^3}, \qquad R_L \frac{\partial w}{\partial x} = -EI \frac{\partial^2 w}{\partial x^2} \quad \text{at } x = L \quad (14, 15)$$

in which $T_0$ and $T_L$ are translational spring constants, and $R_0$ and $R_L$ are rotational spring constants at $x = 0$ and $L$ respectively.

The substitution of equations (4) and (6) into equations (12)–(15) leads to the four simultaneous homogeneous equations

$$-\left(1 + 2\sum_{m=1}^{\infty} \frac{\lambda^4}{\lambda^4 - m^4}\right)\psi''_0 + \left(1 + 2\sum_{m=1}^{\infty} \frac{(-1)^m \lambda^4}{\lambda^4 - m^4}\right)\psi''_L + \left(\bar{T}_0 + 2\pi^2 \sum_{m=1}^{\infty} \frac{m^2 \lambda^4}{\lambda^4 - m^4}\right)\frac{\psi_0}{L^2}$$
$$-\left(2\pi^2 \sum_{m=1}^{\infty} \frac{(-1)^m m^2 \lambda^4}{\lambda^4 - m^4}\right)\frac{\psi_L}{L^2} = 0, \quad (16)$$

$$\left(1 + 2\sum_{m=1}^{\infty} \frac{(-1)^m \lambda^4}{\lambda^4 - m^4}\right)\psi''_0 - \left(1 + 2\sum_{m=1}^{\infty} \frac{\lambda^4}{\lambda^4 - m^4}\right)\psi''_L - \left(2\pi^2 \sum_{m=1}^{\infty} \frac{(-1)^m m^2 \lambda^4}{\lambda^4 - m^4}\right)\frac{\psi_0}{L^2}$$
$$+\left(\bar{T}_L + 2\pi^2 \sum_{m=1}^{\infty} \frac{m^2 \lambda^4}{\lambda^4 - m^4}\right)\frac{\psi_L}{L^2} = 0, \quad (17)$$

$$\left(1 - \frac{2\bar{R}_0}{\pi^2}\sum_{m=1}^{\infty} \frac{m^2}{\lambda^4 - m^4}\right)\psi''_0 + \left(\frac{2\bar{R}_0}{\pi^2}\sum_{m=1}^{\infty} \frac{(-1)^m m^2}{\lambda^4 - m^4}\right)\psi''_L + \left(\bar{R}_0 + 2\bar{R}_0 \sum_{m=1}^{\infty} \frac{\lambda^4}{\lambda^4 - m^4}\right)\frac{\psi_0}{L^2}$$
$$-\left(\bar{R}_0 + 2\bar{R}_0 \sum_{m=1}^{\infty} \frac{(-1)^m \lambda^4}{\lambda^4 - m^4}\right)\frac{\psi_L}{L^2} = 0, \quad (18)$$

$$\left(\frac{2\bar{R}_L}{\pi^2}\sum_{m=1}^{\infty} \frac{(-1)^m m^2}{\lambda^4 - m^4}\right)\psi''_0 + \left(1 - \frac{2\bar{R}_L}{\pi^2}\sum_{m=1}^{\infty} \frac{m^2}{\lambda^4 - m^4}\right)\psi''_L - \left(\bar{R}_L + 2\bar{R}_L \sum_{m=1}^{\infty} \frac{(-1)^m \lambda^4}{\lambda^4 - m^4}\right)\frac{\psi_0}{L^2}$$
$$+\left(\bar{R}_L + 2\bar{R}_L \sum_{m=1}^{\infty} \frac{\lambda^4}{\lambda^4 - m^4}\right)\frac{\psi_L}{L^2} = 0, \quad (19)$$

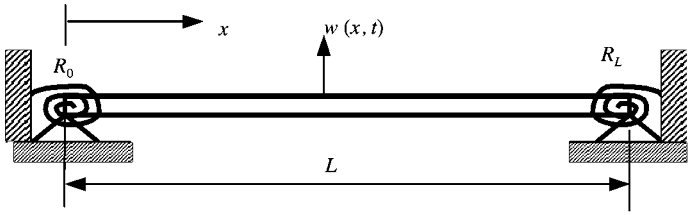


Figure 2. SS beam with rotational restraints.

where

$$\bar{T}_0 = \frac{T_0 L^3}{EI}, \quad \bar{T}_L = \frac{T_L L^3}{EI}, \quad \bar{R}_0 = \frac{R_0 L}{EI}, \quad \bar{R}_L = \frac{R_L L}{EI},$$
$$\omega_n^2 = \frac{EI}{\rho A}\left(\frac{m^4\pi^4}{L^4}\right), \quad \lambda^4 = \frac{\rho A L^4}{\pi^4 EI}\,\omega^2 = \frac{\beta^4}{\pi^4}. \tag{20}$$

For a non-trivial solution, the determinant of the coefficient matrix of equations (16)–(19) must vanish, i.e.,

$$|S_{i,j}| = 0 \quad (i, j = 1, 2, 3, 4). \tag{21}$$

Each element of this determinant is shown in Appendix A. From the determinant frequency, the equation of an elastic Bernoulli–Euler beam with rotational and translational restraints is obtained.

