Journal of Sound and Vibration (2001) **246**(1), 190–197
doi:10.1006/jsvi.2001.3579, available online at http://www.idealibrary.com on IDEAL®

COMPLEX AND DEFECTIVE ZEROS IN CROSS RECEPTANCES

J. E. Mottershead

Department of Engineering, Mechanical Engineering Division, The University of Liverpool, Liverpool, L69 3GH, England, E-mail: j.e.mottershead@liverpool.ac.uk

(*Received* 3 *August* 2000, *and in final form* 3 *January* 2001)

1. INTRODUCTION

Structural dynamacists generally pay more attention to the natural frequencies (poles) than the antiresonances (zeros) of vibrating systems, which is in one sense unfortunate because the special characteristic of the zeros is that they define the frequencies at which vibrations vanish. This information is clearly valuable in many engineering applications. In fact, the dynamic vibration absorber is a device for the assignment of point-receptance zeros, which lie on the imaginary axis of the eigenvalue plane for the classical undamped case and become complex when the absorber includes a damper. The zeros of different point and cross receptances generally occur at different frequencies. In design, there might be the requirement to assign a zero to a particular spatial location at the frequency of a sinusoidal excitation applied at a different co-ordinate. This means that we want to specify a zero of a particular cross receptance. Although it is an important issue, we will not dwell on the problem of assigning zeros in this letter. Instead, we report on numerical results which provide new understanding for the interpretation of measured cross receptances.

The zeros of point and cross receptances can be investigated by solving symmetric and asymmetric generalized eigenvalue problems respectively [1]. In numerical studies, the matrices are formed from the stiffness and mass (**K**, **M**) matrices of the system by deleting a single row and column. When the row and column have the same index, the resulting matrix system will be symmetric and its eigenvalues will be the zeros of a point receptance, which (by the interlacing rules of real symmetric matrices) will lie in an uninterrupted sequence between the system poles. When the deleted rows and columns have different indices the resulting matrices will be asymmetric, interlacing rules will not apply, the eigenvalues may become complex and the system of matrices may be defective. Numerical examples of complex zeros and repeated zeros due to defective matrix-systems are presented and explained.

2. THEORY

We write the system stiffness and mass matrices, $\mathbf{K} = \mathbf{K}^{\mathrm{T}} \geqslant$ (or > 0) and $\mathbf{M} = \mathbf{M}^{\mathrm{T}} > 0$, in the form,

$$\left[\begin{array}{c|c} k_{pq} & \mathbf{k}_p^{\mathrm{T}} \\ \hline \mathbf{k}_q & \mathbf{K}_{pq} \end{array}\right] \in \Re^{n \times n}, \tag{1}$$

0022-460X/01/360190 + 08 $35.00/0

$$\left[\begin{array}{c|c} m_{pq} & \mathbf{m}_p^{\mathrm{T}} \\ \hline \mathbf{m}_q & \mathbf{M}_{pq} \end{array}\right] \in \Re^{n\times n}, \tag{2}$$

by the rearrangement of rows and columns.

$\mathbf{K}_{pq}$ is the matrix formed from $\mathbf{K}$ by deleting the pth row and qth column, k_{pq} is the pqth term of $\mathbf{K}$, $\mathbf{k}_p^{\mathrm{T}}$ is the pth row of $\mathbf{K}$ (except for k_{pq}) and $\mathbf{k}_q$ is the qth column of $\mathbf{K}$ (except for k_{pq}). Similar definitions apply to the terms in the partitioned mass matrix. When $q \neq p$ then $\mathbf{K}_{pq} \neq \mathbf{K}_{pq}^{\mathrm{T}}$, $\mathbf{M}_{pq} \neq \mathbf{M}_{pq}^{\mathrm{T}}$ so that the eigenvalues $\bar{\lambda}_i$, $i = 1, \ldots, n-1$,

$$(\mathbf{K}_{pq} + \bar{\lambda}\mathbf{M}_{pq})\boldsymbol{\psi}_i = 0, \tag{3}$$

may be complex or defective.

