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DYNAMICS OF AUTOPARAMETRIC VIBRATION ABSORBERS USING MULTIPLE PENDULUMS

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This paper analyzes the dynamics of a resonantly excited single-degree-of-freedom linear system coupled to an array of non-linear autoparametric vibration absorbers (pendulums). The case of a 1:1:...:2 internal resonance between pendulums and the primary oscillator is studied. The method of averaging is used to obtain first order approximations to the non-linear response of the system. The stability and bifurcations of equilibria of the averaged equations are computed. It is shown that the frequency interval of the unstable single-mode response, or the absorber bandwidth, can be enlarged substantially compared to that of a single pendulum absorber by adjusting individually the internal mistunings of the pendulums. Use of multiple pendulums is also shown to engender degenerate bifurcations as the double-mode response "switches" from one pendulum to the other with changing external excitation frequency. The effect of various parameters on the performance is discussed and a strategy is developed to find the most effective parameters for maximum bandwidth of operation. This results in a significant enhancement of the performance of autoparametric vibration absorbers.

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1. INTRODUCTION

Non-linear vibration absorbers based on autoparametric coupling between the system requiring reduced response and the absorber element have been studied extensively. Haxton and Barr [1] introduced and studied the autoparametric vibration absorber. The autoparametric vibration absorber exploits the transfer of energy between modes, and the saturation phenomenon, that is known to occur in quadratically coupled multi-degree-of-freedom systems subjected to primary excitation and possessing a 1:2 internal resonance [2, 3]. Haxton and Barr implemented the vibration absorber by attaching a cantilever beam with a tip mass to the primary system. They reported that the autoparametric vibration absorber did not always out-perform the more conventional linear tuned and damped absorber [4] due to the narrow effective bandwidth of performance. Hatwal *et al.* [5–7] studied the periodic and chaotic motions of a two-degree-of-freedom autoparametric system under harmonic excitation. In these works, a pendulum with and without an additional linear torsional spring was used as the vibration absorber. The pendulum was also used by Cuvalci and Ertas [8] as a vibration absorber to reduce response of a flexible cantilever beam. Cartmell and Lawson [9] implemented a computer-controlled sliding mass to improve the performance of the pendulum vibration absorber.

In recent years, some active non-linear control approaches have been proposed to improve the performance effectiveness of the autoparametric absorbers. These utilize the non-linear modal coupling between a structure and a 'virtual' pendulum vibration absorber

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implemented through a computer. A review of these developments along with various experimental investigations can be found in the work of Oueini *et al.* [10]. The most recent studies on non-linear absorbers include the works of Pai and Schulz [11], who have investigated ways to improve the performance of the standard 1:2 internal resonance absorber, and Pai *et al.* [12], who investigated the use of higher order internal resonances. In both studies, some experimental active control implementations were also accomplished for the suppression of structural vibrations.

An extensive stability and bifurcation analysis of the averaged equations for a two-degree-of-freedom autoparametric system has been carried out by Bajaj *et al.* [13]. Two kinds of motions were studied: (a) single-mode motions, wherein only the directly excited primary system vibrates, and (b) coupled-mode motions in which at least one absorber element vibrates. They showed the existence of non-trivial periodic coupled-mode responses which arise as a result of instability of the single-mode response. The coupled-mode equilibrium responses for the amplitude equations can undergo a Hopf bifurcation to limit cycle oscillations. Eventually, for some parameter combinations, the limit cycles lead to chaos by period-doubling bifurcations. These results were further extended by a higher order averaging analysis by Banerjee *et al.* [14], and a Melnikov analysis by Banerjee and Bajaj [15].

One of the main factors limiting the performance of any vibration absorber is its bandwidth. In the case of linear tuned absorbers, many approaches have been utilized to improve the performance. These include a recent work on a 'wideband' vibration neutralizer [16] where an array of neutralizers that are all tuned to slightly different natural frequencies is introduced. Motivated by this work, the present study considers an array of n non-linear autoparametric pendulum vibration absorbers that is used to suppress the response of a resonantly excited primary oscillator. The weakly non-linear resonant response of the $(n+1)$-degree-of-freedom system is studied for $1:1:\ldots:2$ internal resonance. The method of averaging is used to obtain the first order averaged equations that govern the evolution of the amplitudes and phases of response of the various degrees-of-freedom. The coupled system's dynamic response is studied using analytical as well as numerical bifurcation techniques [17]. The performance of the system with n pendulums as absorbers is compared to that of the system with a single autoparametric vibration absorber with identical total mass. It is shown that the effective frequency range of operation of the absorber can be increased significantly by properly adjusting the internal mistunings between the natural frequency of the primary system and the frequencies of the individual pendulums. The effect of other system parameters on the absorbers' performance is also investigated.

It should be noted that we have restricted our analysis to weak excitation. The effect of increase in the amplitude of excitation can be addressed by keeping additional terms in the approximation to the periodic solution or by a higher order averaging analysis, as was done for the autoparametric vibration absorber with one pendulum [6, 14]. Banerjee *et al.* [14] showed that higher order averaging eliminates the structurally unstable saturation phenomenon predicted by first order analysis. Furthermore, more complex non-linear responses can arise for the system, as was shown in the study of Hatwal *et al.* [6].

2. SYSTEM DESCRIPTION AND EQUATIONS OF MOTION

The $(n+1)$-degree-of-freedom autoparametric system is shown in Figure 1. The primary system, whose vibration is to be attenuated, consists of a linear spring–mass–damper system

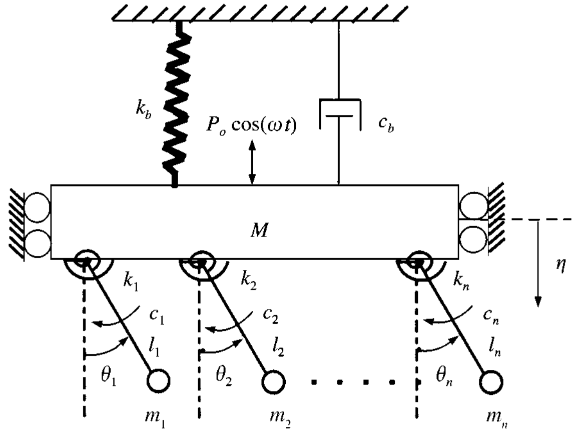

Figure 1. The autoparametric vibratory system depicting the primary mass whose vibration is to be attenuated, along with the n attached pendulums.

undergoing translational motions in the vertical direction. The block in the primary system has mass M, the linear spring has stiffness k_b, and the damper has damping coefficient c_b. The block is excited by the harmonic external force $P_0 \cos(\omega t)$. The secondary system or the vibration absorber consists of an array of n pendulums attached to the block. The equations of motion for this system are derived here using the Lagrangian formulation. The resulting equations of motion are

$$(M + m)\ddot{x} + c_b\dot{x} + k_b x - \sum_{i=1}^{n} r_i v_i m l(\ddot{\theta}_i \sin\theta_i + \dot{\theta}_i^2 \cos\theta_i) = P_0 \cos(\omega t), \tag{1}$$

$$r_i m v_i^2 l^2 \ddot{\theta}_i + c_i\dot{\theta}_i + k_i\theta_i + r_i v_i(mgl - ml\ddot{x}) \sin\theta_i = 0, \quad i = 1, \ldots, n, \tag{2}$$

where

$$m = \sum_{i=1}^{n} m_i, \quad l^2 = \sum_{i=1}^{n} l_i^2, \quad r_k = m_k/m, \quad v_k = l_k/l, \quad k = 1, \ldots, n. \tag{3}$$

Here, m_i denotes the mass of the ith pendulum, θ_i denotes its angular displacement, l_i is its length, k_i is the stiffness of the torsional spring associated with the ith pendulum, and c_i is the damping coefficient of the corresponding linear velocity proportional torsional damper. Also, m is the total mass of the pendulums array and r_k denotes the "mass fraction" of the kth pendulum. The parameter l can be identified as the "root mean square" length (r.m.s. length) of the pendulum system. Then v_k represents a "length fraction" associated with the kth pendulum. Further note that $\sum_{i=1}^{n} r_i = 1$ and $\sum_{i=1}^{n} v_i^2 = 1$. The non-dimensional equations of motion for the system can now be obtained by using the non-dimensional parameters introduced by Hatwal *et al.* [5] and Bajaj *et al.* [13]. The resulting equations of motion are

$$\eta'' + 2\hat{\xi}_b\alpha\eta' + \alpha^2\eta - \sum_{i=1}^{n} r_i v_i \frac{R}{1+R}(\theta_i'' \sin\theta_i + \theta_i'^2 \cos\theta_i) = F\alpha^2 \cos\tau, \tag{4}$$

$$\theta_i'' + 2\xi_i\alpha\hat{\beta}_i\theta_i' + \alpha^2\beta_{3i}^2\theta_i + \left(\alpha^2\beta_{2i}^2 - \frac{\eta''}{v_i}\right)\sin\theta_i = 0, \quad i = 1, \ldots, n, \tag{5}$$

where

$$\tau = \omega t, \quad \eta = x/l, \quad F = P_0/k_b l, \quad R = m/M, \tag{6}$$

$$\alpha = \omega_1/\omega, \quad \omega_1 = \sqrt{k_b/(M+m)}, \quad \beta_{2i} = \omega_{2i}/\omega_1, \quad \beta_{3i} = \omega_{3i}/\omega_1,$$

$$\omega_{2i} = \sqrt{g/l_i}, \quad \omega_{3i} = \sqrt{k_i/m_i l_i^2},$$

$$\hat{\beta}_i = \sqrt{\beta_{2i}^2 + \beta_{3i}^2}, \quad \xi_i = \frac{c_i}{2m_i l_1^2}\sqrt{\omega_{2i}^2 + \omega_{3i}^2}, \quad i = 1, \dots, n,$$

$$\xi_b = \frac{c_b}{2M\Omega_1}, \quad \Omega_1 = \sqrt{\frac{k_b}{M}}, \quad \hat{\xi}_b = \frac{\xi_b}{\sqrt{1+R}}$$

and where a prime denotes derivative with respect to the non-dimensional time τ. In these equations, Ω_1 is the natural frequency of the primary mass system, ω_1 is the natural frequency of the locked-pendulum (all pendulums at rest) system, α is the ratio of the locked-pendulum frequency to the excitation frequency, and $\hat{\beta}_i$ denotes the ratio of the linear natural frequency of the ith pendulum to the frequency of the locked-pendulum system. R defines the ratio of the total mass of the absorbers to that of the primary system, also known as the "mass ratio" for the array. Note also that a pendulum's natural frequency is controlled both by its length and by the torsional spring associated with it.

