Journal of Sound and Vibration (2001) **246**(2), 281–296
doi:10.1006/jsvi.2001.3635, available online at http://www.idealibrary.com on IDEAL®

FEEDFORWARD AND FEEDBACK CONTROL OF SOUND AND VIBRATION—A WIENER FILTER APPROACH

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(*Received* 10 *July* 2000, *and in final form* 18 *January* 2001)

This paper considers the active control of stationary random disturbances in single-input–single-output vibrating and acoustical systems. Both feedforward and feedback control are studied using a deterministic approach, where white noise is replaced by an impulse so that the active control of a stationary random disturbance is transformed to the active control of an impulse response. A theoretical framework using the Wiener filter to study both systems is described where internal model control is used to convert a feedback system into feedforward architecture. The effects of the constraint of causality on the performances of feedforward and feedback control of random disturbances are discussed using two simple examples; a minimum phase single-degree-of-freedom vibrating system and a non-minimum phase acoustical system.

1. INTRODUCTION

In recent years, the active control of sound and vibration has become a reality. There has been significant success in controlling harmonic disturbances and more recently, random disturbances. Feedforward control has generally been used for harmonic disturbances, and feedback control for random disturbances if a suitable reference signal is not available. There have tended to be different approaches to these problems, but there is some value in studying them in a unified framework that facilitates physical insight into control mechanisms and performance limitations. Using Wiener filter theory helps in this quest. The theory offers the optimal filter which minimizes the mean square error in a stationary random signal environment, and was first developed by a mathematician, Wiener [1]. Its engineering applications and analytical solutions are well established; see for example, Van Trees [2] and Popoulius [3] for analogue filters, and Orfanidis [4] for digital filters. As discussed by Kailath [5], the theory has been most influential in the areas of signal processing and control, and its applications cover the areas of signal estimation, communication, and stochastic control theories etc.

A critical issue in the design of random disturbance controllers, which does not occur for deterministic disturbances, is the constraint of causality. The constraint should be satisfied for the controller to be physically realizable (causal stable), and is the key to understanding the mechanisms and limitations of the active control of random noise and vibration. The optimal controller for random disturbances is the causally constrained Wiener filter and the

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optimal controller for harmonic disturbances is the causally unconstrained Wiener filter [6, 7]. Nelson *et al.* [7] presented a theoretical framework to design the optimal controller to minimize stationary random sound in the mean square sense. A numerical model has also been presented by Joplin and Nelson [8], who minimized the acoustic potential energy inside a cavity excited by white noise. An analytic solution procedure in the discrete time domain was later presented by Nelson [9] using the spectral factorization technique. Although white-noise excitation was generally assumed for such analytical studies [7–9], the problems were treated, rather complicatedly, as stochastic optimal control problems.

This paper describes a theoretical study into the feedforward and feedback active control of single-input–single-output (SISO) vibrating and acoustical systems subject to white-noise excitation. The unified theoretical framework uses Wiener filter and internal model control to convert the feedback systems into a feedforward architecture [10]. Although only SISO systems are discussed here, the methodology can be applied to multi-input–multi-output vibrating and acoustical systems [11]. A deterministic approach is presented by replacing white noise by an impulse [2, 12], so that the active control of stationary random disturbances becomes the problem of the active control of an impulse response. This facilitates analysis using deterministic rather than stochastic methods enabling better interpretation of the control mechanisms. Even though this approach may be limited in terms of practical applications, because most disturbances are neither stationary nor white, it enables an easier investigation into the effects of the constraint of causality. Moreover, the use of an impulse instead of white noise is advantageous since a numerical realization of white noise is time consuming and thus it is particularly useful in the investigation of MIMO active control systems [11, 12].

The aim of the paper is to use the combination of the Wiener filter and the deterministic approach in a unified framework to study the feedforward and feedback control of a simple minimum phase vibrating system and a non-minimum phase acoustical system. The effects of the constraint of causality are investigated using this approach and insight into control mechanisms and limitations of stochastic optimal control are presented. The paper is organized as follows. Following this introduction, Wiener filter theory is reviewed briefly in section 2, and the Wiener filter approach to feedforward and feedback active control techniques is presented in section 3. Two simple examples of minimum and non-minimum phase systems are presented in sections 4 and 5, respectively; a single-degree-of-freedom vibrating system and an acoustic duct. The paper is concluded in section 5, and Appendix A describes the discrete-time formulation of the Wiener filter problem.

