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A REVERSE FLOW THEOREM AND ACOUSTIC RECIPROCITY IN COMPRESSIBLE POTENTIAL FLOWS IN DUCTS

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A reverse flow theorem for acoustic propagation in compressible potential flow has been obtained directly from the field equations without recourse to energy conservation arguments. A reciprocity theorem for the scattering matrix for the propagation of acoustic modes in a duct with either acoustically rigid walls or acoustically absorbing walls follows. It is found that for a source at a specific end of the duct, suitably scaled reflection matrices in direct and reverse flow have a reciprocal relationship. Scaled transmission matrices obtained for direct flow and reversed flow with simultaneous switching of source location from one end to the other also have a reciprocal relationship. A related reverse flow theorem specialized to one-dimensional acoustic propagation has also been obtained. Reciprocity relationships for the scattering coefficients for propagation are derived, and are found to be similar though much simpler than in the case of multi-mode propagation. In one-dimensional flow, reciprocal relations and power conservation arguments are used to show that scaled power reflection and transmission coefficients are invariant to flow reversal and switching of source location from one end of the duct to the other. Numerical verification of the reciprocal relationships is given in a companion paper.

1. INTRODUCTION

The general principle of acoustic reciprocity in a medium at rest is well known and is derived in reference [1] by direct manipulation of the field equations in the case of harmonic time dependence. By essentially using this starting point, Eversman [2] has demonstrated and numerically verified reciprocal properties of the scattering matrix for acoustic modes incident, reflected, and transmitted in a non-uniform duct in the absence of mean flow. Moehring [3], by using an approach based on energy conservation, has arrived at the same result when differences in definitions of the normalization of acoustic modes are considered. Moehring's approach depends on a suitable definition of acoustic energy density and acoustic energy flux, which are well known in the case of propagation in a stationary medium. Since the result of Moehring [3] depends on the normal derivative of the acoustic energy flux vanishing on the walls of the duct, it would appear to exclude dissipative walls. However, the classical result [1] concludes that reciprocal relations still hold, provided the duct walls have a locally reacting impedance model (whether dissipative or not) where for harmonic disturbances, acoustic particle velocity is proportional to acoustic pressure. Reciprocity based on energy conservation is more restrictive than necessary.

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An appropriate definition of acoustic energy and energy flux also exists for propagation in a compressible potential flow [4]. On this basis, Moehring [3] extended his observations on properties of the scattering matrix to include non-uniform ducts with rigid walls and potential mean flow with the result that the reciprocal properties also depend on flow reversal. Godin [5] has studied extensively issues of acoustic energy, acoustic reciprocity, and flow reversal theorems in a highly generalized sense directly from the field equations. He has not specifically addressed the simpler case of propagation in non-uniform ducts with compressible irrotational flow, citing Moehring [3] as having demonstrated reverse flow reciprocity in this case [6]. Godin's citations to the literature can be consulted for an extensive survey of the field.

Here, the goal is to approach the acoustic reciprocity problem in a compressible potential flow in non-uniform ducts directly from the field equations in much the same way as the classical formulation in the case of a stationary medium [1]. Furthermore, it is intended to show that reciprocity holds for a finite length dissipative lining imbedded in an otherwise rigid wall. A foundation for such a formulation in the case of uniform flows was given by Flax [7, 8] in connection with unsteady lifting surface theory. The application to potential flows in non-uniform ducts given here yields a reciprocity relationship (perhaps more appropriately referred to as a flow reversal theorem) which is in terms of acoustic potential and acoustic density perturbations on an irrotational compressible mean flow. It can also be given entirely in terms of acoustic potential perturbations. The reverse flow reciprocity formulation which is obtained does not begin with an energy conservation law and leads to a form similar to Moehring [3]. This investigation is not restricted to a duct with rigid walls. The reverse flow reciprocity theorem is then used to establish reciprocal properties of the scattering matrix for propagation of acoustic modes in a non-uniform duct with acoustically absorbing walls.

The special case of one-dimensional propagation is also considered and it is shown that the invariance property for acoustic power transmission in converging–diverging ducts found by Davis [9] is the result of reciprocity and acoustic power conservation arguments. His result is extended to more general duct configurations.

In the present paper, the theoretical framework is established for the reciprocity (flow reversal) relationship, and it is specialized for examining the reciprocal relations which exist between duct modes propagating in compressible mean flow. In a companion paper [10], the results derived here are substantiated by numerical experiments based on a finite element simulation using a new implementation of the boundary condition introduced by the presence of acoustic treatment, the subject of a second companion paper [11].

2. ACOUSTIC PROPAGATION IN A COMPRESSIBLE POTENTIAL FLOW

An extensive discussion of both linear and non-linear formulations for acoustic propagation in potential flows has been given by Campos [12]. His work includes a number of citations to previous work and is the basis for contributions directed mainly toward analytic or semi-analytic solutions for propagation in ducts (see for example references [13, 14]). The investigation reported here has been part of the development of numerical modelling methods for acoustic propagation in non-uniform ducts and therefore, the final form of the governing equations is specialized for that purpose.

The acoustic field equations are obtained by the consideration of unsteady perturbations on a steady compressible potential flow. Accuracy in calculation of both the steady and unsteady flow fields is necessary for computational verification of the theoretical results obtained. The starting point for the formulation of both the steady mean flow and the acoustic perturbation consists of the mass and momentum equations and the energy equation in the form of the isentropic equation of state

$$\frac{\partial \hat{\rho}}{\partial t} + \nabla\cdot(\hat{\rho}\mathbf{V}) = 0, \quad \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = -\frac{1}{\hat{\rho}}\nabla\hat{p}, \tag{1, 2}$$

$$\frac{\hat{p}}{p_0} = \left(\frac{\hat{\rho}}{\rho_0}\right)^{\gamma}. \tag{3}$$

$\hat{p}$, $\hat{\rho}$, $\mathbf{V}$ are fluid properties pressure, density, and velocity, at this point in dimensional form. p_0, ρ_0 are reference values of pressure and density. It is assumed that the mean flow and acoustic perturbations are irrotational and that a potential $\hat{\phi}$ exists such that $\mathbf{V} = \nabla\hat{\phi}$. Acoustic perturbations are assumed on the steady mean flow such that $\hat{\phi} = \phi_r + \phi$, $\hat{\rho} = \rho_r + \rho$ and $\hat{p} = p_r + p$. The linearized continuity equation is

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho_r\nabla\phi + \rho\nabla\phi_r) = 0. \tag{4}$$

The linearized momentum equation, for irrotational acoustic perturbations, is

$$\rho = -\frac{\rho_r}{c_r^2}\left(\frac{\partial\phi}{\partial t} + \nabla\phi_r\cdot\nabla\phi\right). \tag{5}$$

This is used to replace ρ in equation (4) and the linearized equation of state,

$$p = c_r^2\rho, \tag{6}$$

is used to produce an alternative form of the momentum equation in terms of acoustic pressure,

$$p = -\rho_r\left(\frac{\partial\phi}{\partial t} + \nabla\phi_r\cdot\nabla\phi\right). \tag{7}$$

Equation (7) is used to post-process the field solution for ϕ to obtain the acoustic pressure field. The acoustic particle velocity and acoustic velocity potential are related according to

$$\mathbf{v} = \nabla\phi. \tag{8}$$

The perturbation process also produces the conservation equation for the steady flow,

$$\nabla\cdot(\rho_r\nabla\phi_r) = 0, \tag{9}$$

and the steady flow momentum equation in terms of the speed of sound,

$$c_r^2 = 1 - \frac{(\gamma-1)}{2}[\nabla\phi_r\cdot\nabla\phi_r - M_\infty^2], \tag{10}$$

and in terms of the steady flow density,

$$\rho_r = \left[1 - \frac{(\gamma-1)}{2}(\nabla\phi_r\cdot\nabla\phi_r - M_\infty^2)\right]^{1/(\gamma-1)}. \tag{11}$$

Equations (4)–(11) are now in non-dimensional form where ϕ is the acoustic potential, ϕ_r is the local mean flow (reference) potential, ρ is the acoustic density, ρ_r is the local mean flow density, and c_r is the local speed of sound in the mean flow. All quantities are made non-dimensional by using density ρ_∞ and the speed of sound c_∞ at some point, in this case at the plane $x = 0$. Stagnation conditions could also serve as the reference. A reference length R is defined as some characteristic dimension at the plane $x = 0$. In the case of a circular duct, the reference length is the duct radius at $x = 0$. The acoustic potential is non-dimensional with respect to $c_\infty R$, and the acoustic pressure with respect to $\rho_\infty c_\infty^2$. Lengths are made non-dimensional with respect to R. Time is scaled with R/c_∞. In the case of harmonic time dependence, this leads to the definition of non-dimensional frequency $\eta_r = \omega R/c_\infty$. ω is the dimensional source frequency. M_∞ is the Mach number at the reference point.

Equation (9) is the field equation for the calculation of the compressible potential mean flow. Equations (10) and (11) are subsidiary relations that can be used in an iterative solution which at each stage uses a density field derived from the previous iteration step. $\nabla\phi_r$, c_r, ρ_r are required data for the formulation of the acoustic problem.

3. REVERSE FLOW RECIPROCITY PRINCIPLE

The application of a reciprocity relationship for acoustic propagation in potential flows in non-uniform ducts has lagged behind the exploitation of the comparable results for propagation in a quiescent medium, not because of the difficulty in posing the principle, but probably because of the difficulty in producing solutions which could be used to test it. Numerical solutions for duct propagation using the finite element method (FEM) are now achievable [15] and provide the capability of determining the complete acoustic field in a duct (and in the far field of a duct of finite length) as well as the scattering matrix for a non-uniform duct inserted in an otherwise uniform duct of infinite length. This provides an opportunity for testing the reciprocity principle and suggests the development of such a principle for benchmarking of FEM calculations.

The intent here is to approach the reciprocity principle independent of considerations of energy conservation. A counterpart exists in the literature of unsteady lifting surface theory in the Reverse Flow Theorem of Flax [7, 8]. A reciprocity principle for acoustic propagation in non-uniform potential flows in ducts can be obtained by an extension of the formulation of Flax.

Consider the volume Ω shown in Figure 1, which is the interior of a non-uniform duct of arbitrary cross-section. In examples, the duct will be assumed to be axisymmetric (circular or annular), but the principle derived is independent of the duct cross-section. The duct walls can be rigid or locally reacting. The unit normal $\mathbf{n}$ is directed out of the volume at each surface. The source plane S_s is where the acoustic source nominally is specified and the exit plane S_e terminates the duct and may have a reflection matrix specified. For computations, the exit plane will be assumed to be non-reflecting. A typical computational problem would seek to specify the acoustic field within the duct and the scattering matrix at the source plane for incident acoustic modes. Equations (4) and (5) specify the acoustic field within Ω subject to appropriate boundary conditions on S, the surface of Ω.