Defective eigenvalues occur whenever the algebraic multiplicity, $(g+1)$, of the repeated eigenvalues $\bar{\lambda}_j, \bar{\lambda}_{j+1}, \ldots, \bar{\lambda}_{j+g}$ exceeds the geometric multiplicity defined by the dimension of the subspace spanned by the vectors $\boldsymbol{\psi}_j, \boldsymbol{\psi}_{j+1}, \boldsymbol{\psi}_{j+g}$ [2]. It can be shown [1] that the eigenvalues $\bar{\lambda}_i$ are the zeros of the cross receptance $h_{pq} = h_{qp}$ in linear dynamic systems.

3. COMPLEX ZEROS

Since the matrices $\mathbf{K}_{pq}$ and $\mathbf{M}_{pq}$ are real, those eigenvalues that are complex will occur in conjugate pairs. In fact it is the roots of the $\bar{\lambda}_i$, $i = 1, \ldots, n-1$, that determine the frequency and damping of the zeros and can be plotted on the complex eigenvalue plane. The square roots of the complex $\bar{\lambda}_i$ and its complex conjugate $\bar{\lambda}_i^*$ from a set of two pairs of complex conjugate zeros on a circle centred at the origin of the complex plane. Thus, four zeros are defined according to the different combinations of $\pm$ signs,

$$z_1, z_2, z_1^*, z_2^* = \pm\,\sigma \pm \mathrm{i}\bar{\omega}$$

and σ and $\bar{\omega}$ take their usual meanings. It is straightforward to show that the phase shift due to the two left-hand side eigenvalues is $\tan^{-1}(-2\sigma\omega/(-\omega^2 + (\sigma^2 + \bar{\omega}^2)))$, and due to the two right-hand side eigenvalues is $\tan^{-1}(2\sigma\omega/(-\omega^2 + (\sigma^2 + \bar{\omega}^2)))$. The phase shift due to (z_1, z_1^*) is exactly opposite to the phase shift introduced by (z_2, z_2^*). Consequently, in an undamped $\mathbf{K}$, $\mathbf{M}$ system no phase change is observed in the cross receptances at the frequency of a complex zero. The phase shifts are all 180° changes at the poles and the zeros on the imaginary axis.

3.1. NUMERICAL EXAMPLE

The following system is considered:

$$\mathbf{K} = \begin{bmatrix} 2{\cdot}85 & -1 & -0{\cdot}5 & -0{\cdot}2 & -0{\cdot}1 & -0{\cdot}05 \\ -1 & 3{\cdot}3 & -1 & -0{\cdot}5 & -0{\cdot}2 & -0{\cdot}1 \\ -0{\cdot}5 & -1 & 3{\cdot}4 & -1 & -0{\cdot}5 & -0{\cdot}2 \\ -0{\cdot}2 & -0{\cdot}5 & -1 & 3{\cdot}4 & -1 & -0{\cdot}5 \\ -0{\cdot}1 & -0{\cdot}2 & -0{\cdot}5 & -1 & 3{\cdot}3 & -1 \\ -0{\cdot}05 & -0{\cdot}1 & -0{\cdot}2 & -0{\cdot}5 & -1 & 2{\cdot}85 \end{bmatrix},$$

$$\mathbf{M} = \begin{bmatrix} 1{\cdot}2 & -0{\cdot}1 & & & & \\ -0{\cdot}1 & 1{\cdot}2 & -0{\cdot}1 & & & \\ & -0{\cdot}1 & 1{\cdot}2 & -0{\cdot}1 & & \\ & & -0{\cdot}1 & 1{\cdot}2 & -0{\cdot}1 & \\ & & & -0{\cdot}1 & 1{\cdot}2 & -0{\cdot}1 \\ & & & & -0{\cdot}1 & 1{\cdot}2 \end{bmatrix}.$$