## 3. DEGENERATE CASES

Several degenerate cases are considered to show that the determinant derived in this paper can apply to classical boundary conditions and non-classical boundary conditions restrained by general springs.

### 3.1. SS BEAM WITH ROTATIONAL RESTRAINTS

For a simply supported beam with rotational springs at both ends as shown in Figure 2, the resulting boundary conditions are

$$\psi_0 = 0, \qquad R_0\frac{\partial\psi}{\partial x} = EI\frac{\partial^2\psi}{\partial x^2} \quad \text{at } x = 0,$$
$$\psi_L = 0, \qquad R_L\frac{\partial\psi}{\partial x} = -\,EI\frac{\partial^2\psi}{\partial x^2} \quad \text{at } x = L. \tag{22}$$

Hence, the frequency determinant is found from equations (18) and (19) by retaining the rows and columns associated with $\psi_0''$ and $\psi_L''$ of rotational restraints. The resulting frequency determinant is

$$\begin{vmatrix} s_{3,1} & s_{3,2} \\ s_{4,1} & s_{4,2} \end{vmatrix} = 0 \tag{23}$$

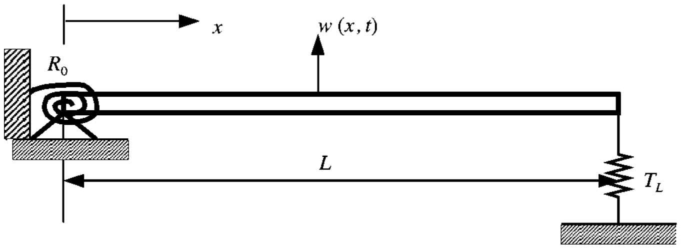


Figure 3. SF beam with rotational and translational springs.

or we can get same result from equation (21) by just putting $\bar{T}_0 \to \infty$ and $\bar{T}_L \to \infty$ instead of equation (23).

### 3.2. SF BEAM WITH ROTATIONAL SPRING AT SIMPLY SUPPORTED END AND SUBJECTED TO A TRANSLATIONAL RESTRAINT AT THE FREE END

For an SF beam with a rotational spring at the simply supported end and subjected to a translational restraint at the free end as shown in Figure 3, the resulting boundary conditions are

$$\begin{aligned} &\psi_0 = 0, \qquad R_0 \frac{\partial \psi}{\partial x} = EI \frac{\partial^2 \psi}{\partial x^2} \quad \text{at } x = 0, \\ &\frac{\partial^2 \psi}{\partial x^2} = 0, \qquad T_L \psi_L = EI \frac{\partial^3 \psi}{\partial x^3} \quad \text{at } x = L. \end{aligned} \tag{24}$$

In this case the frequency determinant is found from equations (17) and (18) by retaining the rows and columns associated with $\psi_0''$ and $\psi_L''$. The resulting frequency determinant is

$$\begin{vmatrix} s_{2,1} & s_{2,4} \\ s_{3,1} & s_{3,4} \end{vmatrix} = 0 \tag{25}$$

or we can get the same result from equation (21) by just putting $\bar{T}_0 \to \infty$ and $\bar{R}_L = 0$ instead of equation (25).

### 3.3. SF BEAM WITH ROTATIONAL SPRING AT SIMPLY SUPPORTED END AND POINT MASS AT FREE END

For an SF beam with a rotational spring at the simply supported end and point mass at the free end as shown in Figure 4, the resulting boundary conditions are

$$\begin{aligned} &\psi_0 = 0, \qquad R_0 \frac{\partial \psi}{\partial x} = EI \frac{\partial^2 \psi}{\partial x^2} \quad \text{at } x = 0, \\ &\frac{\partial^2 \psi}{\partial x^2} = 0, \qquad -M \frac{\partial^2 \psi}{\partial t^2} = EI \frac{\partial^3 \psi}{\partial x^3} \quad \text{at } x = L. \end{aligned} \tag{26}$$

In this case the frequency determinant is also found from equations (17) and (18) by retaining the rows and columns associated with $\psi_0''$ and $\psi_L''$ similar as in the previous

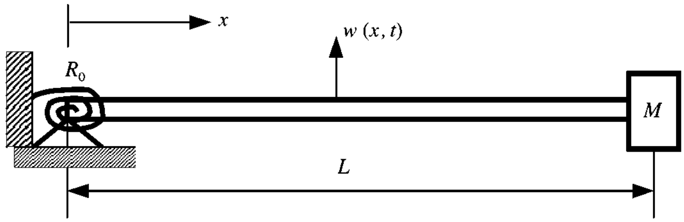


Figure 4. SF beam with rotational spring and point mass.