Let $\phi_1 e^{i\eta_r t}$ be a harmonic solution for the acoustic velocity potential for the case of a mean flow specified everywhere in the duct by its reference Mach number $\mathbf{M}_r = \nabla\phi_r$ and with specified boundary conditions. Let $\phi_2 e^{i\eta_r t}$ be a second harmonic solution for exactly the same duct with different source conditions, but with the flow reversed, $-\mathbf{M}_r = -\nabla\phi_r$. It is important to note that in the reversed flow the reference density ρ_r and reference speed

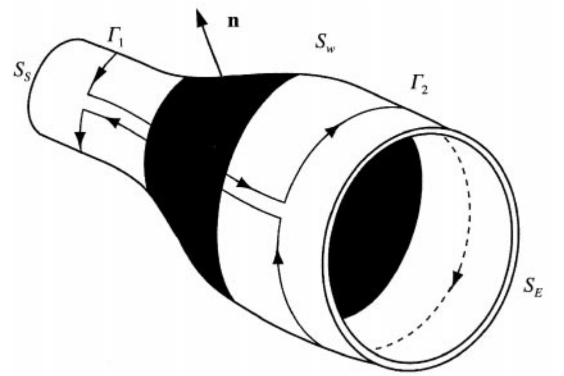

Figure 1. A general duct configuration showing volume and surface important in the development.

of sound c_r are unaltered. Due to equation (4), in the case of a harmonic source at non-dimensional frequency η_r, it follows that

$$\iiint_\Omega \{\phi_2[\mathrm{i}\eta_r\rho_1 + \nabla\cdot(\rho_r\nabla\phi_1 + \nabla\phi_r\rho_1)] - \phi_1[\mathrm{i}\eta_r\rho_2 + \nabla\cdot(\rho_r\nabla\phi_2 - \nabla\phi_r\rho_2)]\}\,\mathrm{d}\Omega = 0. \quad (12)$$

With application of the divergence theorem, equation (12) is reconfigured to

$$\iint_S \{\phi_2[(\rho_r\nabla\phi_1 + \nabla\phi_r\rho_1)] - \phi_1[(\rho_r\nabla\phi_2 - \nabla\phi_r\rho_2)]\}\cdot\mathbf{n}\,\mathrm{d}S$$

$$- \iiint_\Omega \{\nabla\phi_2\cdot\nabla\phi_r\rho_1 + \nabla\phi_1\cdot\nabla\phi_r\rho_2 - \mathrm{i}\eta_r(\phi_2\rho_1 - \phi_1\rho_2)\}\,\mathrm{d}\Omega = 0. \quad (13)$$

The acoustic density in the two solutions is defined according to

$$\rho_1 = -\frac{\rho_r}{c_r^2}(\mathrm{i}\eta_r\phi_1 + \nabla\phi_r\cdot\nabla\phi_1), \qquad \rho_2 = -\frac{\rho_r}{c_r^2}(\mathrm{i}\eta_r\phi_2 - \nabla\phi_r\cdot\nabla\phi_2). \quad (14)$$

Equations (14) are used to eliminate the remaining volume integral in equation (13), leaving the reciprocity principle in terms of acoustic density and potential in a form convenient for subsequent development:

$$\iint_S \{\phi_2[(\rho_r\nabla\phi_1 + \nabla\phi_r\rho_1)] - \phi_1[(\rho_r\nabla\phi_2 - \nabla\phi_r\rho_2)]\}\cdot\mathbf{n}\,\mathrm{d}S = 0. \quad (15a)$$

An alternate form, in terms of the acoustic pressure, obtained by using equation (7), is

$$\iint_S \left\{\left(\frac{p_2}{\rho_r} - \nabla\phi_r\cdot\nabla\phi_2\right)[(\rho_r\nabla\phi_1 + \nabla\phi_r\rho_1)] - \left(\frac{p_1}{\rho_r} + \nabla\phi_r\cdot\nabla\phi_1\right)[(\rho_r\nabla\phi_2 - \nabla\phi_r\rho_2)]\right\}\cdot\mathbf{n}\,\mathrm{d}S = 0. \quad (15b)$$

In establishing reciprocal relationships for the scattering matrix, it is important in this development that equation (15a) or (15b) have contributions only on the source and exit planes, S_s and S_e. This requires that the integrand vanish on the duct walls. For a duct with rigid walls, this occurs because $\nabla\phi_r \cdot \mathbf{n} = 0$, $\nabla\phi_1 \cdot \mathbf{n} = 0$, and $\nabla\phi_2 \cdot \mathbf{n} = 0$ on the walls. In the case when mean flow is absent, $\nabla\phi_r$ vanishes and ρ_r is constant, leading to

$$\iint_S \{p_2\mathbf{v}_1 - p_1\mathbf{v}_2\} \cdot \mathbf{n}\, \mathrm{d}S = 0. \tag{16}$$

For a locally reacting wall impedance in the absence of mean flow, acoustic pressure is proportional to the normal component of particle velocity. This makes the integrand vanish on the walls of the duct, and equation (16) has contributions only on the source and exit planes. Reciprocal properties of the scattering matrix are therefore valid in the absence of flow for normally reacting impedance walls. This is in spite of the fact that acoustic power is not conserved.

When mean flow is present and the walls are normally reacting, further examination is required to establish that equation (15a) has no contribution on the duct walls. Myers [16] has shown that the boundary condition at a normally reacting acoustic lining which relates the boundary displacement ζ to the component of the acoustic particle velocity normal to the undisplaced surface is

$$\nabla\phi \cdot \mathbf{n} = \mathrm{i}\eta_r\zeta + \mathbf{M}_r \cdot \nabla\zeta - \zeta\mathbf{n} \cdot (\mathbf{n} \cdot \nabla)\mathbf{M}_r. \tag{17}$$

With $\nabla\phi_r \cdot \mathbf{n} = 0$ on the duct wall surfaces S_w, lined or unlined, and $\mathbf{M}_r = \nabla\phi_r$ the integral of equation (15a) on S_w becomes

$$I_w = \iint_{S_w} \{\rho_r\phi_2[\mathrm{i}\eta_r\zeta_1 + \mathbf{M}_r \cdot \nabla\zeta_1 - \zeta_1\mathbf{n} \cdot (\mathbf{n} \cdot \nabla)\mathbf{M}_r]$$
$$- \rho_r\phi_1[\mathrm{i}\eta_r\zeta_2 - \mathbf{M}_r \cdot \nabla\zeta_2 + \zeta_2\mathbf{n} \cdot (\mathbf{n} \cdot \nabla)\mathbf{M}_r]\}\, \mathrm{d}S. \tag{18}$$

The following vector relations are introduced (taking into account the special circumstances of the present problem):

$$\rho_r\phi\mathbf{M}_r \cdot \nabla\zeta = \rho_r\mathbf{M}_r \cdot \nabla\phi\zeta - \rho_r\zeta\mathbf{M}_r \cdot \nabla\phi,$$

$$\nabla \cdot \rho_r\mathbf{M}_r = 0, \quad \rho_r\mathbf{M}_r \cdot \nabla\phi\zeta = \nabla \cdot \rho_r\phi\zeta\mathbf{M}_r, \quad \mathbf{n} \cdot \mathbf{M}_r = 0,$$

$$\rho_r\phi\zeta\mathbf{n} \cdot (\mathbf{n} \cdot \nabla)\mathbf{M}_r = \mathbf{n} \cdot (\mathbf{n} \cdot \nabla)\rho_r\phi\zeta\mathbf{M}_r,$$

$$\mathbf{n} \cdot \nabla \times (\mathbf{n} \times \rho_r\phi\zeta\mathbf{M}_r) = \nabla \cdot \rho_r\phi\zeta\mathbf{M}_r - \mathbf{n} \cdot (\mathbf{n} \cdot \nabla)\rho_r\phi\zeta\mathbf{M}_r.$$

Equation (18) can then be reformulated as

$$I_w = \iint_{S_w} \{\rho_r\zeta_1[\mathrm{i}\eta_r\phi_2 - \mathbf{M}_r \cdot \nabla\phi_2] - \rho_r\zeta_2[\mathrm{i}\eta_r\phi_1 + \mathbf{M}_r \cdot \nabla\phi_2]\}\, \mathrm{d}S$$
$$+ \iint_{S_w} \{\mathbf{n} \cdot \nabla \times (\mathbf{n} \times \rho_r\phi_2\zeta_1\mathbf{M}_r) + \mathbf{n} \cdot \nabla \times (\mathbf{n} \times \rho_r\phi_1\zeta_2\mathbf{M}_r)\}\, \mathrm{d}S. \tag{19}$$

The case considered here is an impedance wall imbedded in an otherwise rigid wall. Therefore, as shown in Figure 2, acoustic treatment imbedded in S_w extends less than the full length of the duct. This is physically realistic, and is also consistent with the type of

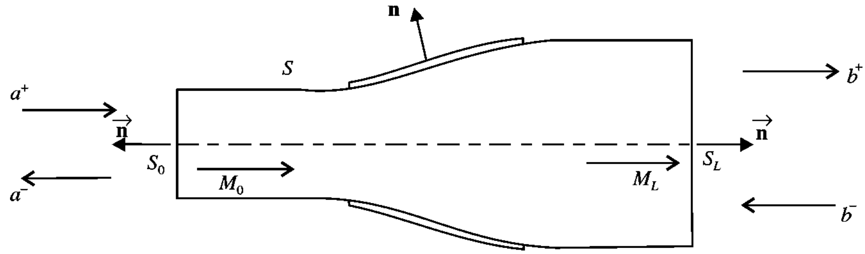

Figure 2. A duct configuration showing uniform extensions and appropriate surface normals.

reciprocity relation between duct modal amplitudes which is sought. Following the development of Moehring [17], the last integral can be written as a line integral on the boundary Γ of the surface S_w by using Stokes' theorem. Since the surface S_w consists of the wall of the duct (in general of varying cross-section), Γ is chosen to consist of closed curves Γ_1 and Γ_2 circumscribed on the duct wall outside of the region of the lining, and therefore, where the duct wall is rigid. There is also a portion of Γ which runs along the duct wall between Γ_1 and Γ_2 to complete the closed curve of Stokes' theorem, but this curve is traversed twice, once in each direction, and has no net contribution. The contour Γ is shown in Figure 1. To make use of Stokes' theorem, it is required that the acoustic field and the wall displacement be continuous on S_w. Hence, since the acoustic treatment is of limited length imbedded in an otherwise rigid wall duct, the transition from rigid wall to admittance wall, as well as the variation of admittance along the treated wall, must be continuous. If this condition is met, Stokes' theorem can be cast in the form

$$\iint_{S_w} \{\mathbf{n}\cdot \nabla\times(\mathbf{n}\times\rho_r\phi\zeta\mathbf{M}_r)\}\,\mathrm{d}S = \int_{\Gamma_1}(\mathbf{n}\times\rho_r\phi\zeta\mathbf{M}_r)\,\mathrm{d}\boldsymbol{\Gamma} + \int_{\Gamma_2}(\mathbf{n}\times\rho_r\phi\zeta\mathbf{M}_r)\,\mathrm{d}\boldsymbol{\Gamma}.$$

The integral on the surface S_w vanishes if the line integrals vanish. On a hard wall the line integrals vanish because the boundary displacement vanishes. This means that if the condition for the use of Stokes' theorem is met, then the integral of equation (19) is

$$I_w = \iint_{S_w} \{\rho_r\zeta_1[\mathrm{i}\eta_r\phi_2 - \mathbf{M}_r\cdot\nabla\phi_2] - \rho_r\zeta_2[\mathrm{i}\eta_r\phi_1 + \mathbf{M}_r\cdot\nabla\phi_1]\}\,\mathrm{d}S. \tag{20}$$

With equation (7) this becomes

$$I_w = \iint_{S_w} \{\zeta_2 p_1 - \zeta_1 p_2\}\,\mathrm{d}S. \tag{21}$$

At a wall of admittance A, there is a relation between pressure and wall velocity which is frequency dependent and of the form

$$\mathrm{i}\eta_r\zeta = Ap. \tag{22}$$

The integral I_w vanishes and equation (15a) or (15b) have contributions only on the portion of the surface area S which corresponds to duct cross-sections beyond the impedance wall, on the surfaces S_s and S_e. With appropriate restrictions on the impedance wall, the reciprocity principle is therefore unchanged for hard and soft wall ducts.