The zeros of $h_{16} = h_{61}$ are given by the eigenvalues $\bar{\lambda}_i$, $i = 1, \ldots, 5$, and may be determined by solving, $(\mathbf{K}_{16} - \bar{\lambda}_i \mathbf{M}_{16})\, \boldsymbol{\psi}_i = 0$,
where,

$$\mathbf{K}_{16} = \begin{bmatrix} -1 & 3{\cdot}3 & -1 & -0{\cdot}5 & -0{\cdot}2 \\ -0{\cdot}5 & -1 & 3{\cdot}4 & -1 & -0{\cdot}5 \\ -0{\cdot}2 & -0{\cdot}5 & -1 & 3{\cdot}4 & -1 \\ -0{\cdot}1 & -0{\cdot}2 & -0{\cdot}5 & -1 & 3{\cdot}3 \\ -0{\cdot}005 & -0{\cdot}1 & -0{\cdot}2 & -0{\cdot}5 & -1 \end{bmatrix},$$

$$\mathbf{M}_{16} = \begin{bmatrix} -0{\cdot}1 & 1{\cdot}2 & -0{\cdot}1 & & \\ & -0{\cdot}1 & 1{\cdot}2 & -0{\cdot}1 & \\ & & -0{\cdot}1 & 1{\cdot}2 & -0{\cdot}1 \\ & & & -0{\cdot}1 & 1{\cdot}2 \\ & & & & -0{\cdot}1 \end{bmatrix}.$$

The zeros are given in Table 1 and the receptance h_{16} is shown in Figure 1. The two purely imaginary zeros between 2 and 2·5 rad/s are 'sharp' whereas the complex zeros at 1·846 rad/s appear to be less well defined in the figure. Figure 2 shows more details in the frequency range of the complex zeros. It is clear that the vibration is not completely eliminated by the complex zeros and no phase change can be detected at the 1·846 rad/s frequency.

4. DEFECTIVE ZEROS

The occurrence of repeated zeros was considered by Mottershead *et al.* [3] but the possibility of a defective system ($\mathbf{K}_{pq}$, $\mathbf{M}_{pq}$) was not included. To correct the omission in the

TABLE 1

Complex zeros (rad/s)

	Real	Imaginary
z_1, z_1^*	0·037	1·846
z_2, z_2^*	− 0·037	1·846
z_3, z_3^*		2·133
z_4, z_4^*		2·362
z_5, z_5^*		121·2

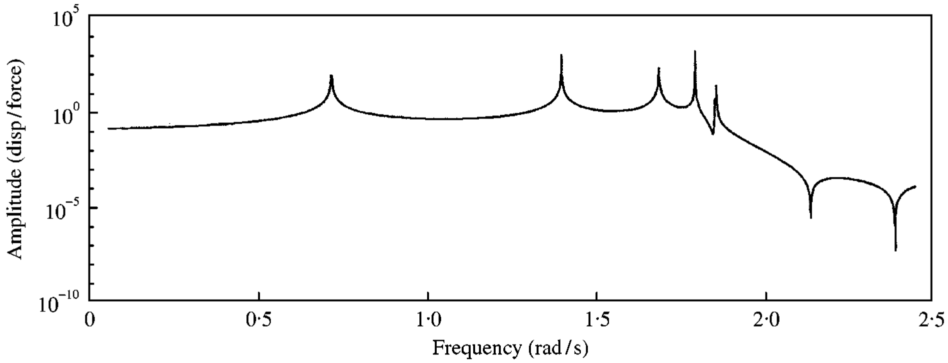

Figure 1. Frequency response h_{16}.