3. FORMULATION AND THE AVERAGED EQUATIONS

We define the following scale changes:

$$\eta = \varepsilon\hat{\eta}, \quad \theta_i = \varepsilon\hat{\theta}_i, \quad F = \varepsilon^2\hat{F}, \quad \hat{\xi}_b = \varepsilon\bar{\xi}_b, \quad \xi_i = \varepsilon\hat{\xi}_i, \quad i = 1, \dots, n, \tag{7}$$

where ε is some arbitrary scaling parameter such that $0 < \varepsilon \ll 1$. This scaling restricts the system to small motions (thus the trigonometric functions are replaced by their Taylor series expansions), small damping and small forcing when it is resonantly excited with α near 1. Substituting scalings (7) into equations (4) and (5), using Taylor series expansion upto $O(\varepsilon^2)$, and diagonalizing the mass matrix yields

$$\hat{\eta}'' + \alpha^2\hat{\eta} = \varepsilon\left[\hat{F}\alpha^2\cos\tau - 2\xi\alpha^2\hat{\eta}' + \sum_{i=1}^{n} 8r_i v_i(\bar{\theta}_i'^2 - p_i^2\bar{\theta}_i^2)\right] + O(\varepsilon^2),$$

$$\bar{\theta}_i'' + p_i^2\bar{\theta}_i = \varepsilon\left(-\alpha^2\hat{\eta}\frac{\bar{\theta}_i}{v_i} - 4\bar{\xi}_i p_i\bar{\theta}_i'\right) + O(\varepsilon^2), \quad i = 1, \dots, n, \tag{8}$$

where

$$\bar{\theta}_i = \frac{1}{2}\sqrt{\frac{R}{2(1+R)}}\,\hat{\theta}_i, \quad p_i = \alpha\hat{\beta}_i, \quad \xi = \frac{\bar{\xi}_b}{\alpha}, \quad \bar{\xi}_i = \frac{1}{2}\hat{\xi}_i, \quad i = 1, \dots, n. \tag{9}$$

In equations (8), α and p_i, $i = 1, \cdots, n$, are the $(n+1)$ non-dimensional linear natural frequencies with the excitation frequency being equal to unity. The angular variables $\overline{\theta}_i$, and the damping ξ and $\overline{\xi}_i$, $i = 1, \ldots, n$, have been introduced for brevity of expressions [13]. These $(n+1)$ equations can now be written in first order vector form as

$$\mathbf{z}' = \mathbf{A}\,\mathbf{z} + \varepsilon\,\mathbf{h}_1(\mathbf{z}, \tau) + O(\varepsilon^2), \tag{10}$$

where

$$\mathbf{z} = [\hat{\eta}_1, \hat{\eta}_2, \overline{\theta}_{11}, \overline{\theta}_{21}, \overline{\theta}_{12}, \overline{\theta}_{22}, \ldots, \overline{\theta}_{1n,}, \overline{\theta}_{2n}]^{\mathrm{T}},$$

$$\hat{\eta}_1 = \hat{\eta}, \quad \hat{\eta}_2 = \hat{\eta}', \quad \overline{\theta}_{1i} = \overline{\theta}_i, \quad \overline{\theta}_{2i} = \overline{\theta}'_i, \quad i = 1, \ldots, n,$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{P}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{P}_n \end{bmatrix}_{2(n+1)\times 2(n+1)},$$

$$\mathbf{B} = \begin{bmatrix} 0 & 1 \\ -\alpha^2 & 0 \end{bmatrix}_{2\times 2}, \quad \mathbf{P}_i = \begin{bmatrix} 0 & 1 \\ -p_i^2 & 0 \end{bmatrix}_{2\times 2}, \quad \mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2} \tag{11}$$

and

$$\mathbf{h}_1 = \begin{Bmatrix} 0 \\ \hat{F}\alpha^2\cos\tau - 2\xi\alpha^2\hat{\eta}_2 + \sum\limits_{i=1}^{n} 8r_i v_i(\overline{\theta}_{2i}^2 - p_i^2\overline{\theta}_{1i}^2) \\ 0 \\ -4\overline{\xi}_1 p_1 \overline{\theta}_{21} - \dfrac{\alpha^2}{v_1}\overline{\theta}_{11}\hat{\eta}_1 \\ 0 \\ -4\overline{\xi}_2 p_2 \overline{\theta}_{22} - \dfrac{\alpha^2}{v_2}\overline{\theta}_{12}\hat{\eta}_1 \\ \vdots \\ 0 \\ -4\overline{\xi}_n p_n \overline{\theta}_{2n} - \dfrac{\alpha^2}{v_n}\overline{\theta}_{1n}\hat{\eta}_1 \end{Bmatrix}. \tag{12}$$

Now, using the transformation

$$\mathbf{z} = \mathbf{\Phi}\,\mathbf{u}, \tag{13}$$

where

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{C} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{D}_n \end{bmatrix}_{2(n+1)\times 2(n+1)},$$

$$\mathbf{C} = \begin{bmatrix} \cos \alpha\tau & \sin \alpha\tau \\ -\alpha \sin \alpha\tau & \alpha \cos \alpha\tau \end{bmatrix}_{2\times 2}, \quad \mathbf{D}_i = \begin{bmatrix} \cos p_i\tau & \sin p_i\tau \\ -p_i \sin p_i\tau & p_i \cos p_i\tau \end{bmatrix}_{2\times 2},$$

$$i = 1, \ldots, n, \quad \mathbf{u} = [u_1, u_2, u_{11}, u_{21}, u_{12}, u_{22}, \ldots, u_{1n}, u_{2n}], \tag{14}$$

equations (10) are transformed into the ‘standard form’ for first order averaging [18, 3]. The transformation matrix $\mathbf{\Phi}$ is a fundamental matrix solution of the linear system given in equations (10) for $\varepsilon = 0$. The transformed equations (10) in standard form are

$$\mathbf{u}' = \varepsilon \mathbf{g}_1(\mathbf{u}, \tau) + O(\varepsilon^2), \tag{15}$$

where

$$\mathbf{g}_1(\mathbf{u}, \tau) = \mathbf{\Phi}^{-1}(\tau)\mathbf{h}_1(\mathbf{\Phi}\,\mathbf{u}, \tau). \tag{16}$$

To leading order, the averaged equations corresponding to the original non-autonomous system (15) are given by [13]

$$\mathbf{u}' = \varepsilon \mathbf{g}_{10}(\mathbf{u}), \tag{17}$$

where $\mathbf{g}_{10}(\mathbf{u})$ is the mean value of $\mathbf{g}_1(\mathbf{u}, \tau)$, and is defined as

$$\mathbf{g}_{10}(\mathbf{u}) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{g}_1(\mathbf{u}, \tau)\, \mathrm{d}\tau. \tag{18}$$

We are interested in the study of the autoparametric vibration absorber when both internal and external resonances exist. This requirement is made explicit by the introduction of external mistunings:

$$\alpha^2 = 1 + 2\varepsilon\sigma_b, \quad p_i^2 = \tfrac{1}{4} + \varepsilon\sigma_i, \quad i = 1, \ldots, n, \tag{19}$$

where σ_b is the external de-tuning between the excitation frequency and the locked-pendulum natural frequency, and σ_i, $i = 1, \ldots, n$, are the external mistunings from perfect 1:2 resonance between the linear natural frequencies of the pendulums and the frequency of excitation. The function $\mathbf{g}_{10}(\mathbf{u})$ for the $(n + 1)$-degree-of-freedom system under both the external resonance conditions is given by

$$\mathbf{g}_{10}(\mathbf{u}) = \begin{Bmatrix} -\xi u_1 + \sigma_b u_2 + \sum_{i=1}^{n} 2 r_i v_i u_{1i} u_{2i} \\ \frac{1}{2}\hat{F} - \sigma_b u_1 - \xi u_2 + \sum_{i=1}^{n} r_i v_i (u_{2i}^2 - u_{1i}^2) \\ -\bar{\xi}_1 u_{11} + \sigma_1 u_{21} + \frac{1}{2v_1}(-u_1 u_{21} + u_2 u_{11}) \\ -\bar{\xi}_1 u_{21} - \sigma_1 u_{11} - \frac{1}{2v_1}(u_1 u_{11} + u_2 u_{21}) \\ -\bar{\xi}_2 u_{12} + \sigma_2 u_{22} + \frac{1}{2v_2}(-u_1 u_{22} + u_2 u_{12}) \\ -\bar{\xi}_2 u_{22} - \sigma_2 u_{12} - \frac{1}{2v_2}(u_1 u_{12} + u_2 u_{22}) \\ \vdots \\ -\bar{\xi}_n u_{1n} + \sigma_n u_{2n} + \frac{1}{2v_n}(-u_1 u_{2n} + u_2 u_{1n}) \\ -\bar{\xi}_n u_{2n} - \sigma_n u_{1n} - \frac{1}{2v_n}(u_1 u_{1n} + u_2 u_{2n}) \end{Bmatrix}. \tag{20}$$