The restriction on the impedance wall is interesting. If admittance is indeed discontinuous along the wall, then ζ has to be discontinuous. Moehring [17] has noted that for discontinuous admittance variation, there is no clear condition to be imposed on the acoustic field or wall displacement at the discontinuity. Rebel and Ronneberger [18] have shown that the condition of admittance discontinuity and the assumption of potential flow at the wall (no boundary layer) causes a problem with the underlying physics of the flow. What is known is that a numerical procedure such as the FEM restricts the solution for acoustic potential to continuous functions, and the lining displacement never appears in the final field equations and boundary condition. Thus one suspects that the restriction to continuously varying admittance is not a critical issue in numerical comparisons, but it may be in comparison with experiment.

Equation (15a) can also be written entirely in terms of acoustic potential:

$$\iint_S \phi_2\left\{\rho_r\left(\nabla\phi_1 - \frac{1}{c_r^2}\mathbf{M}_r\mathbf{M}_r\cdot\nabla\phi_1\right) - \mathrm{i}\eta_r\frac{\rho_r}{c_r^2}\mathbf{M}_r\phi_1\right\}\cdot\mathbf{n}\,\mathrm{d}S$$

$$-\iint_S \phi_1\left\{\rho_r\left(\nabla\phi_2 - \frac{1}{c_r^2}\mathbf{M}_r\,\mathbf{M}_r\cdot\nabla\phi_2\right) + \mathrm{i}\eta_r\frac{\rho_r}{c_r^2}\mathbf{M}_r\phi_2\right\}\cdot\mathbf{n}\,\mathrm{d}S = 0, \quad (23)$$

where the area S is now understood to include only the source and exit planes S_s and S_e, shown in Figure 2.

4. APPLICATION TO A NON-UNIFORM DUCT

The discussion here is presented for a duct with a straight x-axis, but is more complicated only in the notation if the axis is not straight. Figure 2 shows a representative non-uniform duct with cross-section $S(x)$ defined on $0 \leqslant x \leqslant L$. The nominal source plane, identified as S_s in Figure 1, is placed at $x = 0$ and specifically identified as $S(0) = S_0$. Similarly, the nominal exit plane S_e in Figure 1 is placed at $x = L$ and identified as $S(L) = S_L$. With source reversal, the roles of the source and exit planes reverse, but S_0 and S_L are always related to the co-ordinate system. There is a steady mean potential flow in the duct $\mathbf{M}(x, \mathbf{r})$, defined as

$$\mathbf{M}(x,\mathbf{r}) = \frac{c_\infty}{c(x,\mathbf{r})}\mathbf{M}_r(x,\mathbf{r}) = \frac{1}{c_r(x,\mathbf{r})}\nabla\phi_r. \quad (24)$$

$\mathbf{M}_r(x, \mathbf{r})$ is a non-dimensional velocity based on the reference speed of sound c_∞. $M(x, \mathbf{r})$ is the local Mach number defined in the usual way and $c(x, \mathbf{r})$ is the local (dimensional) speed of sound. $(x, \mathbf{r})$ denotes a point in a cross-section at x in a co-ordinate system appropriate for the duct geometry. The corresponding reversed flow is $-\mathbf{M}$. The non-dimensional frequency η can also be defined locally (local speed of sound but with reference length R) according to

$$\eta_r = \frac{\omega R}{c_\infty} = \frac{\omega R}{c}\frac{c}{c_\infty} = c_r\eta. \quad (25)$$

η is a local non-dimensional frequency based on the local speed of sound. In terms of local Mach number and non-dimensional frequency, equation (23) becomes

$$\iint_{S_0+S_L} \rho_r \phi_2 \{[\nabla\phi_1 - \mathbf{M}(\mathbf{M}\cdot\nabla\phi_1)] - \mathrm{i}\eta\mathbf{M}\phi_1\}\cdot\mathbf{n}\,\mathrm{d}S$$

$$- \iint_{S_0+S_L} \rho_r \phi_1 \{[\nabla\phi_2 - \mathbf{M}(\mathbf{M}\cdot\nabla\phi_2)] + \mathrm{i}\eta\mathbf{M}\phi_2\}\cdot\mathbf{n}\,\mathrm{d}S = 0. \tag{26}$$

It is assumed that at the nominal inflow end of the duct at $x = 0$ the steady mean flow is uniform on the cross-section with $\mathbf{M}(x, \mathbf{r}) = M_0\mathbf{i}$, and at the nominal outflow end $x = L$ the steady mean flow is uniform with $\mathbf{M}(x, \mathbf{r}) = M_L\mathbf{i}$. Reference density ρ_{r_0} and ρ_{r_L} and local non-dimensional frequency η_0 and η_L are defined similarly. The assumption of uniform conditions implies that the inflow and outflow planes are well removed from the non-uniform region of the duct. In computational examples, it is found that for ducts with circular or annular cross-sections, uniform inlet and outlet ducts of length two duct radii ahead of and beyond the non-uniformity are sufficient. At $x = 0$ the outward unit normal $\mathbf{n} = -\mathbf{i}$ and at $x = L$ the normal is $\mathbf{n} = \mathbf{i}$. With these observations the reciprocity principle of equation (26) becomes

$$\rho_{r_0}\iint_{S_0} \phi_2\left\{(1 - M_0^2)\frac{\partial\phi_1}{\partial x} - \mathrm{i}\eta_0 M_0\phi_1\right\}\mathrm{d}S$$

$$- \rho_{r_0}\iint_{S_0} \phi_1\left\{(1 - M_0^2)\frac{\partial\phi_2}{\partial x} + \mathrm{i}\eta_0 M_0\phi_2\right\}\mathrm{d}S$$

$$= \rho_{r_L}\iint_{S_L} \phi_2\left\{(1 - M_L^2)\frac{\partial\phi_1}{\partial x} - \mathrm{i}\eta_L M_L\phi_1\right\}\mathrm{d}S$$

$$- \rho_{r_L}\iint_{S_L} \phi_1\left\{(1 - M_L^2)\frac{\partial\phi_2}{\partial x} + \mathrm{i}\eta_L M_L\phi_2\right\}\mathrm{d}S. \tag{27}$$

5. CIRCULAR DUCT EXAMPLE

In the regions of uniform flow at the ends of the duct at $x = 0$ and L acoustic potential can be approximated by an N term eigenfunction expansion in terms of duct modes (Figure 2). In the case of a circular duct, the expansion can be expressed at $x = 0$ in vector-matrix form for angular dependence $\mathrm{e}^{-\mathrm{i}m\theta}$ as

$$\phi_m(x, r, \theta) = [\Phi_m(r)][e_m^+(x)]\{a_m^+\}\mathrm{e}^{-\mathrm{i}m\theta} + [\Phi_m(r)][e_m^-(x)]\{a_m^-\}\mathrm{e}^{-\mathrm{i}m\theta}. \tag{28}$$

The derivative is

$$\frac{\partial\phi_m}{\partial x}(x, r, \theta) = [\Phi_m(r)][e_m^+(x)][-\mathrm{i}k_x^+]\{a_m^+\}\mathrm{e}^{-\mathrm{i}m\theta} + [\Phi_m(r)][e_m^-(x)][-\mathrm{i}k_x^-]\{a_m^-\}\mathrm{e}^{-\mathrm{i}m\theta}. \tag{29}$$

$[\Phi(r)]$ is a $1\times N$ row matrix of duct radial modes, the same for both right and left propagating modes. $[e_m^+(x)]$ and $[e_m^-(x)]$ are $N\times N$ diagonal matrices with typical elements

$e^{-ik^{\pm}_{x_{mn}}x} \cdot [-ik^{+}_{x}]$ and $[-ik^{-}_{x}]$ are $N \times N$ diagonal matrices with typical elements $-ik^{\pm}_{x_{mn}} \cdot \{a^{+}_{m}\}$ and $\{a^{-}_{m}\}$ are $N \times 1$ vectors of modal amplitude coefficients for right (positive x) and left (negative x) modes. $k^{\pm}_{x_{mn}}$ are axial wave numbers corrresponding to angular mode number m and radial mode number n. Axial wave numbers for acoustic modes in the context of the present problem are discussed in Appendix A.1.

With these observations, deleting for simplicity of notation the implied dependence on the mode number m of the axial wave number, and taking the reversed flow solution to have angular dependence $e^{im\theta}$, it is possible to express the solution ϕ_1 and its counterpart reversed flow solution ϕ_2 as

$$\phi_1 = [\Phi]\{a^{+}_{1}\}e^{-im\theta} + [\Phi]\{a^{-}_{1}\}e^{-im\theta}, \tag{30}$$

$$\frac{\partial \phi_1}{\partial x} = [\Phi][-ik^{+}_{x_1}]\{a^{+}_{1}\}e^{-im\theta} + [\Phi][-ik^{-}_{x_1}]\{a^{-}_{1}\}e^{-im\theta}. \tag{31}$$

At $x = 0$ the diagonal matrices $[e^{+}_{m}(0)]$ and $[e^{-}_{m}(0)]$ become identity matrices. The corresponding reversed flow solution is

$$\phi_2 = [\Phi]\{a^{+}_{2}\}e^{im\theta} + [\Phi]\{a^{-}_{2}\}e^{im\theta}, \tag{32}$$

$$\frac{\partial \phi_2}{\partial x} = [\Phi][-ik^{+}_{x_2}]\{a^{+}_{2}\}e^{im\theta} + [\Phi][-ik^{-}_{x_2}]\{a^{-}_{2}\}e^{im\theta}. \tag{33}$$

Similar expansions with modal amplitudes $\{b^{+}_{m}\}$ and $\{b^{-}_{m}\}$ apply at plane $x = L$. Here, the phase information in the matrices $[e^{+}_{m}(L)]$ and $[e^{-}_{m}(L)]$ is absorbed into the amplitude coefficients.