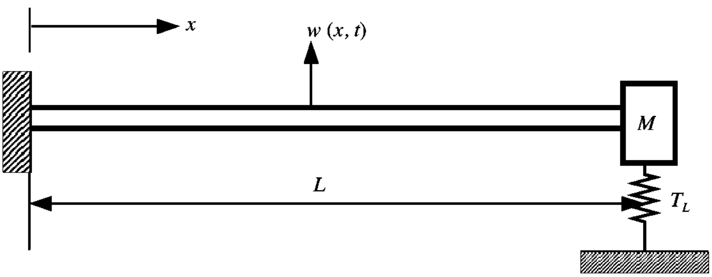


Figure 5. CF beam with point mass and translational spring.

example except for replacing $\bar{T}_L$ with $-\bar{M}(\pi^4\lambda^4)$ in equation (17). The resulting frequency determinant is

$$\begin{vmatrix} \left(1+2\sum_{m=1}^{\infty}\frac{(-1)^m\lambda^4}{\lambda^4-m^4}\right) & \left(-\bar{M}(\pi^4\lambda^4)+2\pi^2\sum_{m=1}^{\infty}\frac{m^2\lambda^4}{\lambda^4-m^4}\right) \\ \left(1-\frac{2\bar{R}_0}{\pi^2}\sum_{m=1}^{\infty}\frac{m^2}{\lambda^4-m^4}\right) & -\left(\bar{R}_0+2\bar{R}_0\sum_{m=1}^{\infty}\frac{(-1)^m\lambda^4}{\lambda^4-m^4}\right) \end{vmatrix}=0, \tag{27}$$

where

$$\bar{M}=\frac{M}{\rho AL}. \tag{28}$$

Of course we can get the same result from equation (21) by just putting $\bar{T}_0\to\infty$, $\bar{R}_L=0$ and $\bar{T}_L\to-\bar{M}(\pi^4\lambda^4)$ instead of equation (27).

### 3.4. CF BEAM WITH POINT MASS AND TRANSLATIONAL SPRING RESTRAINT AT FREE END

For an CF beam with point mass and translational spring restraint at the free end as shown in Figure 5, the boundary conditions are

$$\begin{gathered} \psi_0=0, \qquad \frac{\partial\psi}{\partial x}=0, \quad \text{at } x=0, \\ \frac{\partial^2\psi}{\partial x^2}=0, \qquad T_L\psi_L-M\frac{\partial^2\psi}{\partial t^2}=EI\frac{\partial^3\psi}{\partial x^3}, \quad \text{at } x=L. \end{gathered} \tag{29}$$

TABLE 1

*The first natural frequency parameter ($\beta_1$) for FF beam with symmetric spring boundary conditions ($R_0 = R_L = R$, $T_0 = T_L = T$, $\bar{R} = RL/EI$, $\bar{T} = TL^3/EI$)*

| $\bar{T}$ | $\bar{R}$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 0·01 | 0·1 | 1 | 10 | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
| 0·01 | 0·377 | 0·377 | 0·377 | 0·377 | 0·377 | 0·377 | 0·377 | 0·377 | 0·377 |
| 0·1 | 0·670 | 0·670 | 0·670 | 0·670 | 0·670 | 0·670 | 0·670 | 0·670 | 0·670 |
| 1 | 1·187 | 1·187 | 1·188 | 1·190 | 1·191 | 1·191 | 1·191 | 1·191 | 1·191 |
| 10 | 2·036 | 2·038 | 2·057 | 2·093 | 2·103 | 2·105 | 2·105 | 2·105 | 2·105 |
| $10^2$ | 2·879 | 2·897 | 3·031 | 3·369 | 3·510 | 3·528 | 3·530 | 3·531 | 3·531 |
| $10^3$ | 3·114 | 3·141 | 3·355 | 4·056 | 4·489 | 4·558 | 4·566 | 4·566 | 4·567 |
| $10^4$ | 3·142 | 3·169 | 3·395 | 4·158 | 4·656 | 4·739 | 4·747 | 4·748 | 4·748 |
| $10^5$ | 3·144 | 3·172 | 3·399 | 4·169 | 4·674 | 4·757 | 4·766 | 4·767 | 4·767 |
| $10^6$ | 3·145 | 3·173 | 3·400 | 4·170 | 4·676 | 4·759 | 4·768 | 4·769 | 4·769 |