Substitution of the functional form in equation (20) into equation (17) yields the averaged equations in Cartesian co-ordinates. Transformation of the averaged equations into polar co-ordinates via the change of variables $(u_1, u_2, u_{1i}, u_{2i}) \to (a_b \cos\beta_b, -a_b \sin\beta_b, a_i \cos\beta_i, -a_i \sin\beta_i)$ leads to the averaged equations

$$\begin{aligned}
a_b' &= \varepsilon\left[-\frac{1}{2}\hat{F}\sin\beta_b - a_b\xi - \sum_{i=1}^{n} r_i v_i a_i^2 \sin(2\beta_i - \beta_b)\right], \\
a_b\beta_b' &= \varepsilon\left[-\frac{1}{2}\hat{F}\cos\beta_b + a_b\sigma_b + \sum_{i=1}^{n} r_i v_i a_i^2 \cos(2\beta_i - \beta_b)\right], \\
a_1' &= \varepsilon\left[\frac{1}{2v_1} a_b a_1 \sin(2\beta_1 - \beta_b) - a_1\bar{\xi}_1\right], \\
a_1\beta_1' &= \varepsilon\left[\frac{1}{2v_1} a_b a_1 \cos(2\beta_1 - \beta_b) + a_1\sigma_1\right], \\
a_2' &= \varepsilon\left[\frac{1}{2v_2} a_b a_2 \sin(2\beta_2 - \beta_b) - a_2\bar{\xi}_2\right], \\
a_2\beta_2' &= \varepsilon\left[\frac{1}{2v_2} a_b a_2 \cos(2\beta_2 - \beta_b) - a_2\sigma_2\right], \\
&\vdots
\end{aligned}$$

$$a_n' = \varepsilon\left[\frac{1}{2\nu_n} a_b a_n \sin(2\beta_n - \beta_b) - a_n\bar{\xi}_n\right],$$

$$a_n\beta_n' = \varepsilon\left[\frac{1}{2\nu_n} a_b a_n \cos(2\beta_n - \beta_b) + a_n\sigma_n\right]. \tag{21}$$

The internal mistuning d_i from the exact internal resonance between the linear natural frequency of the locked-pendulum motion and the natural frequency of the ith pendulum is defined as

$$d_i = \sigma_b - 2\sigma_i, \quad i = 1, 2, \ldots, n. \tag{22}$$

Equations (21) are the polar form of the first order averaged equations for the $(n+1)$-degree-of-freedom autoparametric system. The variables a_b and $a_1, a_2, \ldots, a_n$ are, respectively, the amplitudes of the locked-pendulum block motion and the pendulum motions. They can also be interpreted to represent first order approximations to the Poincaré map of the original non-autonomous system [19].

4. STEADY STATE SOLUTIONS OF THE AVERAGED SYSTEM

Steady state constant solutions are easy to obtain from the polar form of the averaged equations (equations (21)). The steady state constant solutions for the locked-pendulum motion, i.e., $a_b \neq 0$ and $a_1 = a_2 = \cdots = a_n = 0$, are given by

$$\bar{a}_b = \frac{1}{2}\frac{\hat{F}}{\sqrt{\xi^2 + \sigma_b^2}}, \quad \tan\bar{\beta}_b = -\frac{\xi}{\sigma_b}. \tag{23}$$

Here, overbarred quantities correspond to single-mode solutions.

The steady state constant solutions for the case of only one pendulum and the primary mass in motion, and other pendulums being stationary, i.e., $a_b \neq 0$, $a_j \neq 0$, $a_i = 0$, $i = 1, 2, \ldots, n$, $i \neq j$, are given by

$$\bar{\bar{a}}_b = 2\nu_j\sqrt{\bar{\xi}_j^2 + \sigma_j^2} \tag{24}$$

and some $\bar{\bar{a}}_j \neq 0$. The corresponding amplitude of the oscillating pendulum satisfies the following quadratic equation in $\bar{\bar{a}}_j^2$:

$$\nu_j^2 r_j^2 \bar{\bar{a}}_j^4 + 4\nu_j^2 r_j(\xi\bar{\xi}_j - \sigma_b\sigma_j)\bar{\bar{a}}_j^2 + 4\nu_j^2(\sigma_b^2 + \xi^2)(\bar{\xi}_j^2 + \sigma_j^2) - 0{\cdot}25\hat{F}^2 = 0. \tag{25}$$

These solutions will be termed 'double-mode' solutions, and they are signified by double overbarred quantities. Clearly, real solutions for the quadratic equation (25) exist only if

$$\hat{F}^2 \geqslant 16\nu_j^2(\sigma_b\bar{\xi}_j + {}^{\xi}\sigma_j)^2. \tag{26}$$

The solutions described so far are identical to the locked-pendulum and coupled-mode motions exhibited by the two-degree-of-freedom autoparametric system [13].

The steady state constant solutions for the case of two pendulums in motion present non-trivial dynamics. To study these motions, let $a_b \neq 0$, $a_1 \neq 0$, $a_2 \neq 0$ and $a_3 = a_4 = \cdots = a_n = 0$. The equations for a_1' and β_1' in equations (21) then give

$$a_b = 2\nu_1\sqrt{\bar{\xi}_1^2 + \sigma_1^2}. \tag{27}$$

Similarly, the equations for a_2' and β_2' give

$$a_b = 2\nu_2\sqrt{\bar{\xi}_2^2 + \sigma_2^2}. \tag{28}$$

For the steady state solutions to exist, a_b obtained from either of the above two equations must be the same. Thus, the necessary condition for the existence of steady state solutions for the case of two pendulums in motion is

$$2\nu_1\sqrt{\bar{\xi}_1^2 + \sigma_1^2} = 2\nu_2\sqrt{\bar{\xi}_2^2 + \sigma_2^2}. \tag{29}$$

This suggests that steady state constant solutions for the case of two pendulums in motion exist only for parameters satisfying the condition given in equation (29). The conditions for the case of any m pendulums in motion ($m < n$) can be obtained in a similar way, and are given by

$$2\nu_1\sqrt{\bar{\xi}_1^2 + \sigma_1^2} = 2\nu_2\sqrt{\bar{\xi}_2^2 + \sigma_2^2} = \cdots = 2\nu_m\sqrt{\bar{\xi}_m^2 + \sigma_m^2}. \tag{30}$$

It can be observed that the conditions in equation (30) for the existence of solutions for more than two pendulums in motion are very stringent. These conditions will certainly be satisfied if all the pendulums are identical. It can be easily shown that in such a case the individual pendulum amplitudes (a_j's) are not determined uniquely. The phases of pendulum motions are, however, identical. The first two of equations (21) can be solved to obtain a locus of steady state solutions for the amplitudes of the m pendulums in motion as follows:

$$\left(a_b^2\xi + \sum_{i=1}^{m} r_i\nu_i(2\nu_i\bar{\xi}_i)a_i^2\right)^2 + \left(a_b^2\sigma_b - \sum_{i=1}^{m} r_i\nu_i(2\nu_i\sigma_i)a_i^2\right)^2 - \frac{1}{4}a_b^2\hat{F}^2 = 0. \tag{31}$$

The amplitudes of the individual pendulums are determined by initial conditions. An explicit solution can be obtained by assuming all the pendulums to have identical amplitude (say, a_j). This is certainly a plausible solution as all the pendulums are identical.

Consider now the case when the pendulums are not all identical, i.e., the parameters associated with the pendulums, ν_j, r_j, $\bar{\xi}_j$, d_j, $j = 1, 2, \ldots, m$, are not the same. Then, the conditions in equation (30) are no longer satisfied for all values of the external frequency mistuning σ_b. Note that the external mistunings $\sigma_1, \sigma_2, \ldots, \sigma_n$ can be written in terms of the internal mistunings, $d_1, d_2, \ldots, d_n$, respectively, using equation (22). Let us focus on the case when the external mistuning σ_b is varied to find its effect on the system response while the other system parameters are held constant. The condition in equation (29) for the motion of two pendulums gives a quadratic equation for σ_b. This suggests that the two pendulums can be in motion for at most two values of σ_b. Thus, except for those two particular values of σ_b, only one pendulum (if a real and positive solution exists for equation (25)) can be in motion, i.e., double-mode motions (in which the primary system, and one of the pendulums has non-zero response) are the most likely responses for the $(n + 1)$-degree-of-freedom system. The stability conditions, derived in the next section, determine which one of the n pendulums will be performing the non-trivial motion.

It should be noted that the satisfaction of equation (29) for a specific σ_b only ensures the existence of solutions for the two pendulums for that particular value of σ_b. Moreover, the individual pendulum motion amplitudes cannot be obtained explicitly. Equation (31) can be used to obtain the locus of steady state solutions for the amplitudes of the two pendulums in motion.