The wave numbers and eigenfunctions are independent of whether angular dependence is $e^{-im\theta}$ or $e^{im\theta}$. The choice of angular dependence in the two solutions eliminates the angular dependence in integration arising in equation (27). The physical implication is that of a spinning mode which has the same vector sense with respect to the flow direction. In calculations, scattering coefficients do not depend on the sign of m.

If equations [30–33] are introduced into equation (27) and the indicated integrations are carried out taking advantage of orthogonality of the acoustic modes, the reverse flow theorem yields

$$\{a^{-}_{1}\}^{T}[J_0][\alpha_0]\{a^{+}_{2}\} + \{b^{+}_{1}\}^{T}[J_L][\alpha_L]\{b^{-}_{2}\} = \{a^{-}_{2}\}^{T}[J_0][\alpha_0]\{a^{+}_{1}\} + \{b^{+}_{2}\}^{T}[J_L][\alpha_L]\{b^{-}_{1}\}. \tag{34}$$

$[J_0]$, $[J_L]$, $[\alpha_0]$, $[\alpha_L]$ are diagonal matrices with elements dependent on the mode of propagation. The composition of these matrices is given in Appendix A.2.

Equation (34) can be written in partitioned form as

$$\begin{Bmatrix} a^{-}_{1} \\ b^{+}_{1} \end{Bmatrix}^{T} \begin{bmatrix} [J_0][\alpha_0] & \\ & [J_L][\alpha_L] \end{bmatrix} \begin{Bmatrix} a^{+}_{2} \\ b^{-}_{2} \end{Bmatrix} = \begin{Bmatrix} a^{-}_{2} \\ b^{+}_{2} \end{Bmatrix}^{T} \begin{bmatrix} [J_0][\alpha_0] & \\ & [J_L][\alpha_L] \end{bmatrix} \begin{Bmatrix} a^{+}_{1} \\ b^{-}_{1} \end{Bmatrix}. \tag{35}$$

Modal amplitudes a^{+}_{n} and a^{-}_{n} and b^{+}_{n} and b^{-}_{n} are related by the acoustic potential scattering matrix according to

$$\begin{Bmatrix} a^{-} \\ b^{+} \end{Bmatrix} = [S] \begin{Bmatrix} a^{+} \\ b^{-} \end{Bmatrix}, \tag{36}$$

where the scattering matrix is defined as

$$[S] = \begin{bmatrix} [R] & [\tilde{T}] \\ [T] & [\tilde{R}] \end{bmatrix}. \tag{37}$$

Contained in $[S]$ are the usual reflection matrix $[R]$ and transmission matrix $[T]$ for acoustic modes incident at $x = 0$ and reflection and transmission matrices $[\tilde{R}]$ and $[\tilde{T}]$ for modes incident at $x = L$. There will be a scattering matrix $[S_1]$ for nominal mean flow and a second one $[S_2]$ for reversed flow. The relationship between $[S_1]$ and $[S_2]$ can be obtained using the reciprocity theorem.

Equation (35) is rewritten by introducing the definition of the scattering matrix from equation (36) and by using the definition

$$\begin{bmatrix} [J_0][\alpha_0] & \\ & [J_L][\alpha_L] \end{bmatrix} = [J][\alpha]. \tag{37$'$}$$

The result is

$$\begin{Bmatrix} a_1^+ \\ b_1^- \end{Bmatrix}^{\mathrm{T}} [S_1]^{\mathrm{T}}[J][\alpha] \begin{Bmatrix} a_2^+ \\ b_2^- \end{Bmatrix} = \begin{Bmatrix} a_2^+ \\ b_2^- \end{Bmatrix}^{\mathrm{T}} [S_2]^{\mathrm{T}}[J][\alpha] \begin{Bmatrix} a_1^+ \\ b_1^- \end{Bmatrix}. \tag{38}$$

Equation (38) reveals that

$$[S_1]^{\mathrm{T}}[J][\alpha] = [J][\alpha][S_2] \tag{39}$$

or

$$[J][\alpha][S_1] = ([J][\alpha][S_2])^{\mathrm{T}}. \tag{40}$$

Equations (39) and (40) show that a weighted version of the nominal flow acoustic potential scattering matrix and similarly weighted version of the reversed flow acoustic potential scattering matrix are transposes of one another. In terms of the acoustic potential reflection and transmission coefficient matrices, the result is

$$[R_1]^{\mathrm{T}}[J_0][\alpha_0] = [J_0][\alpha_0][R_2], \quad [\tilde{R}_1]^{\mathrm{T}}[J_L][\alpha_L] = [J_L][\alpha_L][\tilde{R}_2], \tag{41, 42}$$

$$[T_1]^{\mathrm{T}}[J_L][\alpha_L] = [J_0][\alpha_0][\tilde{T}_2], \quad [\tilde{T}_1]^{\mathrm{T}}[J_0][\alpha_0] = [J_L][\alpha_L][T_2]. \tag{43, 44}$$

The reciprocal relationships of equations (41)–(44) involve acoustic potential reflection and transmission coefficient matrices, with diagonal elements representing reflection and transmission coefficients in the incident modes (here referred to as direct reflection or transmission) and off diagonal reflection and transmission coefficients from the incident mode to another mode. Equations (41) and (42) show that direct acoustic potential reflection coefficients are invariant in reversed flow. The transmission coefficient matrix pairs $[T_1]$, $[T_2]$ and $[\tilde{T}_1]$, $[\tilde{T}_2]$ are not reciprocally related but the pairs $[T_1]$, $[\tilde{T}_2]$ and $[\tilde{T}_1]$, $[T_2]$ are related by equations (43) and (44). These results for acoustic potential reflection and transmission coefficient matrices are more interesting than those obtained for acoustic pressure reflection and transmission coefficient matrices in the absence of flow [2] because they identify a relationship between reflection coefficient matrices in nominal and reversed flow which includes the observation that the direct reflection coefficient (in the incident mode) is invariant to flow direction.

6. RECIPROCITY IN TERMS OF ACOUSTIC PRESSURE

The entire development to this point has been carried out in terms of acoustic potential because the field equations (4) and (5) favor this formulation. Equation (7) provides a direct relationship between acoustic pressure and acoustic potential which can be used to restructure the reciprocity results in terms of acoustic pressure modal amplitudes. In terms of local non-dimensional frequency and local Mach number in a uniform section of duct with uniform flow, equation (7) can be rewritten as

$$p = -\rho_r c_r \left(\mathrm{i}\eta\phi + M \frac{\partial \phi}{\partial x} \right). \tag{45}$$

With equation (45) a connection between acoustic potential modal amplitudes and acoustic pressure modal amplitudes can be established. Consider an acoustic mode propagating in the uniform section with the axial wave number given in Appendix A.1 by equations (A1)–(A3). By referring, for example, to equation (33) which evaluates the potential derivative, the pressure amplitude can be found in terms of the potential amplitude in nominal flow as

$$p_1^{\pm} = -\mathrm{i}\eta\rho_r c_r \left(1 - M \frac{k_x^{\pm}}{\eta} \right) \phi_1^{\pm} = \frac{1}{\beta^{\pm}} \phi_1^{\pm} \tag{46}$$

and in reversed flow by

$$p_2^{\pm} = -\mathrm{i}\eta\rho_r c_r \left(1 + M \frac{k_x^{\pm}}{\eta} \right) \phi_2^{\pm} = \frac{1}{\beta^{\mp}} \phi_2^{\pm}. \tag{47}$$

The coefficients $\beta^{\pm}$, $\beta^{\mp}$ are defined in detail in Appendix A.3.

These relations between acoustic pressure modal coefficients and acoustic potential modal coefficients are evaluated at $x = 0$ and L to produce transformations between acoustic potential modal amplitudes $\{a^{\pm}\}$, $\{b^{\pm}\}$ and acoustic pressure modal amplitudes $\{c^{\pm}\}$, $\{d^{\pm}\}$:

$$\begin{Bmatrix} a_1^- \\ b_1^+ \end{Bmatrix} = \begin{bmatrix} \beta_0^- & \\ & \beta_L^+ \end{bmatrix} \begin{Bmatrix} c_1^- \\ d_1^+ \end{Bmatrix} = [B^{\mp}] \begin{Bmatrix} c_1^- \\ d_1^+ \end{Bmatrix}, \tag{48}$$

$$\begin{Bmatrix} a_1^+ \\ b_1^- \end{Bmatrix} = \begin{bmatrix} \beta_0^+ & \\ & \beta_L^- \end{bmatrix} \begin{Bmatrix} c_1^+ \\ d_1^- \end{Bmatrix} = [B^{\pm}] \begin{Bmatrix} c_1^+ \\ d_1^- \end{Bmatrix}, \tag{49}$$

$$\begin{Bmatrix} a_2^- \\ b_2^+ \end{Bmatrix} = \begin{bmatrix} \beta_0^+ & \\ & \beta_L^- \end{bmatrix} \begin{Bmatrix} c_2^- \\ d_2^+ \end{Bmatrix} = [B^{\pm}] \begin{Bmatrix} c_2^- \\ d_2^+ \end{Bmatrix}, \tag{50}$$

$$\begin{Bmatrix} a_2^+ \\ b_2^- \end{Bmatrix} = \begin{bmatrix} \beta_0^- & \\ & \beta_L^+ \end{bmatrix} \begin{Bmatrix} c_2^+ \\ d_2^- \end{Bmatrix} = [B^{\mp}] \begin{Bmatrix} c_2^+ \\ d_2^- \end{Bmatrix}. \tag{51}$$

The $2N \times 2N$ diagonal matrices $[B^{\pm}]$ and $[B^{\mp}]$ have coefficients defined by equations (46) and (47) for each mode, evaluated at the appropriate end of the duct, arranged along the diagonal. Equations (48)–(51) and equation (36) provide a relationship between the scattering matrices for acoustic potential and the scattering matrices for acoustic pressure:

$$[\bar{S}_1] = [B^{\mp}]^{-1}[S_1][B^{\pm}], \qquad [\bar{S}_2] = [\mathrm{B}^{\pm}]^{-1}[S_2][B^{\mp}]. \tag{52, 53}$$

$[S_1]$, $[S_2]$ are the scattering matrices in nominal and reversed flow for acoustic potential modal amplitudes and $[\bar{S}_1]$, $[\bar{S}_2]$ are the scattering matrices for acoustic pressure modal amplitudes. Equations (48)–(53) and equation (38) are used to arrive at the reciprocity relationship for acoustic pressure modal amplitudes in terms of the nominal flow and reverse flow acoustic pressure scattering matrices:

$$[B^{\pm}]^{-1}[\bar{S}_1]^{\mathrm{T}}[J][\alpha][B^{\mp}] = [B^{\pm}][J][\alpha][\bar{S}_2][\mathrm{B}^{\mp}]^{-1}. \tag{54}$$