TABLE 2

*The second natural frequency parameter ($\beta_2$) for FF beam with symmetric spring boundary conditions ($R_0 = R_L = R$, $T_0 = T_L = T$, $\bar{R} = RL/EI$, $\bar{T} = TL^3/EI$)*

| $\bar{T}$ | $\bar{R}$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 0·01 | 0·1 | 1 | 10 | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
| 0·01 | 0·744 | 1·253 | 2·111 | 2·909 | 3·136 | 3·164 | 3·167 | 3·167 | 3·167 |
| 0·1 | 0·963 | 1·317 | 2·125 | 2·913 | 3·139 | 3·167 | 3·170 | 3·170 | 3·170 |
| 1 | 1·589 | 1·706 | 2·247 | 2·954 | 3·168 | 3·195 | 3·198 | 3·198 | 3·198 |
| 10 | 2·785 | 2·804 | 2·950 | 3·290 | 3·424 | 3·442 | 3·444 | 3·444 | 3·444 |
| $10^2$ | 4·681 | 4·681 | 4·681 | 4·681 | 4·681 | 4·681 | 4·681 | 4·681 | 4·681 |
| $10^3$ | 6·039 | 6·049 | 6·137 | 6·540 | 6·892 | 6·956 | 6·963 | 6·964 | 6·964 |
| $10^4$ | 6·260 | 6·274 | 6·398 | 7·026 | 7·671 | 7·801 | 7·815 | 7·817 | 7·817 |
| $10^5$ | 6·282 | 6·296 | 6·425 | 7·078 | 7·755 | 7·891 | 7·906 | 7·908 | 7·908 |
| $10^6$ | 6·285 | 6·299 | 6·427 | 7·084 | 7·763 | 7·900 | 7·915 | 7·917 | 7·917 |

TABLE 3

*The third natural frequency parameter ($\beta_3$) for FF beam with symmetric spring boundary conditions ($R_0 = R_L = R$, $T_0 = T_L = T$, $\bar{R} = RL/EI$, $\bar{T} = TL^3/EI$)*

| $\bar{T}$ | $\bar{R}$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 0·01 | 0·1 | 1 | 10 | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
| 0·01 | 4·774 | 4·809 | 5·089 | 5·885 | 6·273 | 6·328 | 6·334 | 6·335 | 6·335 |
| 0·1 | 4·775 | 4·811 | 5·090 | 5·885 | 6·274 | 6·329 | 6·334 | 6·335 | 6·335 |
| 1 | 4·793 | 4·828 | 5·103 | 5·892 | 6·278 | 6·332 | 6·338 | 6·338 | 6·339 |
| 10 | 4·960 | 4·990 | 5·232 | 5·953 | 6·316 | 6·368 | 6·374 | 6·374 | 6·374 |
| $10^2$ | 6·130 | 6·139 | 6·216 | 6·508 | 6·694 | 6·723 | 6·726 | 6·726 | 6·726 |
| $10^3$ | 8·583 | 8·585 | 8·603 | 8·692 | 8·781 | 8·799 | 8·801 | 8·801 | 8·801 |
| $10^4$ | 9·341 | 9·350 | 9·430 | 9·921 | 10·612 | 10·780 | 10·799 | 10·801 | 10·801 |
| $10^5$ | 9·417 | 9·427 | 9·515 | 10·066 | 10·849 | 11·038 | 11·060 | 11·062 | 11·062 |
| $10^6$ | 9·425 | 9·434 | 9·524 | 10·080 | 10·872 | 11·063 | 11·085 | 11·087 | 11·087 |

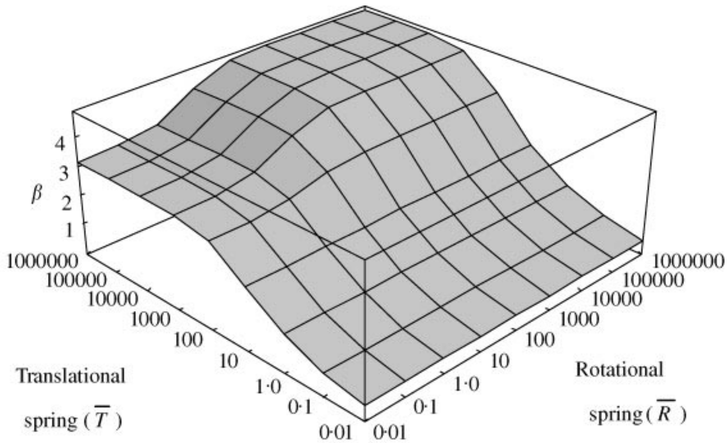


Figure 6. Plot of the first natural frequency parameter ($\beta_1$) for FF beam with symmetric spring boundary conditions ($R_0 = R_L = R$, $T_0 = T_L = T$, $\bar{R} = RL/EI$, $\bar{T} = TL^3/EI$).