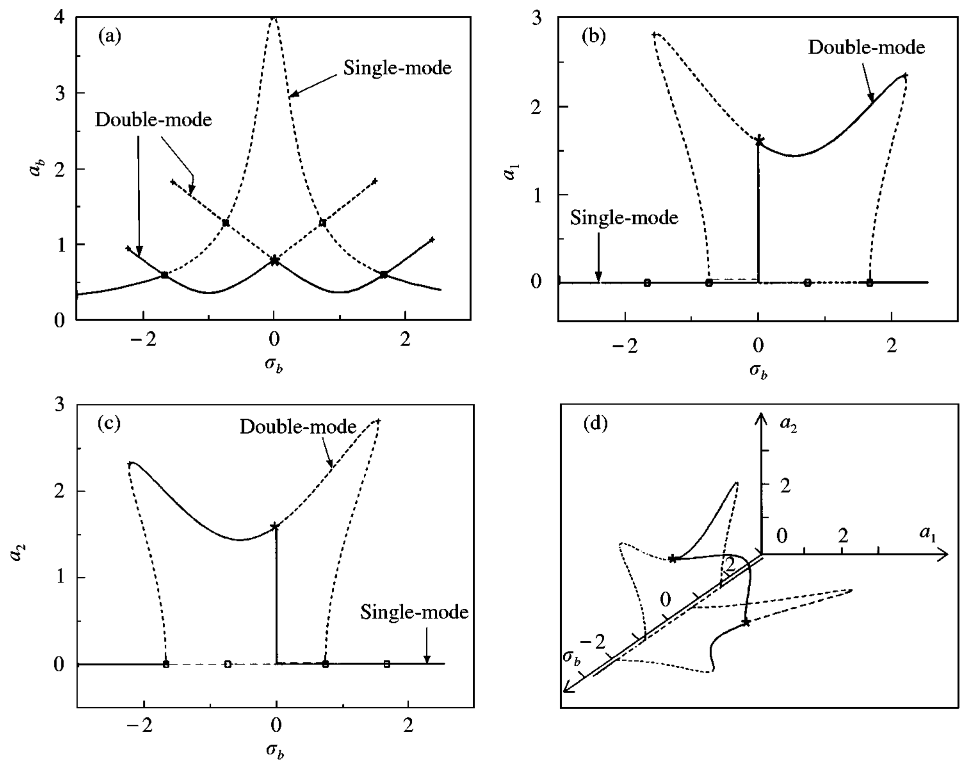

Figure 2. Resonant response of the system with two pendulums as a function of the excitation frequency σ_b. The system parameters are: mass fractions, $r_1 = r_2 = 0{\cdot}5$; internal mistunings, $d_1 = 1{\cdot}0$, $d_2 = -1{\cdot}0$; length fractions, $\nu_1 = \nu_2 = 1/\sqrt{2}$. (a) Amplitude of the primary system, a_b; (b) response amplitude of the first pendulum, a_1; (c) response amplitude of the second pendulum, a_2; (d) set of equilibria branching from the degenerate point '*'.

Steady state canstant solutions for the case of more than one pendulum in motion can be obtained analytically as well as numerically using the continuation and bifurcation analysis software AUTO [17]. Figure 2 shows a representative set of steady state constant solutions a_b, a_1 and a_2 as a function of de-tuning σ_b for a three-degree-of-freedom autoparametric system. This system consists of two pendulums as absorbers. The forcing amplitude $\hat{F}$ is set equal to 2·0 for all the results presented here. The damping constants for the figure and $\xi = \bar{\xi}_1 = \bar{\xi}_2 = 0{\cdot}25$ and the internal mistunings are $d_1 = 1{\cdot}0$ and $d_2 = -1{\cdot}0$. The pendulums have the same lengths and masses, i.e., the "mass fractions" are $r_1 = r_2 = 0{\cdot}5$, and the "length fractions" are $\nu_1 = \nu_2 = 1/\sqrt{2}$. Note that different frequency mistunings can be achieved by using unequal torsional springs. Stable solutions are shown by solid lines and the unstable solutions are shown by dotted lines. Whenever projections of a stable and an unstable solution branch overlap in a figure, both solid and dashed lines are drawn alongside each other. The pitchfork bifurcation points are shown by a 'square' symbol and turning points are indicated by a ' + ' symbol. Note also that only the single-mode motions, where only the primary system oscillates, exist for all values of σ_b. The 'double-mode' motions, where only one pendulum has a non-zero response along with the primary system, exist between the two turning points. These responses have the next widest range of existence in the frequency σ_b. Finally, the response in which both the pendulums have non-trivial motions exists only at $\sigma_b = 0$. For the parameters chosen for the system, condition (29) is satisfied only at this σ_b. The '*' symbol is used to denote the 'degenerate'

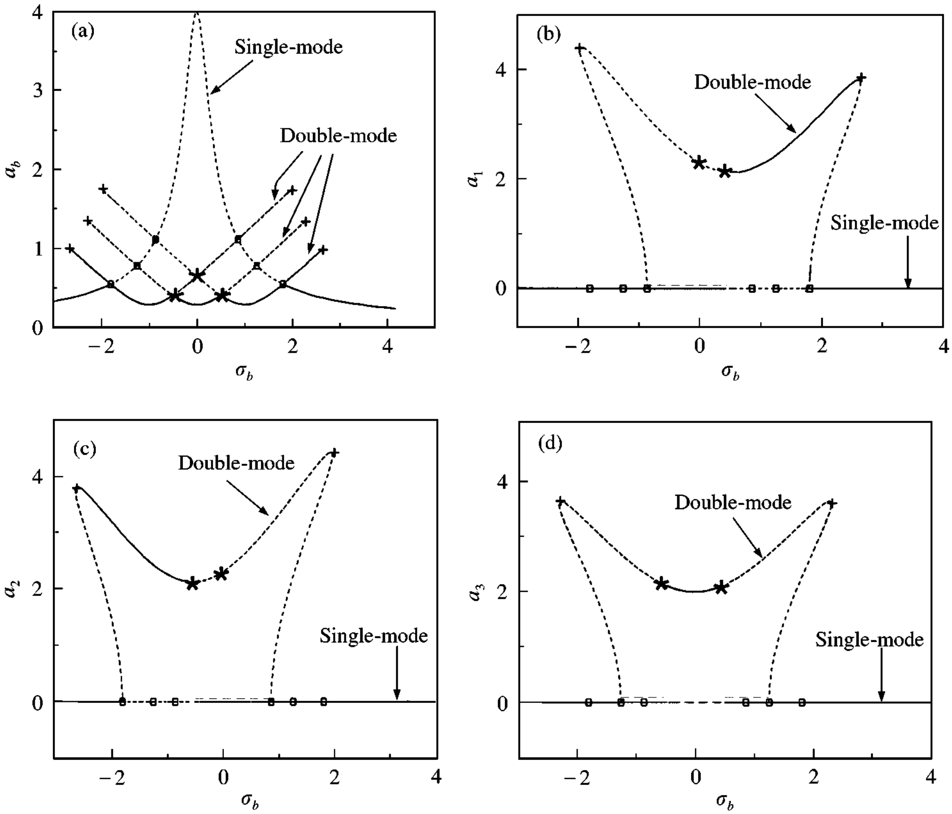

Figure 3. Resonant response of the system with three pendulums as a function of the excitation frequency σ_b. The system parameters are: mass fractions, $r_1 = r_2 = 0{\cdot}3$, $r_3 = 0{\cdot}4$; internal mistunings, $d_1 = 1{\cdot}0$, $d_2 = -1{\cdot}0$, $d_3 = 0{\cdot}0$; length fractions, $v_1 = v_2 = v_3 = 1/\sqrt{3}$. (a) Amplitude of the primary system, a_b; (b) response amplitude of the first pendulum, a_1; (c) response amplitude of the second pendulum, a_2; (d) response amplitude of the third pendulum, a_3.

bifurcation at $\sigma_b = 0$. It gives a manifold of equilibria for a_1 and a_2 which exists precisely at $\sigma_b = 0$.

A representative set of steady state constant solutions for the case of a system with three pendulums is shown in Figure 3. The parameters for this system are $v_1 = v_2 = v_3 = 1/\sqrt{3}$ and $r_1 = r_2 = 0{\cdot}3$, $r_3 = 0{\cdot}4$. The internal mistunings are $d_1 = 1{\cdot}0$, $d_2 = -1{\cdot}0$ and $df_3 = 0{\cdot}0$. All the damping constants are assumed to be 0·25. It can be seen that response for a_b in the vicinity of $\sigma_b = 0$ is smaller than in Figure 2. The Figures 2 and 3 also clearly demonstrate a much greater effective frequency range of vibration absorption as compared to that for a single pendulum vibration absorber [13, 14]. It can also be observed that the amplitude of the main block, $\bar{a}_b$, is a minimum on the corresponding double-mode branch when the excitation frequency equals the given internal mistuning for the pendulum, i.e., when $\sigma_b = d_i$. Furthermore, the maximum for the amplitude $\bar{a}_b$ is at a frequency of excitation for which two pendulums can be in motion, or at a frequency corresponding to the pitchfork bifurcation points.

We note that the steady state solutions for the case of one pendulum in motion are similar to the ones obtained for a two-degree-of-freedom autoparametric vibration absorber. Furthermore, it can be observed that the effects of forcing amplitude $\hat{F}$, primary system damping ξ, the damping ξ_i and the internal mistuning d_i associated with the ith pendulum in motion would be the same as that found in reference [14].