The four reciprocal relationships are

$$[\beta_0^+]^{-1}[\bar{R}_1]^{\mathrm{T}}[J_0][\alpha_0][\beta_0^-] = [\beta_0^+][J_0][\alpha_0][\bar{R}_2][\beta_0^-]^{-1}, \tag{55}$$

$$[\beta_L^-]^{-1}[\bar{\bar{R}}_1]^{\mathrm{T}}[J_L][\alpha_L][\beta_L^+] = [\beta_L^-][J_L][\alpha_L][\bar{\bar{R}}_2][\beta_L^+]^{-1}, \tag{56}$$

$$[\beta_0^+]^{-1}[\bar{T}_1]^{\mathrm{T}}[\mathrm{J}_L][\alpha_L][\beta_L^+] = [\beta_0^+][J_0][\alpha_0][\bar{\bar{T}}_2][\beta_L^+]^{-1}, \tag{57}$$

$$[\beta_L^-]^{-1}[\bar{\bar{T}}_1]^{\mathrm{T}}[J_0][\alpha_0][\beta_0^-] = [\beta_L^-][J_L][\alpha_L][\bar{T}_2][\beta_0^-]^{-1}. \tag{58}$$

The reciprocal relationships of equations (55)–(58) involve acoustic pressure reflection and transmission coefficient matrices, with diagonal elements representing reflection and transmission coefficients in the incident modes (here referred to as direct reflection or transmission) and off diagonal reflection and transmission coefficients from the incident mode to another mode. From equations (55) and (56) it is seen that in terms of acoustic pressure amplitude, direct reflection and transmission coefficients in nominal flow and reversed flow are not invariant but are simply related. Transmission coefficient matrix pairs $[\bar{T}_1]$, $[\bar{T}_2]$ and $[\bar{\bar{T}}_1]$, $[\bar{\bar{T}}_2]$ are not directly related but pairs $[\bar{T}_1]$, $[\bar{\bar{T}}_2]$ and $[\bar{\bar{T}}_1]$, $[\bar{T}_2]$ are related by equations (57) and (58).

Owing to the way in which equations (55)–(58) were developed, the weighted, or scaled, matrices on the right- and left-hand sides are equivalent to their counterparts in equations (41)–(44). This convenient definition of scaled pressure reflection and transmission matrices makes them numerically equal to their scaled acoustic potential counterparts.

7. RECIPROCITY IN ONE-DIMENSIONAL FLOW

In some cases it is possible to use a one-dimensional approximation for steady flow and acoustic perturbations. A reverse flow reciprocity relationship can also be obtained in this case. Figure 1 shows the duct under consideration, but which in this case satisfies the requirements that the flow and propagation can be considered as quasi-one dimensional. Acoustic treatment is not considered in this case because at best it can only be included in an *ad hoc* way not fully consistent with the one-dimensional restrictions. The cross-sectional shape of the duct is not a consideration in this one-dimensional approximation. The one-dimensional acoustic continuity equation is

$$A\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho_r A \frac{\partial \phi}{\partial x} + \rho A \frac{\partial \phi_r}{\partial x}\right) = 0, \tag{59}$$

where A is the local cross-sectional area of the duct. The one-dimensional acoustic momentum equation is

$$\rho = -\frac{\rho_r}{c_r^2}\left(\mathrm{i}\eta_r \phi + M_r \frac{\partial \phi}{\partial x}\right) \tag{60}$$

or

$$p = -\rho_r\left(\mathrm{i}\eta_r\phi + M_r\frac{\partial\phi}{\partial x}\right). \tag{61}$$

The steady flow is obtained from

$$\frac{\partial}{\partial x}\left(\rho_r A\frac{\partial\phi}{\partial x}\right) = 0 \tag{62}$$

and the relations between mean flow velocity and mean flow potential and acoustic particle velocity and acoustic potential are

$$M_r = \frac{\partial\phi_r}{\partial x}, \qquad u = \frac{\partial\phi}{\partial x}. \tag{63, 64}$$

The acoustic relationship between pressure and density is

$$p = c_r^2\rho. \tag{65}$$

The mean flow speed of sound is determined by

$$c_r^2 = 1 - \frac{(\gamma - 1)}{2}[M_r^2 - M_\infty^2] \tag{66}$$

and the mean flow density is

$$\rho_r = \left[1 - \frac{(\gamma-1)}{2}(M_r^2 - M_\infty^2)\right]^{1/(\gamma-1)}. \tag{67}$$

Equations (59)–(67) are non-dimensional as in equations (4)–(11), with the convention here that the reference length R is a (hypothetical) radius corresponding to a (circular) source "plane" at $x = 0$, the reference cross-sectional area. The non-dimensionalization described here differs from the usual one-dimensional development in which some characteristic duct length is generally used as a reference length. The non-dimensional frequency η_r is defined exactly as in the preceeding three-dimensional reciprocity development.

The duct which is locally of arbitrary cross-section, is terminated on each end by a section of uniform duct, of length sufficient to assure uniform mean flow and propagation in terms of acoustic modes. The source plane $x = 0$ is where the acoustic source is nominally specified and the exit plane $x = L$ terminates the duct. The exit plane is assumed as non-reflecting.

The equivalent of equation (12) is

$$\int\{\phi_2[\mathrm{i}\eta_r A\rho_1 + \nabla\cdot(\rho_r A\nabla\phi_1 + \mathbf{M}_r A\rho_1)] - \phi_1[\mathrm{i}\eta_r A\rho_2 + \nabla\cdot(\rho_r A\nabla\phi_2 - \mathbf{M}_r A\rho_2)]\}\,\mathrm{d}x = 0. \tag{68}$$

Here, for ease of notation, and for a degree of similarity with the general development, the gradient operator is used to denote

$$\nabla = \frac{\partial}{\partial x}\mathbf{e}_x \tag{69}$$

and

$$\mathbf{M}_r = M_r \mathbf{e}_x, \tag{70}$$

with $\mathbf{e}_x$ the unit vector in the x direction. With application of integration by parts, which is equivalent to the divergence theorem used to obtain equation (13), and by using equation (60) to partially replace ρ_1 and ρ_2, and to subsequently eliminate the remaining integral on x, equation (68) is reduced to

$$\left[\phi_2\left(\rho_r A\frac{\partial \phi_1}{\partial x} + M_r A \rho_1\right) - \phi_1\left(\rho_r A\frac{\partial \phi_2}{\partial x} - M_r A\rho_2\right)\right]_0^L = 0. \tag{71}$$

When this is written entirely in terms of acoustic potential the result is

$$\left[A\rho_r\left\{\phi_2\left[(1-M^2)\frac{\partial \phi_1}{\partial x} - \mathrm{i}\eta M\phi_1\right] - \phi_1\left[(1-M^2)\frac{\partial \phi_2}{\partial x} + \mathrm{i}\eta M\phi_2\right]\right\}\right]_0^L = 0. \tag{72}$$

For multiple mode propagation, eigenfunction expansions were used to define the acoustic potential field in the uniform flow regions at $x = 0$ and L. This is still appropriate, with the simplification that only the plane waves propagate, and no analysis is required to determine the modes and wave numbers.

There is one "mode" propagating in each direction in the uniform duct sections, so that there is only a superposition of a right and left wave, given for example here for the nominal flow solution at $x = 0$ in the form

$$\phi_1 = a_1^+ + a_1^-, \qquad \frac{\partial \phi_1}{\partial x} = (-\,\mathrm{i}k_{x_1}^+)a_1^+ + (-\,\mathrm{i}k_{x_1}^-)a_1^-. \tag{73, 74}$$

The axial wave numbers are determined from the general relations in Appendix A.1, equations (A1) and (A3), with $\kappa_{mn} = 0$, for the plane wave case,

$$\left(\frac{k_{x_{mn}}^\pm}{\eta}\right)_1 = \frac{1}{1-M^2}(-\,M \pm 1), \qquad \left(\frac{k_{x_{mn}}^\mp}{\eta}\right)_2 = \frac{1}{1-M^2}(M \pm 1). \tag{75, 76}$$

The superscript choice $\pm$ corresponds to right and left waves. η and M represent local values of non-dimensional frequency and Mach number determined by the local speed of sound, with η still based on the reference length. The operator arising in equation (72),

$$L\{\phi\} = (1-M^2)\frac{\partial \phi}{\partial x} - \mathrm{i}\eta M\phi = \alpha\phi, \tag{77}$$

introduces the parameters α^+ and α^- corresponding to right and left waves, for example at $x = 0$,

$$\alpha_1^+ = \rho_r[-\,\mathrm{i}(1-M^2)k_{x_1}^+ - \mathrm{i}\eta M] = -\,\mathrm{i}\rho_r\eta, \tag{78}$$

$$\alpha_1^- = \rho_r[-\,\mathrm{i}(1-M^2)k_x^+ - \mathrm{i}\eta M] = \mathrm{i}\rho_r\eta. \tag{79}$$

Similar expressions exist for $\alpha_2^\pm$, in reversed flow. It is noted that $\alpha_2^+ = \alpha_1^+ = \alpha$ and $\alpha_2^- = \alpha_1^- = -\,\alpha$. α is evaluated at the ends $x = 0$ and L as required.

With these definitions, equation (72) can be rearranged in a form equivalent to equation (38),

$$\begin{Bmatrix} a_1^- \\ b_1^+ \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} A_0\alpha_0 & \\ & A_L\alpha_L \end{bmatrix} \begin{Bmatrix} a_2^+ \\ b_2^- \end{Bmatrix} = \begin{Bmatrix} a_2^- \\ b_2^+ \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} A_0\alpha_0 & \\ & A_L\alpha_L \end{bmatrix} \begin{Bmatrix} a_1^+ \\ b_1^- \end{Bmatrix}. \tag{80}$$

a_n^+ and a_n^- and b_n^+ and b_n^-, acoustic potential modal amplitudes at $x = 0$ and L, are related by the acoustic potential scattering matrix according to

$$\begin{Bmatrix} a^- \\ b^+ \end{Bmatrix} = [S] \begin{Bmatrix} a^+ \\ b^- \end{Bmatrix}, \tag{81}$$

where the scattering matrix is defined as

$$[S] = \begin{bmatrix} R & \tilde{T} \\ T & \tilde{R} \end{bmatrix}. \tag{82}$$

R and T are the reflection and transmission coefficients (scalars) for a source at $x = 0$ with a reflection free termination at $x = L$. $\tilde{R}$ and $\tilde{T}$ are reflection and transmission coefficients for a source at $x = L$ and reflection free at $x = 0$. There will be a scattering matrix $[S_1]$ for nominal mean flow and a second one $[S_2]$ for reversed flow. Equations (81) and (82) can be used in equation (80) to obtain

$$\begin{Bmatrix} a_1^+ \\ b_1^- \end{Bmatrix}^{\mathrm{T}} [S_1]^{\mathrm{T}}[A][\alpha] \begin{Bmatrix} a_2^+ \\ b_2^- \end{Bmatrix} = \begin{Bmatrix} a_2^+ \\ b_2^- \end{Bmatrix}^{\mathrm{T}} [S_2]^{\mathrm{T}}[A][\alpha] \begin{Bmatrix} a_1^+ \\ b_1^- \end{Bmatrix}, \tag{83}$$

where the definition

$$\begin{bmatrix} A_0\alpha_0 & \\ & A_L\alpha_L \end{bmatrix} = [A][\alpha] \tag{84}$$

is introduced. The implications of equation (83) are summarized by the reciprocity relations

$$([S_1]^{\mathrm{T}}[A][\alpha]) = [A][\alpha][S_2] \tag{85}$$

and

$$[A][\alpha][S_1] = ([A][\alpha][S_2])^{\mathrm{T}}. \tag{86}$$

Equations (85) and (86) are used to establish the following relationships for the acoustic potential reflection and transmission coefficients:

$$R_1 = R_2, \quad \tilde{R}_1 = \tilde{R}_2, \quad \tilde{T}_2 A_0\alpha_0 = T_1 A_L\alpha_L, \quad \tilde{T}_1 A_0\alpha_0 = T_2 A_L\alpha_L. \tag{87–90}$$