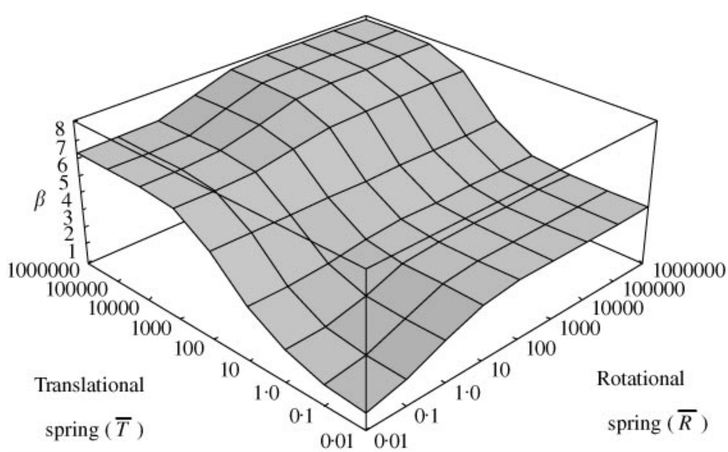


Figure 7. Plot of the second natural frequency parameter ($\beta_2$) for FF beam with symmetric spring boundary conditions ($R_0 = R_L = R$, $T_0 = T_L = T$, $\bar{R} = RL/EI$, $\bar{T} = TL^3/EI$).

The resulting frequency determinant is found from equations (4) and (17) by deleting $\psi_0$ and $\psi''_L$ and replacing $\bar{T}_L$ with $\bar{T}_L - \bar{M}(\pi^4 \lambda^4)$. The resulting frequency determinant is

$$\begin{vmatrix} \left(1 + 2\sum_{m=1}^{\infty} \frac{(-1)^m \lambda^4}{\lambda^4 - m^4}\right) & \left(\bar{T}_L - \bar{M}(\pi^4\lambda^4) + 2\pi^2 \sum_{m=1}^{\infty} \frac{m^2 \lambda^4}{\lambda^4 - m^4}\right) \\ \left(\frac{2}{\pi^2} \sum_{m=1}^{\infty} \frac{m^2}{\lambda^4 - m^4}\right) & \left(1 + 2\sum_{m=1}^{\infty} \frac{(-1)^m \lambda^4}{\lambda^4 - m^4}\right) \end{vmatrix} = 0 \tag{30}$$

or we can get the same results from equation (21) by putting $\bar{T}_0 \to \infty$, $\bar{R}_0 \to \infty$, $\bar{R}_L = 0$ and $\bar{T}_L = (T_L L^3/EI) - \bar{M}(\pi^4 \lambda^4)$.

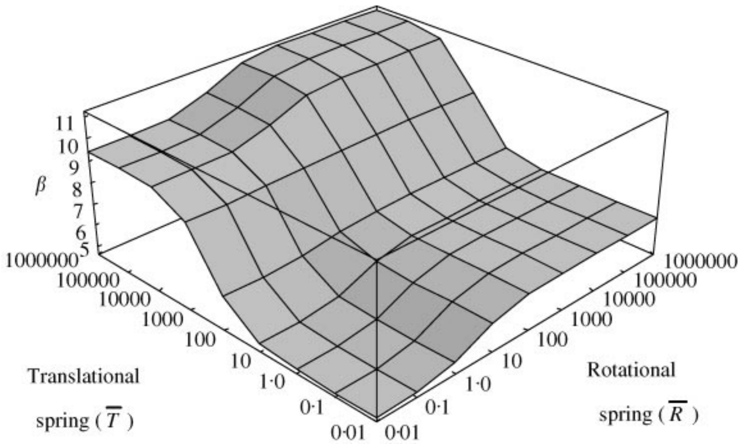


Figure 8. Plot of the third natural frequency parameter ($\beta_3$) for FF beam with symmetric spring boundary conditions ($R_0 = R_L = R$, $T_0 = T_L = T$, $\bar{R} = RL/EI$, $\bar{T} = TL^3/EI$).

TABLE 4

*The first natural frequency parameter ($\beta_1$) for CF beam with translational spring and point mass at free end ($\bar{T}_L = T_L L^3/EI$, $\bar{M} = M/\rho AL$)*

| | Point mass ($\bar{M}$) | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\bar{T}_L$ | 0 | 0·01 | 0·1 | 1 | 10 | $10^2$ | $10^3$ | $10^4$ | $10^5$ |
| 0 | 1·8766 | 1·8583 | 1·7239 | 1·2484 | 0·7360 | 0·4161 | 0·2341 | 0·1316 | 0·0740 |
| 0·01 | 1·8781 | 1·8598 | 1·7253 | 1·2495 | 0·7366 | 0·4164 | 0·2343 | 0·1318 | 0·0741 |
| 0·1 | 1·8916 | 1·8731 | 1·7379 | 1·2587 | 0·7421 | 0·4195 | 0·2360 | 0·1327 | 0·0746 |
| 1 | 2·0115 | 1·9922 | 1·8500 | 1·3413 | 0·7908 | 0·4471 | 0·2515 | 0·1415 | 0·0795 |
| 10 | 2·6403 | 2·6192 | 2·4550 | 1·7992 | 1·0617 | 0·6002 | 0·3377 | 0·1899 | 0·1068 |
| $10^2$ | 3·6418 | 3·6370 | 3·5920 | 2·9847 | 1·7809 | 1·0068 | 0·5665 | 0·3186 | 0·1791 |
| $10^3$ | 3·8993 | 3·8992 | 3·8986 | 3·8907 | 3·1397 | 1·7785 | 1·0007 | 0·5628 | 0·3165 |
| $10^4$ | 3·9253 | 3·9253 | 3·9253 | 3·9252 | 3·9244 | 3·1600 | 1·7783 | 1·0001 | 0·5624 |
| $10^5$ | 3·9278 | 3·9278 | 3·9278 | 3·9278 | 3·9278 | 3·9278 | 3·1620 | 1·7783 | 1·0000 |
| $10^6$ | 3·9281 | 3·9281 | 3·9281 | 3·9281 | 3·9281 | 3·9281 | 3·9281 | 3·1623 | 1·7783 |