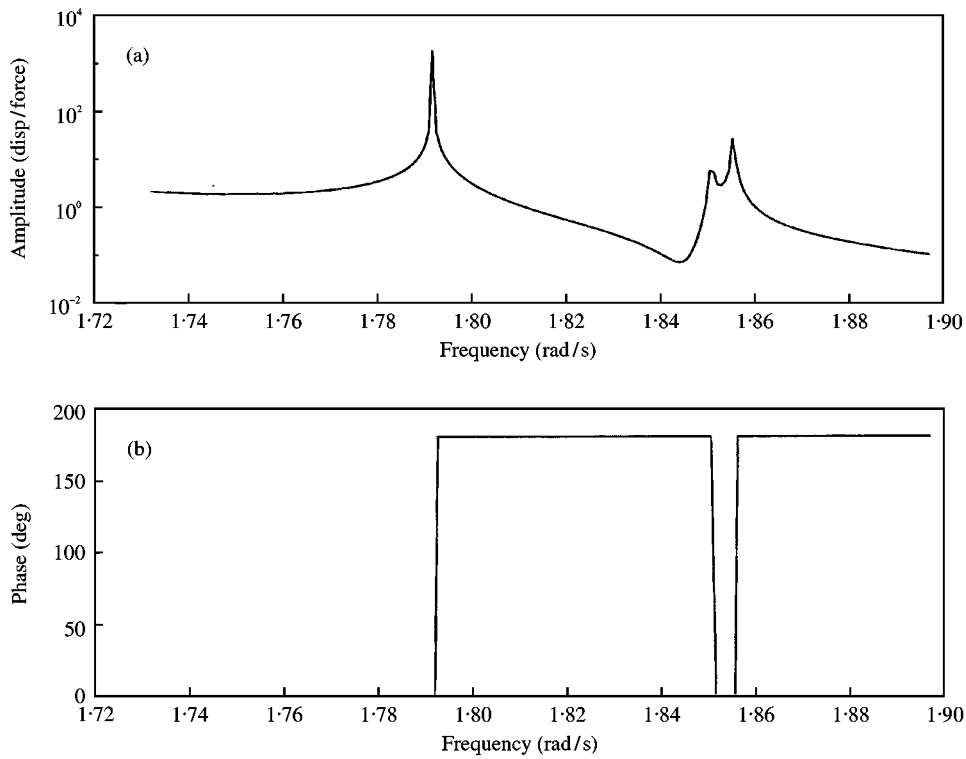

Figure 2. Frequency response h_{16} showing details of complex zeros: (a) Amplitude, (b) Phase.

previous paper, we begin by considering the two systems $(\mathbf{K}_{qq}, \mathbf{M}_{qq})$ and $(\mathbf{K}_{pq}, \mathbf{M}_{pq})$ and without loss of generality set $p = 2$, $q = 1$. The following eigenvalue equations can be written:

$$\begin{bmatrix} k_{22} - \bar{\lambda}_s m_{22} & k_{23} - \bar{\lambda}_s m_{23} & \cdots & k_{2r} - \bar{\lambda}_s m_{2r} \\ k_{32} - \bar{\lambda}_s m_{32} & k_{33} - \bar{\lambda}_s m_{33} & \cdots & k_{3r} - \bar{\lambda}_s m_{3r} \\ \vdots & \vdots & & \vdots \\ k_{r2} - \bar{\lambda}_s m_{r2} & k_{r3} - \bar{\lambda}_s m_{r3} & \cdots & k_{rr} - \bar{\lambda}_s m_{rr} \end{bmatrix} \boldsymbol{\psi}_s = \mathbf{0}, \tag{4}$$

$$\begin{bmatrix} k_{12}-\bar{\lambda}_t m_{12} & k_{13}-\bar{\lambda}_t m_{13} & \cdots & k_{1r}-\bar{\lambda}_t m_{2r} \\ k_{32}-\bar{\lambda}_t m_{32} & k_{33}-\bar{\lambda}_t m_{33} & \cdots & k_{3r}-\bar{\lambda}_t m_{3r} \\ \vdots & \vdots & & \vdots \\ k_{r2}-\bar{\lambda}_t m_{r2} & k_{r3}-\bar{\lambda}_t m_{r3} & \cdots & k_{rr}-\bar{\lambda}_t m_{rr} \end{bmatrix} \tilde{\boldsymbol{\psi}}_t = \mathbf{0}. \tag{5}$$

If $\bar{\lambda}_s = \bar{\lambda}_t$ the two matrices are identical except for the first rows. Furthermore, if $\bar{\lambda}_s$ and $\bar{\lambda}_t$ are both distinct and a pole-zero cancellation $\lambda_p = \bar{\lambda}_s$ takes place [3, 4], then $\boldsymbol{\psi}_s = \tilde{\boldsymbol{\psi}}_t$.