By using the definitions of α_0 and α_L defined by equations (78) and (79) evaluated at $x = 0$ and L, equations (89) and (90) are written as

$$T_1 = \frac{A_0}{A_L}\frac{\rho_{r_L}}{\rho_{r_0}}\frac{c_{r_L}}{c_{r_0}}\left(\frac{\rho_{r_0}}{\rho_{r_L}}\right)^2 \tilde{T}_2, \qquad T_2 = \frac{A_0}{A_L}\frac{\rho_{r_L}}{\rho_{r_0}}\frac{c_{r_L}}{c_{r_0}}\left(\frac{\rho_{r_0}}{\rho_{r_L}}\right)^2 \tilde{T}_1. \tag{91, 92}$$

Equations (87) and (88) produce the interesting result that the acoustic potential reflection coefficients, at the left end and at the right end are invariant to flow direction. No reciprocal relationships link the left to right transmission coefficients T_1 and T_2 or the right to left transmission coefficients $\tilde{T}_1$ and $\tilde{T}_2$ in nominal flow and reversed flow. Equation (91) links the left to right transmission coefficient T_1 in nominal flow to the right to left transmission coefficient $\tilde{T}_2$ in reversed flow. Equation (92) links the left to right transmission coefficient in reversed flow T_2 to the right to left transmission coefficient $\tilde{T}_1$ in nominal flow.

The reciprocal results of equations (87)–(90) are for acoustic potential modal amplitudes. In the case of acoustic pressure modal amplitudes, a transition to acoustic potential modal amplitudes is made through equation (61). By using equations (75) and (76), which are axial wave numbers in the case of plane waves, the transformations from acoustic potential to acoustic pressure for right and left waves in nominal and reversed flow are found to be

$$\phi_1^+ = \beta^+ p_1^+, \quad \phi_1^- = \beta^- p_1^-, \tag{93}$$

$$\phi_2^+ = \beta^- p_2^+, \quad \phi_2^- = \beta^+ p_2^-, \tag{94}$$

where

$$\beta^+ = \frac{1+M}{-\,\mathrm{i}\eta\rho_r c_r}, \quad \beta^- = \frac{1-M}{-\,\mathrm{i}\eta\rho_r \mathrm{c}_r}. \tag{95}$$

Mach number and non-dimensional frequency are based on the local speed of sound. Non-dimensional frequency is still based on the reference length. Non-dimensional density and non-dimensional speed of sound are evaluated locally.

Equations (93) and (94) are used to introduce pressure modal amplitudes in equation (83). The scattering matrix is now in terms of pressure scattering coefficients,

$$[\bar{S}] = \begin{bmatrix} \bar{R} & \tilde{\bar{T}} \\ \bar{T} & \tilde{\bar{R}} \end{bmatrix}. \tag{96}$$

After following the same steps leading to equations (87)–(90), reciprocity relations for acoustic pressure scattering coefficients are found to be

$$\overline{R_1} A_0 \alpha_0 \frac{\beta_0^-}{\beta_0^+} = \overline{R_2} A_0 \alpha_0 \frac{\beta_0^+}{\beta_0^-} \quad \text{or} \quad \overline{R_1} = \frac{(1+M_0)^2}{(1-M_0)^2} \overline{R_2}, \tag{97}$$

$$\overline{\tilde{R}_1} A_L \alpha_L \frac{\beta_L^+}{\beta_L^-} = \overline{\tilde{R}_2} A_L \alpha_L \frac{\beta_L^-}{\beta_L^+} \quad \text{or} \quad \overline{\tilde{R}_1} = \frac{(1-M_L)^2}{(1+M_L)^2} \overline{\tilde{R}_2}, \tag{98}$$

$$\overline{T_1} A_L \alpha_L \frac{\beta_L^+}{\beta_0^+} = \overline{\tilde{T}_2} A_0 \alpha_0 \frac{\beta_0^+}{\beta_L^+} \quad \text{or} \quad \overline{T_1} = \frac{A_0}{A_L} \frac{\rho_{r_L}}{\rho_{r_0}} \frac{c_{r_L}}{c_{r_0}} \frac{(1+M_0)^2}{(1+M_L)^2} \overline{\tilde{T}_2}, \tag{99}$$

$$\overline{\tilde{T}_1} A_0 \alpha_0 \frac{\beta_0^-}{\beta_L^-} = \overline{T_2} A_L \alpha_L \frac{\beta_L^-}{\beta_L^+} \quad \text{or} \quad \overline{T_2} = \frac{A_0}{A_L} \frac{\rho_{r_L}}{\rho_{r_0}} \frac{c_{r_L}}{c_{r_0}} \frac{(1-M_0)^2}{(1-M_L)^2} \overline{\tilde{T}_1}. \tag{100}$$

Equations (97)–(100) are written both implicitly in terms of coefficients defined above and explicitly in terms of local area and flow conditions. The mach number is for the nominal flow.

Pressure reflection coefficients are not invariant in reversed flow but are reciprocally related as given by equations (97) and (98). No reciprocal relationships link the left to right transmission coefficients $\overline{T_1}$ and $\overline{T_2}$ or the right to left transmission coefficients $\overline{\tilde{T}_1}$ and $\overline{\tilde{T}_2}$ in nominal flow and reversed flow. Equation (99) links the left to right transmission coefficient $\overline{T_1}$ in nominal flow to the right to left transmission coefficient $\overline{\tilde{T}_2}$ in reversed flow. Equation (100) links the left to right transmission coefficient in reversed flow $\overline{T_2}$ to the right to left transmission coefficient $\overline{\tilde{T}_1}$ in nominal flow.

8. ACOUSTIC POWER CONSIDERATIONS IN ONE-DIMENSIONAL PROPAGATION

In the case of one-dimensional propagation, additional results can be obtained by the consideration of acoustic power. The definition of acoustic intensity due to Morfey [4] is valid as part of a conservation law in non-uniform ducts for compressible potential flow. This definition of acoustic intensity is used in the uniform flow sections either end of the non-uniformity to obtain acoustic power expressions.

For propagation in uniform flow, the Morfey intensity formulation simplifies to the time-averaged scalar form

$$\frac{I}{\rho_\infty c_\infty^3} = \left\langle (1+M^2)pu + \rho_r c_r M u^2 + \frac{1}{\rho_r c_r} M p^2 \right\rangle. \tag{101}$$

Acoustic power is obtained by integration over a cross-section, to yield

$$\frac{P}{A_{ref}\rho_\infty c_\infty^3} = \iint_S \left\langle (1+M^2)pu + \rho_r c_r M u^2 + \frac{1}{\rho_r c_r} M p^2 \right\rangle \mathrm{d}S. \tag{102}$$

A modal expansion as given by equations (73) and (74), with wave numbers given by equations (75) and (76), is used to obtain an acoustic power in terms of acoustic potential modal amplitudes. The result is

$$\frac{P}{A_{ref}\rho_\infty c_\infty^3} = a^{+*} P_{11}^{++} a^+ + a^{-*} P_{11}^{--} a^-, \tag{103}$$

where, for example, a^* denotes the complex conjugate of a. Power can be represented in terms of acoustic potential amplitudes as in equation (46) in uniform duct sections at $x = 0$ and L. The power coefficients P_{11}^{++} and P_{11}^{--} can be easily obtained by carrying out the operations indicated in equation (102), having made use of the relation between acoustic pressure and acoustic potential of equation (61), and the relation between acoustic particle velocity and acoustic potential in equation (64). For example at $x = 0$,

$$P_{11_0}^{++} = \tfrac{1}{2}\rho_{r_0} c_{r_0} \eta_0^2 A_0, \quad P_{11_0}^{--} = -\tfrac{1}{2}\rho_{r_0} c_{r_0} \eta_0^2 A_0. \tag{104}$$

Energy conservation arguments lead to the conclusion that $\Pi_{inc} + \Pi_{ref} = \Pi_{trans}$, with the observation that power is positive for incident and transmitted and negative for reflected contributions. Incident power at $x = 0$ is given by

$$\Pi_{ref} = a^{+*} P_{11_0}^{++} a^+. \tag{105}$$

Reflected power in nominal and reversed flow at $x = 0$ is given by

$$\Pi_{ref} = a^{+*} R_1^* P_{11_0}^{--} R_1 a^+, \quad \Pi_{ref} = a^{+*} R_2^* P_{11_0}^{--} R_2 a^+. \tag{106}$$

Transmitted power in nominal and reversed flow at $x = L$ is given by

$$\Pi_{trans} = a^{+*} T_1^* P_{11_L}^{++} T_1 a^+, \quad \Pi_{trans} = a^{+*} T_2^* P_{11_L}^{++} T_2 a^+. \tag{107}$$

Subscripts on the power coefficients denote the uniform section in which they are evaluated. From equation (87), $R_1 = R_2$, that is, the acoustic potential reflection coefficient is invariant to flow direction, as is P_{11}. The power reflection coefficients are defined by either

$$R_{\pi_1} = \frac{R_1^* P_{11_0} R_1}{P_{11_0}} = |R_1|^2 \quad \text{or} \quad R_{\pi_2} = \frac{R_2^* P_{11_0} R_2}{P_{11_0}} = |R_2|^2. \tag{108}$$

Use has been made of the fact that $P_{11_0}^{--} = -P_{11_0}^{++}$, and the reflection coefficient is defined without regard to the sign of the reflected power. The power reflection coefficient is therefore invariant to flow direction, $R_{\pi_1} = R_{\pi_2}$. It is concluded that the power transmission coefficients are also invariant to flow direction, $T_{\pi_1} = T_{\pi_2}$. A similar development based on equation (88) would show that $\tilde{R}_{\pi_1} = \tilde{R}_{\pi_2}$ and therefore that $\tilde{T}_{\pi_1} = \tilde{T}_{\pi_2}$.