## 4. NUMERICAL ANALYSIS

Numerical investigations have been performed to confirm the validity of the present formulation for a uniform Bernoulli–Euler beam with elastically restrained ends as well as typical classical boundary conditions.

The frequency determinants derived through the present formulation involve an infinite series of algebraic terms. Therefore, the accuracy of the solution depends on the number of terms in the series. The sensitivity of the number of terms was investigated, but are not presented in this paper. For most cases, 50 terms were enough to get desirable accuracy. The natural frequencies are obtained by monitoring values of the frequency parameter $\beta_i$ until the determinant goes to zero.

The eigenvalues for the FF (free–free) beam with symmetrical spring boundary conditions as shown in Figure 1 are provided in Tables 1, 2 and 3. Figures 6, 7, and 8 are the

TABLE 5

*The second natural frequency parameter ($\beta_2$) for CF beam with translational spring and point mass at free end ($\bar{T}_L = T_L L^3/EI$, $\bar{M} = M/\rho AL$)*

| | Point mass ($\bar{M}$) | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\bar{T}_L$ | 0 | 0·01 | 0·1 | 1 | 10 | $10^2$ | $10^3$ | $10^4$ | $10^5$ |
| 0 | 4·6978 | 4·6532 | 4·4019 | 4·0327 | 3·9400 | 3·9293 | 3·9283 | 3·9281 | 3·9281 |
| 0·01 | 4·6979 | 4·6533 | 4·4019 | 4·0327 | 3·9400 | 3·9293 | 3·9283 | 3·9281 | 3·9281 |
| 0·1 | 4·6988 | 4·6541 | 4·4024 | 4·0327 | 3·9400 | 3·9293 | 3·9283 | 3·9281 | 3·9281 |
| 1 | 4·7075 | 4·6622 | 4·4068 | 4·0331 | 3·9400 | 3·9293 | 3·9283 | 3·9281 | 3·9281 |
| 10 | 4·7977 | 4·7463 | 4·4537 | 4·0362 | 3·9400 | 3·9293 | 3·9283 | 3·9281 | 3·9281 |
| $10^2$ | 5·6201 | 5·5513 | 5·0541 | 4·0803 | 3·9405 | 3·9293 | 3·9283 | 3·9281 | 3·9281 |
| $10^3$ | 6·8787 | 6·8741 | 6·8218 | 5·5014 | 3·9481 | 3·9294 | 3·9283 | 3·9281 | 3·9281 |
| $10^4$ | 7·0532 | 7·0532 | 7·0527 | 7·0473 | 5·6095 | 3·9302 | 3·9283 | 3·9281 | 3·9281 |
| $10^5$ | 7·0693 | 7·0693 | 7·0693 | 7·0693 | 7·0687 | 5·6220 | 3·9283 | 3·9281 | 3·9281 |
| $10^6$ | 7·0709 | 7·0709 | 7·0709 | 7·0709 | 7·0709 | 7·0709 | 5·6233 | 3·9282 | 3·9281 |

TABLE 6

*The third natural frequency parameter ($\beta_3$) for CF beam with translational spring and point mass at free end ($\bar{T}_L = T_L L^3/EI$, $\bar{M} = M/\rho AL$)*