In the case when there are repeated zeros of multiplicity $(g+1)$ in the cross receptances and dim $(\mathscr{S}) = (g+1)$, $\mathscr{S} = \text{span}\ (\tilde{\boldsymbol{\psi}}_t, \tilde{\boldsymbol{\psi}}_{t+1}, \ldots, \tilde{\boldsymbol{\psi}}_{t+g})$ (so that the eigenvalues are not defective), then it is possible to form eigenvectors of the point-receptance zeros so that,

$$(\mathbf{k}_1 - \bar{\lambda}_s \mathbf{m}_1)\, \boldsymbol{\psi}_i = 0,\ i = s, s+1, \ldots, s+g, \tag{6}$$

$$\boldsymbol{\psi}_{s+i-1} = [\tilde{\boldsymbol{\psi}}_t\ \ \tilde{\boldsymbol{\psi}}_{t+1} \cdots \tilde{\boldsymbol{\psi}}_{t+g}]\boldsymbol{\alpha}_i,\ i = 1, \ldots, g, \tag{7}$$

where $\boldsymbol{\alpha}_i$ is a particular vector of scaling factors. The repeated zeros $\bar{\lambda}_s, \bar{\lambda}_{s+1}, \ldots, \bar{\lambda}_{s+g}$ all cancel with poles of the system. If there are more point-receptance zeros than cross-receptance zeros, then a similar argument can be used to show that all the zeros of the cross receptance and all but one zero of the point receptance will cancel with poles. In any case, the poles and zeros will cancel to leave a single pole, a single zero or complete cancellation in the point receptance and all cross receptances that include the same index. Equation (6) is considered in more detail in Appendix A.

When dim $(\mathscr{S}) < (g+1)$ so that the cross-receptance zeros are defective, then the number of uncancelled zeros may be greater than one. Hence, there can be uncancelled coincident zeros in cross-receptance measurements, the multiplicity of which will not exceed by more than one the difference between the algebraic and geometric multiplicity of the repeated zeros. Coincident zeros without cancellation by poles cannot occur in the point receptances since the point-receptance zeros cannot be defective (the matrices $\mathbf{K}_{qq}$, $\mathbf{M}_{qq}$ are symmetric).

4.1. NUMERICAL EXAMPLE

We consider the system defined by

$$\mathbf{K} = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 3 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & 3 & -1 & \\ & & & -1 & 3 & -1 \\ & & & & -1 & 1 \end{bmatrix},$$

$$\mathbf{M} = \mathbf{I}_{6\times 6},$$

so that

$$\mathbf{K}_{45} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 3 & -1 & & \\ & -1 & 2 & -1 & \\ & & & -1 & -1 \\ & & & & 1 \end{bmatrix}$$

$$\mathbf{M}_{45} = \mathrm{diag}\,(1 \;\; 1 \;\; 1 \;\; 0 \;\; 1)$$

and the eigenvalues $\bar{\lambda}_i\,(\mathbf{K}_{45}, \mathbf{M}_{45})$, $i = 1, \ldots, s$, determine the zeros of the cross receptance $h_{45} = h_{54}$. The poles of the system are given together with the zeros of h_{44}, h_{55} and h_{45} in Table 2. The eigenvectors of $(\mathbf{K}_{45}, \mathbf{M}_{45})$ are shown in Table 3, from which it can be observed that the geometric multiplicity of the two repeated cross-receptance zeros is one. Therefore, the system $(\mathbf{K}_{45}, \mathbf{M}_{45})$ (and its eigenvalues) are defective. The frequency response h_{45} is given in Figure 3. It is clear that no phase change occurs at the frequency of the repeated zeros (1 rad/s) which is consistent with there being two coincident zeros present in the receptance.