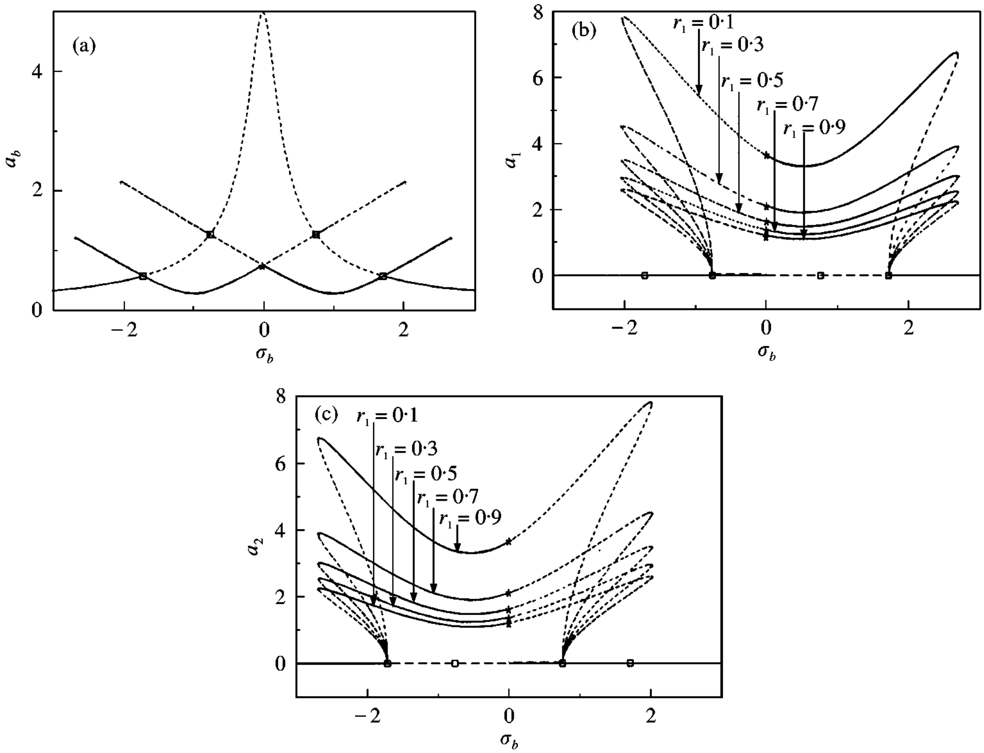

Figure 4. Effect of increasing the mass fraction r_1 on the response of the system with two pendulums. The other system parameters are: dampings, $\xi = \bar{\xi}_1 = \bar{\xi}_2 = 0{\cdot}2$; internal mistunings, $d_1 = 1{\cdot}0$, $d_2 = -1{\cdot}0$; length fractions, $v_1 = v_2 = 1/\sqrt{2}$. (a) Response amplitude of the primary system, a_b; (b) response amplitude of the first pendulum, a_1; (c) response amplitude of the second pendulum, a_2.

Here, we study the effects of the "mass fraction", r_i, and the "length fraction", v_i, associated with the ith pendulum. Note that $\sum_{i=1}^{n} r_i = 1$ and $\sum_{i=1}^{n} v_i^2 = 1$, and thus we cannot change these parameters independently. However, the double-mode response ($\bar{a}_b \neq 0$, $\bar{a}_i \neq 0$) would depend only on the parameters associated with the ith pendulum. Consider a system with two pendulum absorbers. The parameters for this system are $v_1 = v_2 = 1/\sqrt{2}$, $d_1 = 1$ and $d_2 = -1$. All the damping constants are assumed to be 0·2. The steady state response of this system for different values of r_1 is presented in Figure 4. It can be seen that the primary system amplitude, $\bar{a}_b$, does not vary with r_1, the mass fraction for the first pendulum, as it is absent from equation (24). The amplitude of the double-mode response $\bar{a}_1$ for the first pendulum decreases with increase in its mass fraction r_1. As $r_1 \to 1$, the double-mode response $\bar{a}_1$ has a finite amplitude, whereas the amplitude of the other pendulum becomes infinite or meaningless ($r_1 = 1$ means that the second pendulum is massless). Note also that the frequency intervals for existence as well as stability of various solutions are unaffected by the mass fraction.

The effect of the length fraction v_1 is studied for the system with equal mass fractions, $r_1 = r_2 = 0{\cdot}5$ and is shown in Figure 5. The amplitude of the primary system increases with increase in v_1. The double-mode response for the pendulum, $\bar{a}_1$, decreases with increase in v_1. As v_1 approaches 1, the steady state response $\bar{a}_2$ becomes infinite or meaningless (the pendulum with $v_2 \to 0$ is a pendulum with zero length).

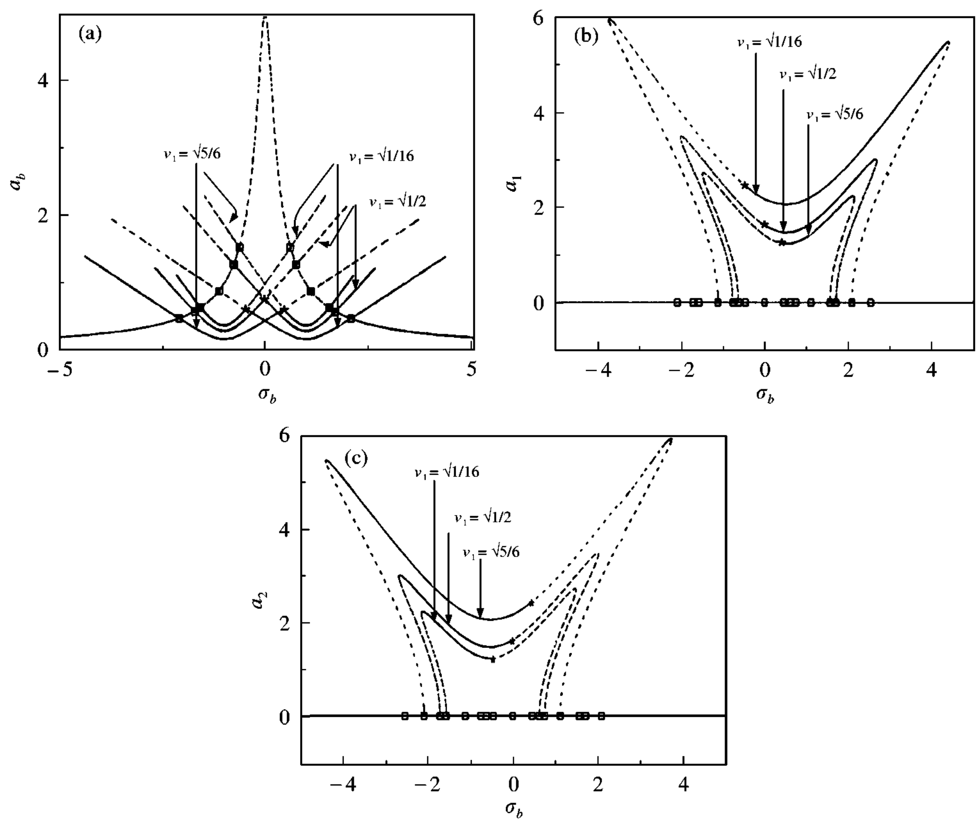

Figure 5. Effect of increasing the length fraction ν_1 on the response of the system with two pendulums. The other system parameters are: dampings, $\xi = \bar{\xi}_1 = \bar{\xi}_2 = 0{\cdot}2$; internal mistunings, $d_1 = 1{\cdot}0$, $d_2 = -1{\cdot}0$; mass fractions, $r_1 = r_2 = 0{\cdot}5$. (a) Response amplitude of the primary system, a_b; (b) response amplitude of the first pendulum, a_1; (c) response amplitude of the second pendulum, a_2.

It is important to note that in the sets of results presented above, the pendulums are differently mistuned relative to the primary system. In the case of identical pendulums, the mistunings would be equal and no difference will exist between the responses for the systems with one or many pendulums. Brennan [16] showed a similar increase in the effective bandwidth by using an array of linear absorbers tuned to slightly different natural frequencies.

5. STABILITY AND BIFURCATION ANALYSIS

The eigenvalues of the Jacobian matrix corresponding to the single-mode solution (equation (23)) of the averaged system in Cartesian co-ordinates (equations (20)) can be shown to satisfy the following $(n+1)$ quadratic equations:

$$\lambda^2 + 2\xi\lambda + \sigma_b^2 + \xi^2 = 0,$$

$$\lambda^2 + 2\bar{\xi}_i\lambda + \sigma_i^2 + \bar{\xi}_i^2 - 0{\cdot}25\,\frac{\bar{a}_b^2}{\nu_i^2} = 0, \quad i = 1, 2, \ldots, n. \tag{32}$$

The first equation here corresponds to perturbations in the primary system motion, whereas the latter n equations arise due to disturbances in the pendulum motions. From equations (32), it can be deduced that no purely imaginary pairs of eigenvalues exist for non-zero dampings and, as a result, Hopf bifurcations [19] cannot arise from the single-mode solution. Therefore, as in reference [13], the single-mode steady state solution can lose its stability only when an eigenvalue becomes zero. Using equations (32) and (23), we get the following conditions for the loss of stability of the single-mode solution:

$$4v_i^2(\sigma_b^2 + d_i^2 - 2\sigma_b d_i + 4\bar{\xi}_i^2)(\xi^2 + \sigma_b^2) = \hat{F}^2, \quad i = 1, 2, \ldots, n. \tag{33}$$

If any of the conditions in equation (33) are satisfied, the single-mode solution loses its stability. The ith condition in equation (33) corresponding to the ith pendulum is the same as the discriminant condition for the existence of 'double-mode' steady state solutions with $\bar{a}_b \neq 0$ and $\bar{a}_i \neq 0$. Thus, the double-mode steady state solutions arise by a pitchfork bifurcation where condition (33) is satisfied. It is also interesting to note that the pitchfork bifurcation set does not depend on the mass fraction r_i for the ith pendulum.