It is now possible to connect the power transmission coefficients when the source is moved from one end of the duct to the other. For a source at $x = 0$ the power transmission coefficient (the ratio of transmitted to incident power) is in nominal flow

$$T_{\pi_1} = \frac{P_{11_L}}{P_{11_0}} |T_1|^2 = \frac{\rho_{r_L}}{\rho_{r_0}} \frac{c_{r_0}}{c_{r_L}} \frac{A_L}{A_0} |T_1|^2. \tag{109}$$

For a source at $x = L$ in reverse flow the power transmission coefficient is defined for right to left power according to

$$\tilde{T}_{\pi_2} = \frac{\tilde{T}_2^* P_{11_0} \tilde{T}_2}{P_{11_L}} = \frac{P_{11_0}}{P_{11_L}} |\tilde{T}_2|^2 = \frac{\rho_{r_0}}{\rho_{r_L}} \frac{c_{r_L}}{c_{r_0}} \frac{A_0}{A_0} |\tilde{T}_2|^2. \tag{110}$$

Equations (109) and (110) are deduced by using the definitions of incident and transmitted power appropriately evaluated at $x = 0$ or L. The reciprocal relationship of equation (89) is used to replace $\tilde{T}_2$ in equation (110) yielding the result

$$\tilde{T}_{\pi_2} = \frac{\rho_{r_L}}{\rho_{r_0}} \frac{c_{r_0}}{c_{r_L}} \frac{A_L}{A_0} |T_1|^2. \tag{111}$$

It has thus been shown that $\tilde{T}_{\pi_2} = T_{\pi_1}$. It is therefore concluded that $T_{\pi_1} = T_{\pi_2} = \tilde{T}_{\pi_1} = \tilde{T}_{\pi_2}$ and thence from power conservation arguments that $R_{\pi_1} = R_{\pi_2} = \tilde{R}_{\pi_1} = \tilde{R}_{\pi_2}$. This completes the interesting result that for the one-dimensional model power reflection and transmission coefficients are invariant to flow reversal and switching of the source location.

The result that $T_{\pi_1} = \tilde{T}_{\pi_1}$ states that the transmission coefficient for the nominal flow direction is the same from either end of the duct. This result contains the invariance theorem of Davis [4], but also admits a generalization. Davis found that for a converging–diverging non-uniformity in an otherwise uniform duct, for equal pressure amplitude input at the upstream and downstream ends of the duct, the transmitted power is related by

$$\frac{\Pi_1}{\tilde{\Pi}_1} = \frac{(1+M)^2}{(1-M)^2}. \tag{112}$$

Here, Π_1 is the transmitted power at the downstream end due to a source at the upstream end and $\tilde{\Pi}_1$ is the transmitted power at the upstream end due to a source at the downstream end, both in the nominal flow. Since the result of Davis is for a converging diverging duct, with both ends of the same area, $M_0 = M_L = M$.

By using equations (103) and (104), which define acoustic power in terms of acoustic potential amplitudes, and by modifying them using equations (93)–(95) to relate acoustic pressure amplitude to acoustic potential amplitude, the incident power at $x = 0$ is

$$\Pi_{inc}\,(x=0) = \frac{1}{2}\,|p_0^+|^2\,\frac{A_0}{\rho_{r_0} c_{r_0}}\,(1 + M_0)^2. \tag{113}$$

Similarly, for an acoustic pressure input at $x = L$, the incident power there is

$$\tilde{\Pi}_{inc}\,(x=L) = \frac{1}{2}\,|\tilde{p}_L^-|^2\,\frac{A_L}{\rho_{r_L} c_{r_L}}\,(1 - M_L)^2. \tag{114}$$

p_0^+ and $\tilde{p}_L^-$ are pressure mode amplitudes incident at the ends $x = 0$ and $x = L$. Superscripts $+$ and $-$ reinforce the idea that at $x = 0$, the incident mode is a right running wave and at $x = L$, the incident mode is a left running wave. The tilde ($\sim$) is a reminder that the source is at the end $x = L$. By using the fact that the transmitted power in each case is the product of the incident power and the appropriate power transmission coefficient, which is the same for either case ($T_{\pi_1} = \tilde{T}_{\pi_1}$), and that the incident modal pressure amplitudes are the same for either source ($|p_0^+| = |\tilde{p}_L^-|$), it follows that

$$\frac{\Pi_{trans}}{\tilde{\Pi}_{trans}} = \frac{A_0}{A_L}\,\frac{\rho_{r_L}}{\rho_{r_0}}\,\frac{c_{r_L}}{c_{r_0}}\,\frac{(1 + M_0)^2}{(1 - M_L)^2}. \tag{115}$$

Equation (115) contains Davis' result [4] in the case of a converging–diverging duct when the duct area, flow density, speed of sound and Mach number are the same at both ends. Other results are possible involving acoustic potential and acoustic pressure amplitudes from the core result that the power transmission coefficients are invariant.

9. CONCLUSION

A reverse flow theorem for acoustic propagation in compressible potential flow has been obtained directly from the field equations without recourse to energy conservation arguments. A reciprocity theorem for the scattering matrix for the propagation of acoustic modes in a duct with either hard walls, or a section of locally reacting absorbing wall imbedded in an otherwise hard wall, follows. It is found that for a source at a specific end of the duct, suitably scaled reflection matrices in direct and reverse flow have a reciprocal relationship. Scaled transmission matrices obtained for direct flow and reversed flow with simultaneous switching of source location from one end to the other also have a reciprocal relationship.

The approach presented here is an alternative to the approach of Moehring [3, 17], with the distinction that no energy conservation condition is used. It has been exploited to provide explicit reciprocal relations which are of theoretical interest, but which also have the more pragmatic significance of providing a convenient means for benchmarking large-scale propagation codes such as those used in the numerical experiments supporting this investigation [10].

A similar reverse flow theorem for acoustic propagation in one-dimensional compressible potential flow also has been established, and the corresponding reciprocity theorem for the scattering coefficients for propagation of incident plane wave acoustic modes obtained. Reciprocal relations and power conservation arguments are used to show that scaled power reflection and transmission coefficients are invariant to flow reversal and switching of source location from one end of the duct to the other.

Numerical verification of the reciprocal relationships is the subject of a companion paper.

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APPENDIX A: MULTIPLE MODE PROPAGATION

A.1. ACOUSTIC MODES AND AXIAL WAVE NUMBERS

Acoustic propagation in uniform duct segments is described as a superposition of duct modes. The characteristics of these modes depend on the axial wave numbers, explained extensively in reference [19]. Results for circular ducts are given explicitly here. Results for annular ducts require only minor modifications.

Axial wave numbers are given in the nominal flow for modes which are cut on by

$$\left(\frac{k^{\pm}_{x_{mn}}}{\eta}\right)_1 = \frac{1}{1-M^2}\left[-M \pm \sqrt{1-(1-M^2)\left(\frac{\kappa_{mn}\sigma}{\sigma\eta}\right)^2}\right] \tag{A1}$$

and for modes which are cut off by

$$\left(\frac{k^{\pm}_{x_{mn}}}{\eta}\right)_1 = \frac{1}{1-M^2}\left[-M \mp \mathrm{i}\sqrt{(1-M^2)\left(\frac{\kappa_{mn}\sigma}{\sigma\eta}\right)^2-1}\right]. \tag{A2}$$

In reversed flow for cut on modes,

$$\left(\frac{k^{\pm}_{x_{mn}}}{\eta}\right)_2 = \frac{1}{1-M^2}\left[M \pm \sqrt{1-(1-M^2)\left(\frac{\kappa_{mn}\sigma}{\sigma\eta}\right)^2}\right], \tag{A3}$$

and for cut off modes,

$$\left(\frac{k^{\pm}_{x_{mn}}}{\eta}\right)_2 = \frac{1}{1-M^2}\left[M \mp \mathrm{i}\sqrt{(1-M^2)\left(\frac{\kappa_{mn}\sigma}{\sigma\eta}\right)^2-1}\right]. \tag{A4}$$

The non-dimensional frequency η is based on the local speed of sound and the reference radius in the uniform duct section in which the reference conditions are defined, in this problem at $x = 0$. $\kappa_{mn}\sigma$ are eigenvalues determined from the uniform duct eigenproblem [19] in a uniform duct with local radius possibly different than the reference radius, for example, at the other end of the duct at $x = L$. In the case of the circular duct, they are determined from $\mathrm{J}'_m(\kappa_{mn}\sigma) = 0$, with J_m being the Bessel function. m is the angular mode number and n, $1 \leqslant n \leqslant N$, is the radial mode number. σ is the ratio of the local duct radius to the reference radius, $\sigma = R_l/R$. $\sigma\eta$ is therefore non-dimensional frequency based on local speed of sound and local duct radius. At $x = 0$ where the reference radius and c_∞ are defined, $\sigma = 1$ and $\eta = \eta_r$. The convention on the sign choice in equations (A1)–(A4) corresponding to $k^+_{x_{mn}}$ is that the positive sign is chosen if the radical is real and the minus sign is chosen if the radical is imaginary. The opposite choices are made for $k^-_{x_{mn}}$. $k^+_{x_{mn}}$ then corresponds to waves propagating in the positive x direction (except for the possibility that with Mach number negative some propagating waves may appear not to propagate in the positive x direction due to convection) and to cut-off modes decaying in the positive x direction. The opposite interpretation applies for $k^-_{x_{mn}}$.

A.2. ACOUSTIC MODES IN RECIPROCITY RELATIONSHIPS

In carrying out the integrals of equation (27), the notation

$$\iint_{S_0} [\Phi]^{\mathrm{T}}[\Phi]\,\mathrm{d}S = [J_0], \quad \iint_{S_L} [\Phi]^{\mathrm{T}}[\Phi]\,\mathrm{d}S = [J_L] \tag{A5, 6}$$

is introduced. $[J_0]$ and $[J_L]$ are $N \times N$ diagonal matrices resulting from the orthogonality of the duct eigenfunctions at $x = 0$ and $x = L$. For a circular duct, the eigenfunctions $\Phi_{mn}(r)$ are Bessel functions of the first kind of order m. In numerical implementations, it is convenient to generate the Bessel functions, solve the related eigenproblem, and generate $[J_0]$ and $[J_L]$ using an FEM formulation.