Another interesting result can be obtained by an inspection of Table 2 and the eigenvectors and left-eigenvectors of the zeros. We observe from the table that zeros occur together in the point and cross receptances h_{44} and h_{45} at 1·414 and 2 rad/s without cancellation with a pole. The coincident frequencies occur because $\boldsymbol{\psi}_s = \tilde{\boldsymbol{\psi}}_t$ (both have zero terms at the fourth coordinate), but the zeros are distinct and $\boldsymbol{\psi}_s \neq \tilde{\boldsymbol{\xi}}_t$ which means that a pole is prohibited and cancellation cannot take place.

Table 2

Poles and zeros

Poles (rad/s)	Zeros (rad/s)		
	h_{44}	h_{55}	h_{45}
0·684	0·765	0·898	1
0·911	1	1	1
1·286	1·414	1·306	1·414
1·969	1·848	1·815	2
2·117	2	2·048	∞

Table 3

Eigenvectors of $(\mathbf{K}_{45}\mathbf{M}_{45})$

$\tilde{\boldsymbol{\psi}}_1$	$\tilde{\boldsymbol{\psi}}_2$	$\tilde{\boldsymbol{\psi}}_3$	$\tilde{\boldsymbol{\psi}}_4$	$\tilde{\boldsymbol{\psi}}_5$
− 0·5774	0·5774	0·7071	0·4082	0
− 0·5774	0·5774	− 0·0000	− 0·8165	0
− 0·5774	0·5774	− 0·7071	0·4082	0
0	0	0	0	1·0000
0	0	0	0	0

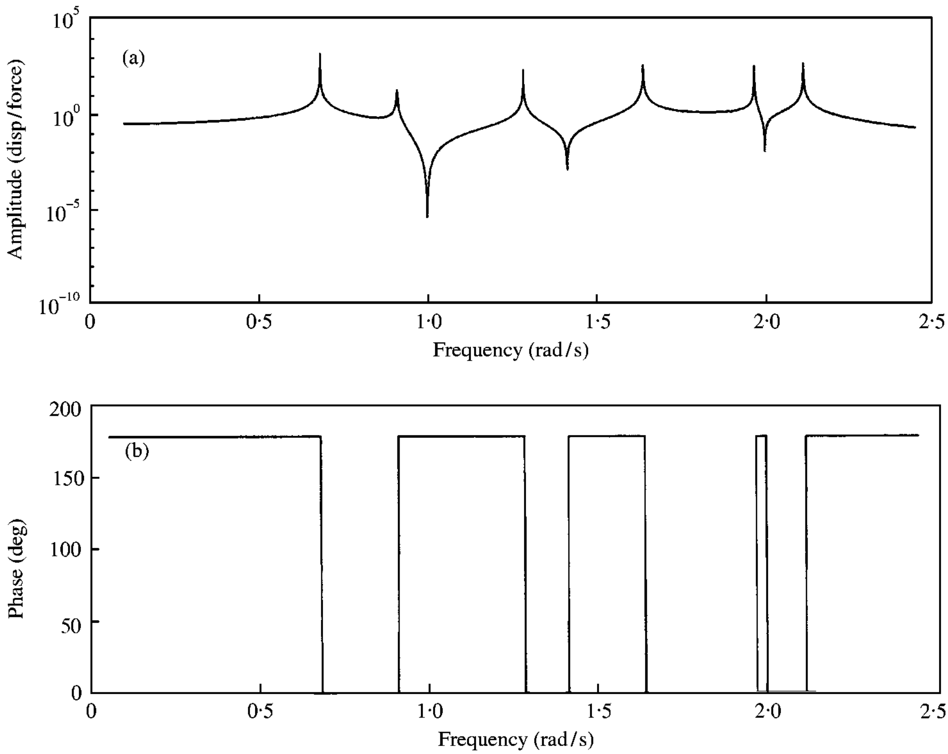

Figure 3. Frequency response h_{45}: (a) Amplitude, (b) Phase.