The eigenvalues of the Jacobian matrix, which determine the stability of the double-mode solutions ($\bar{a}_b \neq 0$, $\bar{a}_j \neq 0$), can be shown to satisfy

$$J_4\lambda^4 + J_3\lambda^3 + J_2\lambda^2 + J_1\lambda + J_0 = 0,$$

$$\lambda^2 + 2\bar{\xi}_i\lambda - \frac{(v_j^2(\bar{\xi}_j^2 + \sigma_j^2) - v_i^2(\bar{\xi}_i^2 + \sigma_i^2))}{v_i^2} = 0, \quad i = 1, 2, \ldots, n, \quad i \neq j, \tag{34}$$

where

$$J_4 = 1, \quad J_3 = 2(\xi + \bar{\xi}_j),$$

$$J_2 = \sigma_b^2 + \xi^2 + 4\xi\bar{\xi}_j + 2\bar{a}_j^2 r_j,$$

$$J_1 = 2\bar{\xi}_j\xi^2 + 2\bar{\xi}_j\sigma_b^2 + 2(\xi + \bar{\xi}_j)\bar{a}_j^2 r_j,$$

$$J_0 = 2\bar{a}_j^2 r_j(0{\cdot}5\bar{a}_j^2 r_j + \xi\bar{\xi}_j - \sigma_b\sigma_j). \tag{35}$$

An analysis of the quartic in equations (34) does not reveal any more information than is given in Bajaj *et al.* [13]. They used the Routh–Hurwitz criterion to determine the stability conditions. The stability conditions show a turning point in the double-mode solutions branch. The double-mode solutions can also undergo a Hopf bifurcation. The equation for the Hopf-bifurcation set for this system is

$$\bar{a}_j^2 r_j(\xi + \bar{\xi}_j)^2(\xi^2 + \sigma_b^2 + 2\sigma_b\sigma_j + 2\xi\bar{\xi}_j) + \xi\bar{\xi}_j(\xi^2 + \sigma_b^2)(\xi^2 + 4\bar{\xi}_j\xi + 4\bar{\xi}_j^2 + \sigma_j^2) = 0, \tag{36}$$

where $\bar{a}_j$ can be obtained from equation (25).

A detailed study of the Hopf bifurcation to amplitude-modulated motions, and then a torus-doubling cascade leading to chaotic amplitude-modulated solutions, was performed by Bajaj *et al.* [13] and Banerjee *et al.* [14]. The point to make here is that for zero internal mistuning between the pendulum in motion in the double-mode response and the primary system, the response cannot lose stability by a Hopf bifurcation. It arises in systems with low damping only for sufficiently large internal mistuning d_i. Thus, it may not be difficult to avoid the complex amplitude-modulated dynamics that can arise in the system response.

In this study we focus mainly on the additional quadratic equations in equations (34). Again, to understand the significance of these equations and the resulting stability conditions, consider the three-degree-of-freedom (or two pendulums) autoparametric system with $\bar{a}_b \neq 0$, $\bar{a}_1 \neq 0$ and $a_2 = 0$. The condition for the stability of the double-mode solution, considering only the quadratic equation in equations (34), is

$$(v_1^2(\bar{\xi}_1^2 + \sigma_1^2) - v_2^2(\bar{\xi}_2^2 + \sigma_2^2)) \leqslant 0. \tag{37}$$

Now consider the case when $\bar{a}_b \neq 0$, $\bar{a}_2 \neq 0$ and $a_1 = 0$. The corresponding stability condition for this case can be shown to be

$$(v_1^2(\bar{\xi}_1^2 + \sigma_1^2) - v_2^2(\bar{\xi}_2^2 + \sigma_2^2)) \geqslant 0. \tag{38}$$

It can be seen from the two conditions (37) and (38) that only one of these conditions can be satisfied at a time, except for the case when the equality holds. This suggests that, as far as this stability condition is concerned, the stable double modes of response are mutually exclusive, i.e., only one pendulum can execute stable non-zero steady state solution if the stability is not lost by the quartic equation in equations (34). Note that the equality in equation (37) is also the condition for the existence of non-zero steady state solutions for both the pendulums (equation (29)). For the case of equality, one eigenvalue becomes zero and the stability of the solutions cannot be determined by the linear analysis. In the neighbourhood of the equality, one of the pendulums loses stability for the non-zero motion while the other gains stability. At the equality, non-degeneracy conditions for simple bifurcations are violated and a one-dimensional manifold of equilibria emerges connecting the non-zero solution for a pendulum to its zero motion and *vice versa*.

The analysis for the $(n+1)$-degree-of-freedom system with motions involving two pendulums reveals $(n-1)$ such quadratic equations and the corresponding stability condition. Using equations (34), these conditions can be written as

$$(v_j^2(\bar{\xi}_j^2 + \sigma_j^2) - v_i^2(\bar{\xi}_i^2 + \sigma_i^2)) \leqslant 0, \quad i = 1, 2, \ldots, n, \quad i \neq j. \tag{39}$$

Considering the situation with $a_i \neq 0$, $i \neq j$ and $a_j = 0$, it can be shown that this condition restricts only one pendulum to possessing a stable non-zero steady state solution. The equality in equation (39) corresponds to the existence of non-zero steady state solutions for the ith and the jth pendulums. This equality can be written in terms of the external mistuning σ_b, and the internal mistunings d_i and d_j, as

$$(v_j^2 - v_i^2)\sigma_b^2 - 2\sigma_b(v_j^2 d_j - v_i^2 d_i) + v_j^2 d_j^2 - v_i^2 d_i^2 + 4(v_j^2\bar{\xi}_j^2 - v_i^2\bar{\xi}_i^2) = 0. \tag{40}$$

As an example, consider the two-pendulum autoparametric system with dampings $\xi = \bar{\xi}_1 = \bar{\xi}_2 = 0{\cdot}2$ and internal mistunings $d_1 = d_2 = 0{\cdot}0$. Let the pendulums have equal masses, $r_1 = r_2 = 0{\cdot}5$, but different lengths such that $v_1 = 1/\sqrt{3}$ and $v_2 = \sqrt{2/3}$. The steady state solutions for a_b, a_1 and a_2 can then be found as a function of the de-tuning σ_b. The stability condition in equation (40) gives no real root for σ_b for the given parameters of the system, and the Routh–Hurwitz criterion for the quartic in equations (34) is always satisfied. This suggests that the double-mode solution with $\bar{a}_2 \neq 0$ will always be unstable and, thus, in any observed steady state motion the response of the second pendulum, a_2, will be zero for all values of σ_b. Figure 6 shows the response amplitudes a_b, a_1 and a_2 for the system with $a_2 = 0$ as the stable solution for all values of σ_b. It is interesting to note also that the pendulum with larger length (v_2) has smaller non-zero amplitudes at every frequency σ_b in

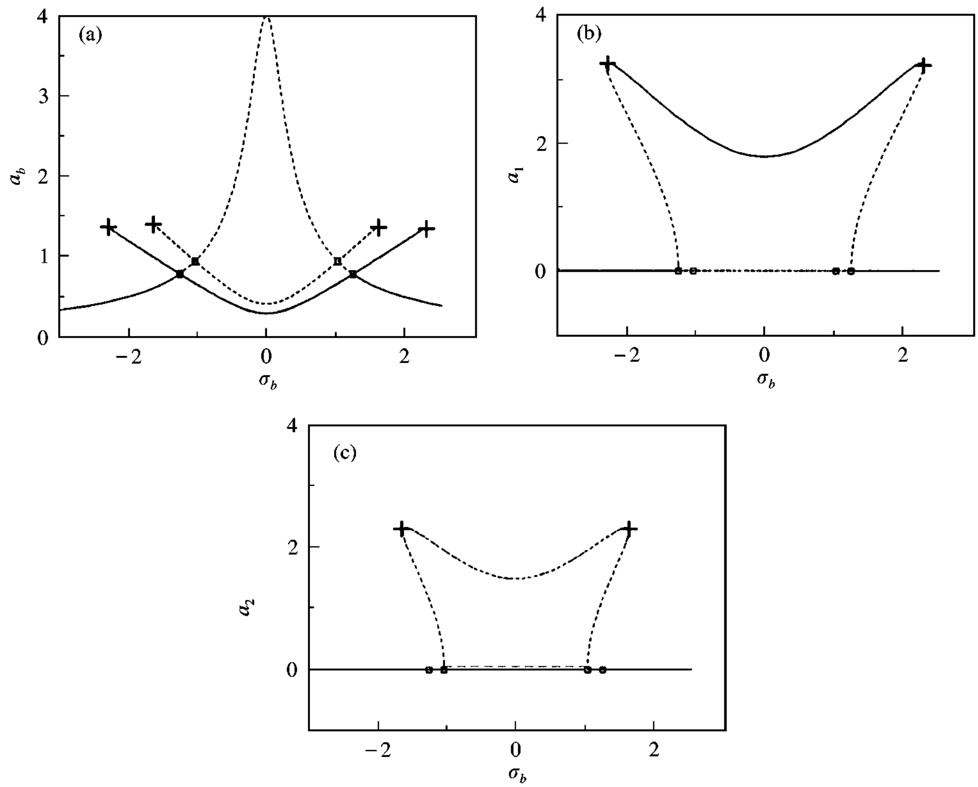

Figure 6. Resonant response of the system with two pendulums as a function of the excitation frequency σ_b. The system parameters are: mass fractions, $r_1 = r_2 = 0{\cdot}5$; internal mistunings, $d_1 = d_2 = 0{\cdot}0$; length fractions, $v_1 = 1/\sqrt{3}$, $v_2 = \sqrt{2/3}$. (a) Amplitude of the primary system, a_b; (b) response amplitude of the first pendulum, a_1; (c) response amplitude of the second pendulum, a_2; note that only solutions with $a_2 = 0$ are stable for any σ_b.

its unstable double-mode motion, while the corresponding amplitude of the primary system is higher.