With the eigenfunction expansions of equations (30)–(33), the integrals of equation (27) can be written as

$$\rho_{r_0} \iint_{S_0} \phi_2 \left\{ (1 - M_0^2) \frac{\partial \phi_1}{\partial x} - \mathrm{i}\eta_0 M_0 \phi_1 \right\} \mathrm{d}S$$
$$= \rho_{r_0}[(1 - M_0^2)(\{a_2^+\}^{\mathrm{T}} + \{a_2^-\}^{\mathrm{T}})[J_0]([-\mathrm{i}k_{x_{01}}^+]\{a_1^+\} + [-\mathrm{i}k_{x_{01}}^-]\{a_1^-\})$$
$$- \mathrm{i}\eta_0 M_0(\{a_2^+\}^{\mathrm{T}} + \{a_2^-\}^{\mathrm{T}})[J_0](\{a_1^+\} + \{a_1^-\})], \tag{A7}$$

$$\rho_{r_0} \iint_{S_0} \phi_1 \left\{ (1 - M_0^2) \frac{\partial \phi_2}{\partial x} + \mathrm{i}\eta_0 M_0 \phi_2 \right\} \mathrm{d}S$$
$$= \rho_{r_0}[(1 - M_0^2)(\{a_1^+\}^{\mathrm{T}} + \{a_1^-\}^{\mathrm{T}})[J_0]([-\mathrm{i}k_{x_{02}}^+]\{a_2^+\} + [-\mathrm{i}k_{x_{02}}^-]\{a_2^-\})$$
$$+ \mathrm{i}\eta_0 M_0(\{a_1^+\}^{\mathrm{T}} + \{a_1^-\}^{\mathrm{T}})[J_0](\{a_2^+\} + \{a_2^-\})], \tag{A8}$$

$$\rho_{r_L} \iint_{S_L} \phi_2 \left\{ (1 - M_L^2) \frac{\partial \phi_1}{\partial x} - \mathrm{i}\eta_L M_L \phi_1 \right\} \mathrm{d}S$$
$$= \rho_{r_L}[(1 - M_L^2)(\{b_2^+\}^{\mathrm{T}} + \{b_2^-\}^{\mathrm{T}})[J_L]([-\mathrm{i}k_{x_{L1}}^+]\{b_1^+\} + [-\mathrm{i}k_{x_{L1}}^-]\{b_1^-\})$$
$$- \mathrm{i}\eta_L M_L(\{b_2^+\}^{\mathrm{T}} + \{b_2^-\}^{\mathrm{T}})[J_L](\{b_1^+\} + \{b_1^-\})], \tag{A9}$$

$$\rho_{r_L} \iint_{S_L} \phi_1 \left\{ (1 - M_L^2) \frac{\partial \phi_2}{\partial x} + \mathrm{i}\eta_L M_L \phi_2 \right\} \mathrm{d}S$$
$$= \rho_{r_L}[(1 - M_L^2)(\{b_1^+\}^{\mathrm{T}} + \{b_1^-\}^{\mathrm{T}})[J_L]([-\mathrm{i}k_{x_{L2}}^+]\{b_2^+\} + [-\mathrm{i}k_{x_{L2}}^-]\{b_2^-\})$$
$$+ \mathrm{i}\eta_L M_L(\{b_1^+\}^{\mathrm{T}} + \{b_1^-\}^{\mathrm{T}})[J_L](\{b_2^+\} + \{b_2^-\})]. \tag{A10}$$

The notation $k_{x_{01}}^{\pm}$ designates the axial wave number evaluated at $x = 0$ for the nominal flow direction. $k_{x_{02}}^{\pm}$ corresponds to reversed flow. $k_{x_{L1}}^{\pm}$ designates the axial wave number evaluated at $x = L$ for the nominal flow direction. $k_{x_{L2}}^{\pm}$ corresponds to reversed flow. In equations (A7)–(A10) amplitude coefficients $\{a^{\pm}\}$ are associated with an eigenfunction expansion at $x = 0$ and $\{b^{\pm}\}$ correspond to $x = L$. With the use of equations (A1) and (A2), introduce the following definitions for the nominal flow direction for propagating modes (k_x real):

$$\alpha_1^+ = \rho_r[-\mathrm{i}(1 - M^2)k_{x_1}^+ - \mathrm{i}\eta M] = -\mathrm{i}\rho_r \eta \sqrt{1 - (1 - M^2)\left(\frac{\kappa\sigma}{\sigma\eta}\right)^2}, \tag{A11}$$

$$\alpha_1^- = \rho_r[-\mathrm{i}(1 - M^2)k_{x_1}^- - \mathrm{i}\eta M] = \mathrm{i}\rho_r \eta \sqrt{1 - (1 - M^2)\left(\frac{\kappa\sigma}{\sigma\eta}\right)^2} \tag{A12}$$

and for cut-off modes (k_x complex):

$$\alpha_1^+ = \rho_r[-\mathrm{i}(1-M^2)k_{x_1}^+ - \mathrm{i}\eta M] = -\rho_r\eta\sqrt{(1-M^2)\left(\frac{\kappa\sigma}{\sigma\eta}\right)^2 - 1}\,, \tag{A13}$$

$$\alpha_1^- = \rho_r[-\mathrm{i}(1-M^2)k_{x_1}^- - \mathrm{i}\eta M] = \rho_r\eta\sqrt{(1-M^2)\left(\frac{\kappa\sigma}{\sigma\eta}\right)^2 - 1}\,. \tag{A14}$$

Analogous definitions can be introduced in reversed flow using equations (A3) and (A4). For example, for a propagating mode

$$\alpha_2^+ = \rho_r[-\mathrm{i}(1-M^2)k_{x_2}^+ + \mathrm{i}\eta M] = -\mathrm{i}\rho_r\eta\sqrt{1-(1-M^2)\left(\frac{\kappa\sigma}{\sigma\eta}\right)^2}, \tag{A15}$$

it becomes apparent that the definitions do not change in reversed flow so the conclusion is made that $\alpha_2^+ = \alpha_1^+ = \alpha^+$ and $\alpha_2^- = \alpha_1^- = \alpha^-$ for both propagating and cut-off modes. Furthermore, it is apparent that $\alpha^- = -\alpha^+$ and these are to be evaluated at $x = 0$ and L as required.

Equations (A7)–(A10) can be rewritten in the form

$$\begin{aligned}\rho_{r_0}\iint\limits_{S_0}\phi_2\left\{(1-M_0^2)\frac{\partial\phi_1}{\partial x} - \mathrm{i}\eta_0 M_0\phi_1\right\}\mathrm{d}S \\ = \{a_2^+\}^{\mathrm{T}}[J_0][\alpha_0^+]\{a_1^+\} + \{a_2^+\}^{\mathrm{T}}[J_0][\alpha_0^-]\{a_1^-\} \\ + \{a_2^-\}^{\mathrm{T}}[J_0][\alpha_0^+]\{a_1^+\} + \{a_2^-\}^{\mathrm{T}}[J_0][\alpha_0^-]\{a_1^-\},\end{aligned} \tag{A16}$$

$$\begin{aligned}\rho_{r_0}\iint\limits_{S_0}\phi_1\left\{(1-M_0^2)\frac{\partial\phi_2}{\partial x} - \mathrm{i}\eta_0 M_0\phi_2\right\}\mathrm{d}S \\ = \{a_1^+\}^{\mathrm{T}}[J_0][\alpha_0^+]\{a_2^+\} + \{a_1^+\}^{\mathrm{T}}[J_0][\alpha_0^-]\{a_2^-\} \\ + \{a_1^-\}^{\mathrm{T}}[J_0][\alpha_0^+]\{a_2^+\} + \{a_1^-\}^{\mathrm{T}}[J_0][\alpha_0^-]\{a_2^-\},\end{aligned} \tag{A17}$$

$$\begin{aligned}\rho_{r_L}\iint\limits_{S_L}\phi_2\left\{(1-M_L^2)\frac{\partial\phi_1}{\partial x} - \mathrm{i}\eta_L M_L\phi_1\right\}\mathrm{d}S \\ = \{b_2^+\}^{\mathrm{T}}[J_L][\alpha_L^+]\{b_1^+] + \{b_2^+\}^{\mathrm{T}}[J_L][\alpha_L^-]\{b_1^-\} \\ + \{b_2^-\}^{\mathrm{T}}[J_L][\alpha_L^+]\{b_1^+\} + \{b_2^-\}^{\mathrm{T}}[J_L][\alpha_L^-]\{b_1^-\},\end{aligned} \tag{A18}$$

$$\begin{aligned}\rho_{r_L}\iint\limits_{S_L}\phi_1\left\{(1-M_L^2)\frac{\partial\phi_2}{\partial x} - \mathrm{i}\eta_L M_L\phi_2\right\}\mathrm{d}S \\ = \{b_1^+\}^{\mathrm{T}}[J_L][\alpha_L^+]\{b_2^+\} + \{b_1^+\}^{\mathrm{T}}[J_L][\alpha_L^-]\{b_2^-\} \\ + \{b_1^-\}^{\mathrm{T}}[J_L][\alpha_L^+]\{b_2^+\} + \{b_1^-\}^{\mathrm{T}}[J_L][\alpha_L^-]\{b_2^-\}.\end{aligned} \tag{A19}$$

The diagonal matrices $[\alpha_0^{\pm}]$ and $[\alpha_L^{\pm}]$ have elements defined by equations (A11)–(A15). Elements α_0 are evaluated at $x = 0$ and α_L are evaluated at $x = L$ and the distinction between nominal flow and reversed flow disappears.

When the integral evaluations of equations (A16)–(A19) are used in equation (27), there is considerable simplification due to the fact that $\alpha_0^- = -\alpha_0^+ = -\alpha_0$ and $\alpha_L^- = -\alpha_L^+ = -\alpha_L$. The diagonal matrices $[\alpha_0]$ and $[\alpha_L]$ are constructed by evaluating equations (42)–(45) at $x = 0$ or L for each acoustic mode included in the acoustic potential expansions of equations (30) or (32). $[J_0][\alpha_0]$ and $[J_L][\alpha_L]$ are diagonal and therefore equal to their transpose.

A.3. ACOUSTIC WAVE NUMBERS IN PRESSURE FORMULATION

Equations (46) and (47) introduce coefficients $\beta^{\pm}$, $\beta^{\mp}$ which are determined from the axial wave numbers $k_x^{\pm}$ defined from equations (A1)–(A30). For nominal flow and propagating modes, this yields

$$\frac{1}{\beta^{\pm}} = -\,\mathrm{i}\eta\rho_r c_r \frac{1 \mp M\sqrt{1-(1-M^2)(\kappa\sigma/\sigma\eta)^2}}{1-M^2} \tag{A20}$$

and for nominal flow and cut-off modes

$$\frac{1}{\beta^{\pm}} = -\,\mathrm{i}\eta\rho_r c_r \frac{1 \pm \mathrm{i}M\sqrt{(1-M^2)(\kappa\sigma/\sigma\eta)^2-1}}{1-M^2}. \tag{A21}$$

For reversed flow and propagating modes,

$$\frac{1}{\beta^{\mp}} = -\,\mathrm{i}\eta\rho_r c_r \frac{1 \pm M\sqrt{1-(1-M^2)(\kappa\sigma/\sigma\eta)^2}}{1-M^2} \tag{A22}$$

and for reversed flow and cut-off modes,

$$\frac{1}{\beta^{\mp}} = -\,\mathrm{i}\eta\rho_r c_r \frac{1 \mp \mathrm{i}M\sqrt{(1-M^2)(\kappa\sigma/\sigma\eta)^2-1}}{1-M^2}. \tag{A23}$$