| | Point mass ($\bar{M}$) | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\bar{T}_L$ | 0 | 0·01 | 0·1 | 1 | 10 | $10^2$ | $10^3$ | $10^4$ | $10^5$ |
| 0 | 7.8601 | 7.7880 | 7.4544 | 7.1367 | 7.0781 | 7.0718 | 7.0712 | 7.0711 | 7.0711 |
| 0.01 | 7.8601 | 7.7880 | 7.4544 | 7.1367 | 7.0781 | 7.0718 | 7.0712 | 7.0711 | 7.0711 |
| 0.1 | 7.8609 | 7.7882 | 7.4545 | 7.1367 | 7.0781 | 7.0718 | 7.0712 | 7.0711 | 7.0711 |
| 1 | 7.8627 | 7.7898 | 7.4551 | 7.1367 | 7.0781 | 7.0718 | 7.0712 | 7.0711 | 7.0711 |
| 10 | 7.8816 | 7.8064 | 7.4611 | 7.1369 | 7.0781 | 7.0718 | 7.0712 | 7.0711 | 7.0711 |
| $10^2$ | 8.0909 | 7.9920 | 7.5294 | 7.1391 | 7.0782 | 7.0718 | 7.0712 | 7.0711 | 7.0711 |
| $10^3$ | 9.5572 | 9.4983 | 8.8105 | 7.1718 | 7.0782 | 7.0718 | 7.0712 | 7.0711 | 7.0711 |
| $10^4$ | 10.1580 | 10.1574 | 10.1512 | 9.6803 | 7.0827 | 7.0718 | 7.0712 | 7.0711 | 7.0711 |
| $10^5$ | 10.2079 | 10.2079 | 10.2078 | 10.2073 | 9.9430 | 7.0723 | 7.0712 | 7.0711 | 7.0711 |
| $10^6$ | 10.2127 | 10.2127 | 10.2127 | 10.2127 | 10.2127 | 9.9932 | 7.0712 | 7.0711 | 7.0711 |

three-dimensional plots of Tables 1, 2 and 3 respectively. These values were obtained from equation (21) using the first 50 terms of the infinite series. The present results obtained with he first 50 terms and those of reference [7] show quite a good agreement. However, if we compare these results in more detail, we find that the present results with the first 50 terms are slightly higher than those of reference [7] with an accuracy of 0–0·88% deviation. From an engineering point of view, the accuracy will be considered sufficient. If we need a more accurate result, however, it will be easily achieved by considering more terms of the series.

The eigenvalues for the CF (clamped-free) beam with point mass and translational spring boundary conditions as shown in Figure 5 are provided in Tables 4, 5 and 6. Figures 9, 10 and 11 are the three-dimensional plots of Tables 4, 5 and 6 respectively. These values were obtained from equation (21) using the first 500 terms of the infinite series. $\bar{T}_0 = 1·0E6$, $\bar{R}_0 = 1·0E6$ were assigned to simulate fixed condition at one end. All of these calculations were performed with MATHEMATICA.

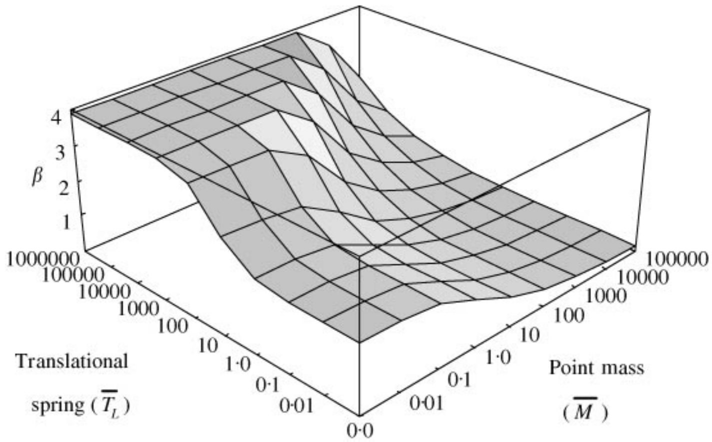


Figure 9. Plot of the first natural frequency parameter ($\beta_1$) for CF beam with translational spring and point mass at free end ($\bar{T}_L$, $\bar{M}$).

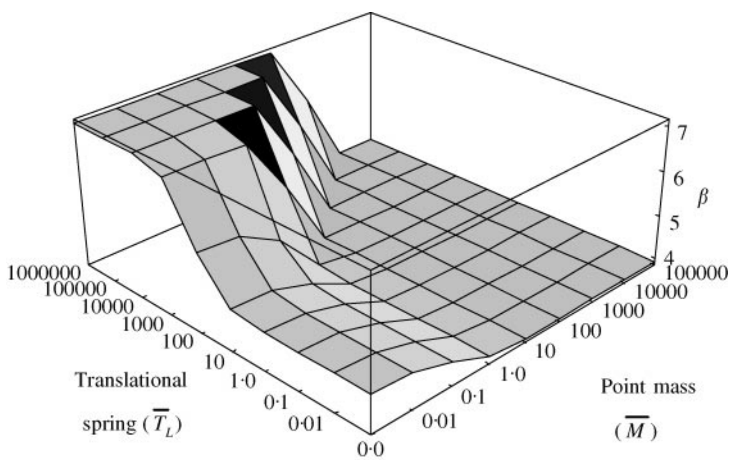


Figure 10. Plot of the second natural frequency parameter ($\beta_2$) for CF beam with translational spring and point mass at free end ($\bar{T}_L$, $\bar{M}$).

Besides the cases whose results have been presented in this paper, the various degenerate cases have been investigated and showed good agreement with the previous results shown in references [12–14], although not presented here.