5. CONCLUSIONS

Complex and defective zeros can occur in cross receptance measurements. The complex zeros always occur in sets of two pairs of complex conjugates so that they are not detectable by a phase change. When complex zeros are present, the vibration is not completely eliminated at the frequency of those zeros. Repeated defective zeros may appear in cross receptances without cancellation by a pole. The number of uncancelled repeated zeros will not exceed by more than one the difference between the algebraic and geometric multiplicity of the repeated eigenvalues.

ACKNOWLEDGMENT

The author wishes to acknowledge discussions with Dr U. Prells of Swansea University.

REFERENCES

1. J. E. Mottershead 1998 *Mechanical Systems and Signal Processing* **12**, 591–597. On the zeros of structural frequency response functions and their sensitivities.
2. G. H. Golub and C. F. Van Loan 1989 *Matrix Computations*. Baltimore and London: John Hopkins University Press.
3. J. E. Mottershead, T. Li and J. He 2000 *Journal of Sound and Vibration* **231**, 219–231. Pole-zero cancellation in structures: repeated roots.
4. J. E. Mottershead and G. Lallement 1999 *Journal of Sound and Vibration* **222**, 833–851. Vibration nodes, and the cancellation of poles and zeros by unit-rank modifications to structures.

APPENDIX A: REPEATED ZEROS

We consider the rearranged stiffness and mass matrices as in equations (1) and (2), and note that the system ($\mathbf{K}_{pq}$, $\mathbf{M}_{pq}$) is symmetric except for the qth row and pth column. Thus, if the qth row and pth column were deleted the result would be a symmetric $(n-2)\times(n-2)$ system: this is a special feature of the zeros eigenvalue problem of the cross receptances. Now let ($\mathbf{K}_{pq}$, $\mathbf{M}_{pq}$) have repeated eigenvalues $\bar{\lambda}_i$ $(i = t, t+1, \ldots, t+g)$, eigenvectors $\tilde{\boldsymbol{\psi}}_i$ and left eigenvectors $\tilde{\boldsymbol{\xi}}_i$. Generally the eigenvalues, eigenvectors and left eigenvectors will be complex. However, if the terms $\tilde{\psi}_{pi} = \tilde{\xi}_{qi} = 0$ then $\tilde{\boldsymbol{\psi}}_i = \tilde{\boldsymbol{\xi}}_i$ since both are eigenvectors of the symmetric $(n-2)\times(n-2)$ system and will contain entirely real numbers. Of course, the eigenvalues will also be real in that case.

If the cross-receptance zeros are non-defective, then it is clear that g eigenvectors, $\boldsymbol{\psi}_{s+i-1}$ $(i = 1, \ldots, g)$, of the point receptance zeros can be formed as in equation (7) so that $\psi_{p,s+i-1} = 0$. Likewise vectors can be formed from the left eigenvectors of the matrix in equation (5) with the same constraint on the qth term, and $\bar{\lambda}_t = \bar{\lambda}_s$. In both the cases, the same eigenvectors of the symmetric $(n-2)\times(n-2)$ submatrix may be obtained. This shows that the first row of equation (4) is satisfied (the first row of the matrix in equation (4) is identically the first column of the matrix in equation (5)). The first row of the matrix in equation (5) is $(\mathbf{k}_1 - \bar{\lambda}_s\mathbf{m}_1)$ when $\bar{\lambda}_s = \bar{\lambda}_t$, so that equation (6) is satisfied by the constrained eigenvector $\boldsymbol{\psi}_{s+i-1}$.

Repeated poles of multiplicity g, at the same frequency, having eigenvectors $(0, \boldsymbol{\psi}_{s+i-1}^{\mathrm{T}})^{\mathrm{T}}$, will cancel the repeated point receptance zeros. The eigenvectors of the poles have vibration nodes at both the pth and qth coordinates.