6. STRATEGY FOR THE CHOICE OF PARAMETERS

The study so far has shown that the effective bandwidth can be increased by using pendulums with slightly different frequencies. In this section we discuss a few guidelines for the choice of optimum parameters giving maximum effective bandwidth. It can be noted that the maximum bandwidth depends on the choice of parameters and not on the number of pendulums. The multiple pendulums, if properly mistuned, would be important in further minimizing the primary system response and avoiding the Hopf bifurcations in the stable region of operation.

The bandwidth depends on the first and the last pitchfork bifurcation points where the locked-mass motion loses stability. The parameters of the pendulum bifurcating to non-zero steady state motion and the primary system determine the pitchfork bifurcation set for the pendulum, see equation (33). This is true also for the Hopf-bifurcation set. Mass fraction affects the response of the pendulum as shown in Figure 4. This indicates that choosing the appropriate mass will be a compromise between the amplitude of oscillation

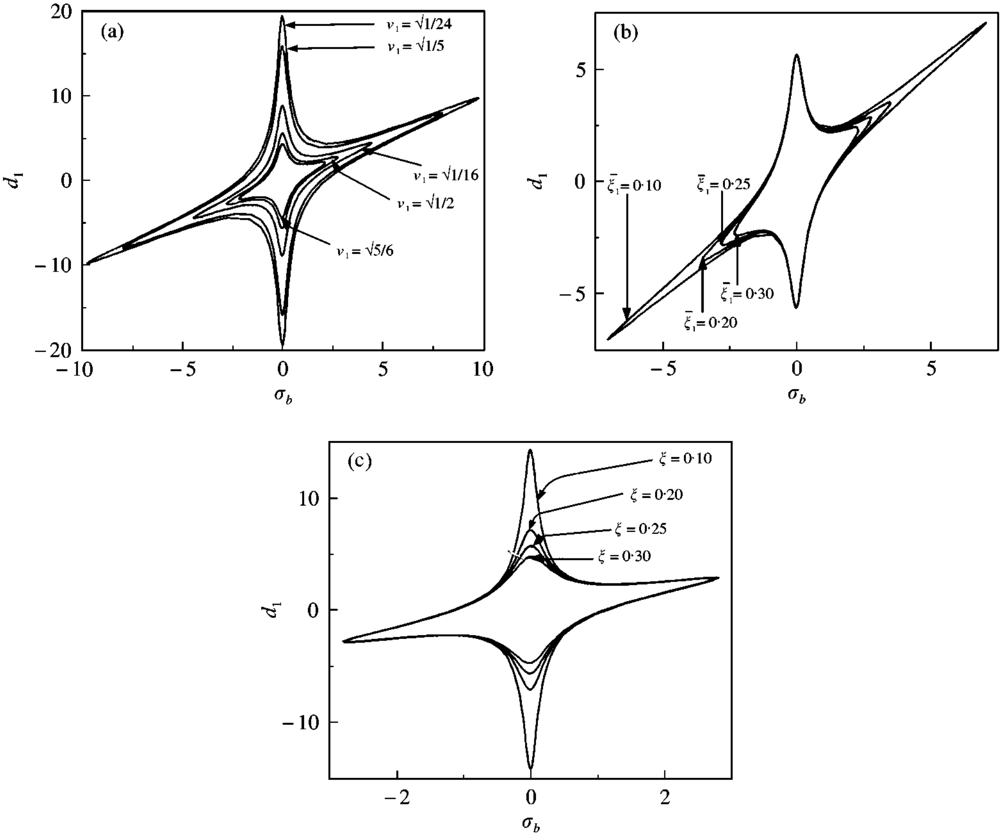

Figure 7. The pitchfork bifurcation sets in $\sigma_b - d_1$ plane. Nominal parameters are: mass fractions, $r_1 = r_2 = 0{\cdot}5$; dampings $\xi = \bar{\xi}_1 = \bar{\xi}_2 = 0{\cdot}25$. (a) Effect of increasing length fraction, v_1; (b) effect of increasing the pendulum damping, $\bar{\xi}_1$; (c) effect of increasing the primary system damping, ξ.

for the pendulum and the cost of material. Lower mass of a pendulum will lead to a larger steady state amplitude of oscillation for that pendulum and *vice versa.*

The pitchfork bifurcation sets for any of the pendulums can be determined irrespective of the total number of pendulums and the parameters of other pendulums. The equation for the pitchfork bifurcation set is a quartic in σ_b, thus suggesting that there could be either two or four pitchfork bifurcations for a given parameter set. This will also be evident in the respective pitchfork bifurcation sets discussed later. From the point of view of applications, it would be better to restrict the system parameters such that there are no more than two pitchfork bifurcation points. When more than two pitchfork bifurcation points arise, the beneficial effect of the pendulum absorber appears over a smaller frequency range and is not significant. This can be observed from the study in reference [14] where the effects of varying the internal mistuning were clearly demonstrated.

Consider now a system with two pendulums. All damping constants are taken to be 0·25 and the mass fractions are $r_1 = r_2 = 0{\cdot}5$. The pitchfork bifurcation sets for the first pendulum are drawn in (σ_b, d_1) plane. The pitchfork bifurcation set is shown for different values of v_1 in Figure 7(a). It can be seen that the range of σ_b for which two bifurcation points exist for any given internal mistuning d_1, increases with decrease in the length fraction v of the corresponding pendulum, i.e., v_1. However, this range reaches a limit for small values of v. The pitchfork bifurcation sets can be used to determine the largest or

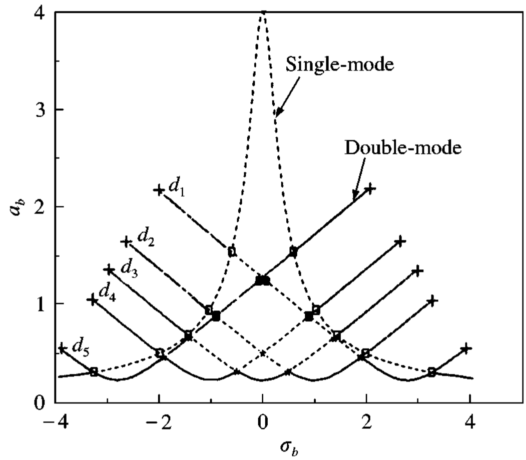

Figure 8. Response amplitude of the primary system, a_b, for a system with five pendulums. The system parameters are: mass fractions, $r_1 = r_2 = \cdots = r_5 = 0{\cdot}2$; length fractions, $\nu_1 = \nu_2 = \cdots = \nu_5 = 1/\sqrt{5}$; internal mistunings, $d_1 = 2{\cdot}8$, $d_2 = 1$, $d_3 = 0$, $d_4 = -1$, $d_5 = -2{\cdot}8$. The Hopf points are shown by solid squares.

smallest mistuning, $\pm d_s$ due to symmetry, for which only two pitchfork bifurcations will arise for that pendulum. The mistuning d_s, $d_s > 0$, signifies the maximum possible mistuning allowed for given parameters without having more than two pitchfork bifurcations. It can be seen in Figure 7(a) that for $\nu_1 = 1/\sqrt{5}$, $d_s \sim 2{\cdot}8$. The pitchfork bifurcation sets for different values of primary system damping ξ are shown in Figure 7(c). It can be seen that the range of σ_b for a fixed d_s increases with decrease in ξ. However, this increase is very insignificant and shows that the effective bandwidth of the autoparametric absorber is to a large extent independent of the primary system damping. Note that the amplitude of the primary system response does depend on this damping. The variation of the pitchfork bifurcation set with the pendulum damping $\bar{\xi}_i$, here $\bar{\xi}_1$, is shown in Figure 7(b). The range of σ_b for a fixed d_s increases with decrease in the pendulum damping, $\bar{\xi}_1$. It can be seen that for these parameters the effect of decrease in damping for low values is insignificant, but it is very prominent for higher pendulum damping values (e.g., see the set for $\bar{\xi}_1 = 0{\cdot}3$).