## 5. CONCLUSIONS

The frequency expressions for Bernoulli-Euler beams with generally restrained boundary conditions have been presented by using Fourier series as a mode function. The expressions are quite general since identical Fourier series expressions may be used for many different physical problems like beams, columns, strings, plates and shells with both classical and non-classical boundary conditions. Several degenerate cases are investigated to confirm that the expressions are valid to solve beam problems with a variety of boundary

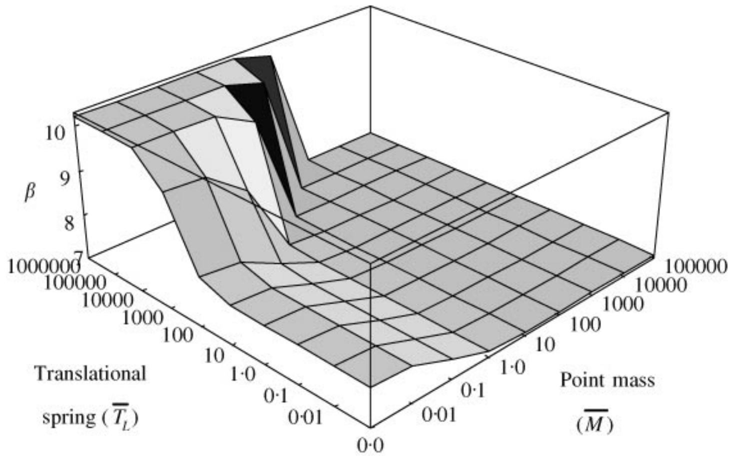


Figure 11. Plot of the third natural frequency parameter ($\beta_3$) for CF beam with translational spring and point mass at free end ($\overline{T}_L$, $\overline{M}$).

conditions. The three-dimensional plots of natural frequency parameters for a beam with both translational and rotational restraints at its ends have been presented to graphically illustrate how the parameters change with the spring constants. And the numerical results and its three-dimensional plots for a CF beam with translational spring and point mass at its free end have also been presented to illustrate the change of frequency parameters with varying conditions.

On the basis of these results, the frequency expressions derived in the present paper can be used in the design of beams with various supporting conditions.

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## APPENDIX A

$$s_{1,1} = -\left(1 + 2\sum_{m=1}^{\infty}\frac{\lambda^4}{\lambda^4 - m^4}\right), \qquad s_{1,2} = \left(1 + 2\sum_{m=1}^{\infty}\frac{(-1)^m\lambda^4}{\lambda^4 - m^4}\right),$$

$$s_{1,3} = \left(\bar{T}_0 + 2\pi^2\sum_{m=1}^{\infty}\frac{m^2\lambda^4}{\lambda^4 - m^4}\right), \qquad s_{1,4} = -\left(2\pi^2\sum_{m=1}^{\infty}\frac{(-1)^m m^2\lambda^4}{\lambda^4 - m^4}\right),$$

$$s_{2,1} = \left(1 + 2\sum_{m=1}^{\infty}\frac{(-1)^m\lambda^4}{\lambda^4 - m^4}\right), \qquad s_{2,2} = -\left(1 + 2\sum_{m=1}^{\infty}\frac{\lambda^4}{\lambda^4 - m^4}\right),$$

$$s_{2,3} = -\left(2\pi^2\sum_{m=1}^{\infty}\frac{(-1)^m m^2\lambda^4}{\lambda^4 - m^4}\right), \qquad s_{2,4} = \left(\bar{T}_L + 2\pi^2\sum_{m=1}^{\infty}\frac{m^2\lambda^4}{\lambda^4 - m^4}\right),$$

$$s_{3,1} = \left(1 - \frac{2\bar{R}_0}{\pi^2}\sum_{m=1}^{\infty}\frac{m^2}{\lambda^4 - m^4}\right), \qquad s_{3,2} = -\left(\frac{2\bar{R}_0}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^m m^2}{\lambda^4 - m^4}\right),$$

$$s_{3,3} = \left(\bar{R}_0 + 2\bar{R}_0\sum_{m=1}^{\infty}\frac{\lambda^4}{\lambda^4 - m^4}\right), \qquad s_{3,4} = -\left(\bar{R}_0 + 2\bar{R}_0\sum_{m=1}^{\infty}\frac{(-1)^m\lambda^4}{\lambda^4 - m^4}\right),$$

$$s_{4,1} = \left(\frac{2\bar{R}_L}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^m m^2}{\lambda^4 - m^4}\right), \qquad s_{4,2} = \left(1 - \frac{2\bar{R}_L}{\pi^2}\sum_{m=1}^{\infty}\frac{m^2}{\lambda^4 - m^4}\right),$$

$$s_{4,3} = -\left(\bar{R}_L + 2\bar{R}_L\sum_{m=1}^{\infty}\frac{(-1)^m\lambda^4}{\lambda^4 - m^4}\right), \qquad s_{4,4} = \left(\bar{R}_L + 2\bar{R}_L\sum_{m=1}^{\infty}\frac{\lambda^4}{\lambda^4 - m^4}\right)$$