We now consider a system with five pendulums having the same masses and lengths, i.e., $\nu_1 = \nu_2 = \cdots = \nu_5 = 1/\sqrt{5}$ and $r_1 = r_2 = \cdots = r_5 = 0{\cdot}2$. All the damping constants are assumed to be 0·25. If the internal mistunings for the system are ordered such that $d_1 > d_2 > \cdots > d_5$, the above analysis suggests that we should take $d_1 = d_s$ and $d_5 = -d_s$. The other mistunings are then set as $d_2 = 1$, $d_3 = 0$ and $d_4 = -1$. The steady state response for the primary system, a_b, is shown in Figure 8. Note that in these response curves, Hopf points, shown by the solid squares, arise in the unstable region and do not effect the system steady state motions.

In general, if Hopf bifurcations do arise at some excitation frequency σ_b for a given internal mistuning d_i associated with a pendulum, they might appear in regions otherwise shown to be stable as far as pitchfork bifurcation conditions are concerned. Thus, it is important to determine the effect of various parameters on the Hopf-bifurcation sets. Then, one can try to ensure that for the optimal parameters chosen, the Hopf bifurcations are limited to the unstable region. To illustrate how this can be achieved, again consider the two-pendulum absorber system. The parameters for the system are $r_1 = r_2 = 0{\cdot}5$ and all damping constants are assumed to be 0·20. The Hopf-bifurcation sets for different values of length fraction ν_1 are shown in Figure 9(a). These sets should be used in conjunction with

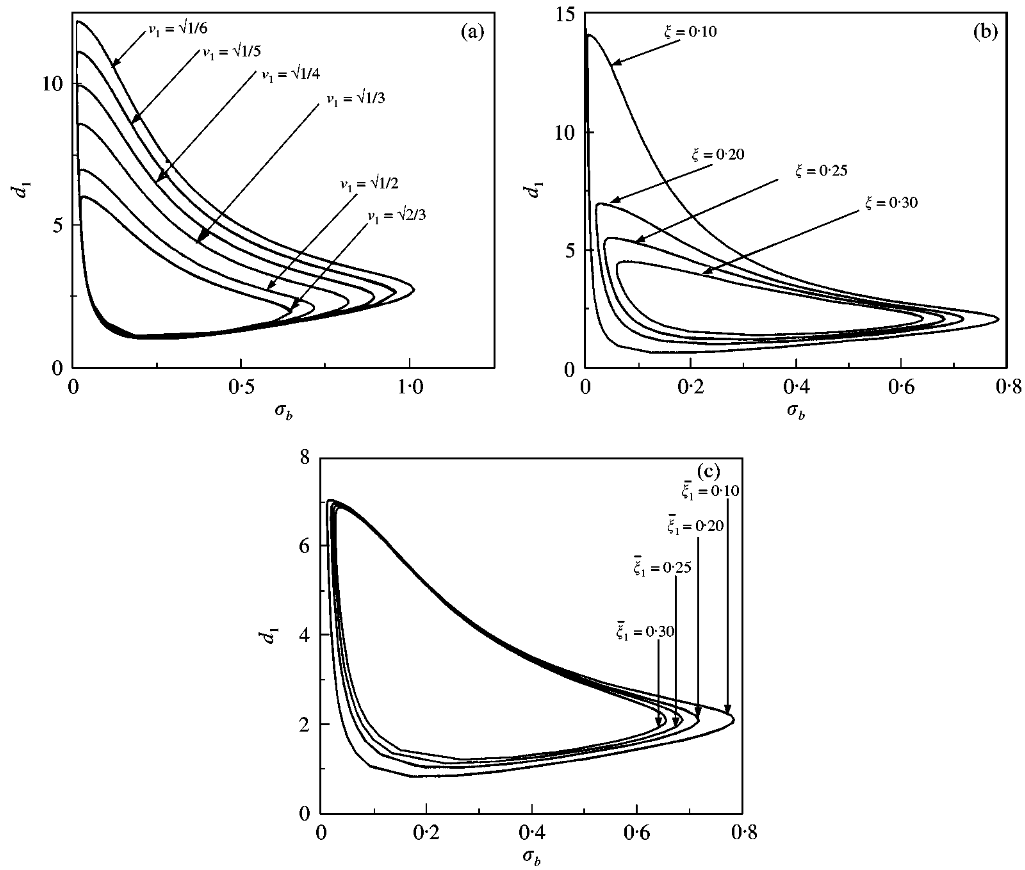

Figure 9. The Hopf-bifurcation sets in $\sigma_b - d_1$ plane. The nominal parameters are: mass fractions, $r_1 = r_2 = 0{\cdot}5$; dampings $\xi = \bar{\xi}_1 = \bar{\xi}_2 = 0{\cdot}2$. (a) Effect of increasing length fraction, ν_1; (b) effect of increasing the primary system damping, ξ; (c) effect of increasing the pendulum damping, $\bar{\xi}_1$.

the pitchfork bifurcation sets drawn earlier, see Figure 7. We would like to first ensure that there are no more than two pitchfork bifurcation points. Then, for the internal mistuning satisfying this requirement, the Hopf bifurcations, if they appear, should not be in the stable region. Figure 7(a) shows that for very low values of length fraction we can come close to $d_i = 6$ without having more than two pitchfork bifurcation points. Using Figure 9(a) we can observe that for low values of length fraction, and for the region of interest, $d_i < 6$, two Hopf-bifurcation points appear. One of the points is close to $\sigma_b = 0$ and does not change much with decrease in length fraction. The other Hopf-bifurcation points appears at a σ_b nearer to the $\sigma_b = d_1$. It can be seen that the right-side boundary of the Hopf-bifurcation set gives the Hopf-bifurcation point getting closer to $\sigma_b = d_1$. Figure 9(b) shows the Hopf-bifurcation sets for different values of ξ. Clearly, a decrease in the primary system damping, ξ, leads to an earlier inception of Hopf bifurcation. Figure 9(c) also shows the same effect for the pendulum damping, here $\bar{\xi}_1$. It can be observed that the effect of the primary system damping is more prominent compared to the secondary system or pendulum damping. Again, the Hopf-bifurcation sets plotted for different values of damping need to be used along with the pitchfork bifurcation sets for the appropriate choice of internal mistunings and length fractions.

We can also determine the number of pendulums required to achieve the desired performance under some simplifying assumptions. Suppose that all the pendulums have same masses and lengths, r, v, and damping constants, $\bar{\xi}$, but different internal mistunings, $d_1 < d_2 < \cdots < d_n$. We further assume that the mistunings are related as follows:

$$d_1 + (i-1)w = d_i, \quad i \leqslant n, \tag{41}$$

where w is the mistuning spacing between the consecutive internal mistunings. Then, the pitchfork- and Hopf-bifurcation sets can be used to determine the optimum values of d_1 and d_n. These two mistunings will determine the effective bandwidth of the absorber. The spacing parameter w can also be expressed as $w = (d_n - d_1)/(n-1)$.

The desired absorber performance can be expressed in terms of an "absorber action" AA, $AA \geqslant 1$, defined as the ratio of the maximum amplitude of the block $\bar{a}_b$ with pendulum in motion to the minimum amplitude of the block with pendulum in motion. Since the maximum primary system amplitude arises at the frequency where two pendulums have non-zero motions, one can show that AA is given by

$$AA = \frac{\sqrt{\bar{\xi}_1^2 + (w^2/16)}}{\bar{\xi}_1}. \tag{42}$$

It can be observed that to have $AA = 1$ we require infinite number of pendulums ($w = 0$). The other extreme value for AA would correspond to having no pendulum in resonance. The maximum value of AA possible for finite number of pendulums is $AA = \frac{1}{2}[\hat{F}/\xi(2v_1\bar{\xi}_1)]$. Equation (42) can be used to find the number of pendulums needed for the desired value of AA. If we require $AA = 1{\cdot}5$ with all damping constants assumed to be 0·20, and d_1 and d_2 being $-2{\cdot}0$ and 2·0, respectively, equation (42) gives $n = 5$ (rounded to the next integer). Thus, for given parameters and required $AA = 1{\cdot}5$, we need to attach five pendulums to the primary system.

7. SUMMARY AND CONCLUSION

This study has described the dynamics of an $(n+1)$-degree-of-freedom autoparametric vibration absorber which consists of an array of n pendulums. A first order asymptotic analysis of the system has been carried out under resonant excitation conditions with 1:1:…:2 internal resonances. The averaged equations are used to obtain steady state solutions of the system.

The stability and bifurcation analysis is carried out for the averaged equations. The single-mode response bifurcates to the double-mode response by pitchfork bifurcations. The double-mode response can undergo a Hopf bifurcation to limit cycles. This behavior is the same as found for the coupled-mode response of two-degree-of-freedom autoparametric systems [13].

The first order analysis shows that non-zero steady state solutions with more than one pendulum oscillating are possible only if condition (30) is satisfied. The stability analysis shows that at this point a one-dimensional manifold of equilibria emerges connecting the non-zero motion of the pendulum to the zero motion and *vice versa* for some other pendulum. Thus, except for a measured zero set of values of the external mistuning σ_b, only one pendulum has a non-zero steady state response over the frequency interval for which the single-mode solution is unstable. In addition, if the pendulums in the array are differently mistuned with respect to the primary system, the overall frequency interval over which various double-mode solutions arise and are stable is much wider than the frequency interval with only one pendulum. Thus, the bandwidth of effectiveness can be increased substantially by using pendulums with slightly different natural frequencies.

To the best of the authors' knowledge, this is the first time in the literature that the idea of a multiple array of autoparametric absorbers has been suggested and its effectiveness in enhancing the absorber bandwidth demonstrated analytically. The authors are currently investigating the implementation of this idea on a flexible beam with electronic circuit-based non-linear controllers 'mimicking' the multiple pendulum array.

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