2. REVIEW OF WIENER FILTER THEORY

The Wiener filter is the optimal controller for minimizing stationary, ergodic random disturbances. The control problem to be solved, which is represented in block diagram form in Figure 1, is to find an optimum causal, stable filter with impulse response $h_o(t)$ which minimizes the error signal $e(t)$ in the mean square sense. The mean square error is written as $J = E[e^2(t)]$ where $E[\bullet]$ denotes the ensemble average and the error signal is $e(t) = d(t) + y(t)$, where $y(t) = \int_0^\infty h_o(\tau) r(t-\tau)\,\mathrm{d}\tau$ and τ is an arbitrary time variable. The impulse response of the optimal filter $h_o(t)$, by which the cost function J is minimized, can be obtained from the Wiener–Hopf equation given by [2, 3]

$$\int_0^\infty h_o(\tau_2) R_{rr}(\tau_1 - \tau_2)\,\mathrm{d}\tau_2 = -R_{rd}(\tau_1), \quad \tau_1 \geqslant 0, \tag{1}$$

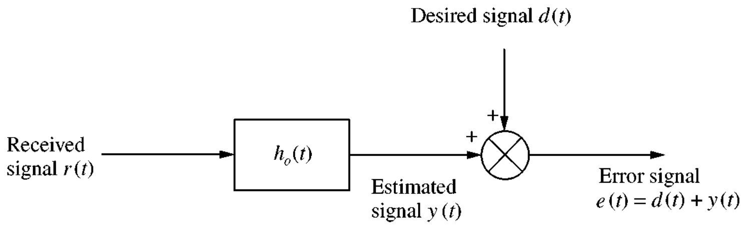

Figure 1. Wiener's problem of finding a filter $h_o(t)$ in order to minimize the mean square error when the desired and received signals are ergodic and stationary random.

where $R_{rr}(\tau) = E[r(t)r(t+\tau)]$ is the auto-correlation function of $r(t)$, $R_{rd}(\tau) = E[r(t)d(t+\tau)]$ is the cross-correlation function of $r(t)$ and $d(t)$, and the symbols τ_1 and τ_2 denote arbitrary time lags. The minimum mean square error is given by

$$J_o = R_{dd}(0) + \int_0^{\infty} h_o(\tau_1) R_{rd}(\tau_1)\,\mathrm{d}\tau_1, \quad \tau_1 \geqslant 0, \tag{2}$$

where $R_{dd}(0) = E[\{d(t)\}^2]$ is the mean square value of $d(t)$. The Wiener–Hopf equation can be solved easily in two special cases; when the constraint of causality is ignored and when the received signal is white noise. If the constraint is ignored, the solution can be obtained from the double-sided Fourier transform of equation (1) to give the unconstrained Wiener filter:

$$H_{uo}(\mathrm{j}\omega) = -\frac{S_{rd}(\mathrm{j}\omega)}{S_{rr}(\omega)}, \tag{3}$$

where $S_{rr}(\omega)$ is the auto-spectrum of the received signal, and $S_{rd}(\mathrm{j}\omega)$ is the cross-spectrum of the received and desired signals. There is no guarantee that the filter's impulse response $h_{uo}(t)$ is causal stable which means that it may not be physically realizable in the time domain. However, the solution does give an upper bound on the reduction that can be achieved by a causally constrained controller. Moreover, the unconstrained Wiener filter is the optimal controller for harmonic sound, and in this case the mean square error becomes zero. When the received signal is white noise with zero mean and unit variance, $r(t) = w(t)$, the integral in equation (1) is eliminated because $R_{rr}(\tau_1 - \tau_2) = \delta(\tau_2 - \tau_1)$ and the impulse response of the Wiener filter is given by

$$h_o(\tau_1) = -R_{wd}(\tau_1), \quad \tau_1 \geqslant 0, \tag{4}$$

where $R_{wd}(\tau) = E[w(t)d(t+\tau)]$, and the minimum mean square error is

$$J_o = R_{dd}(0) - \int_0^{\infty} h_o^2(\tau_1)\,\mathrm{d}\tau_1. \tag{5}$$

It should be noted that, when an impulse $\delta(t)$ instead of white noise $w(t)$ is used as the received signal, the same results are obtained for equations (4) and (5) since again $R_{rr}(\tau_1 - \tau_2) = \delta(\tau_2 - \tau_1)$ [2, 11]. Due to this, the analytical and numerical analysis procedures for the active control of random sound and vibration considered in this paper are significantly simplified.

When the analytical solution of equation (1) is causally constrained it can be obtained by using the spectral factorization method, and the frequency response of the optimal filter is

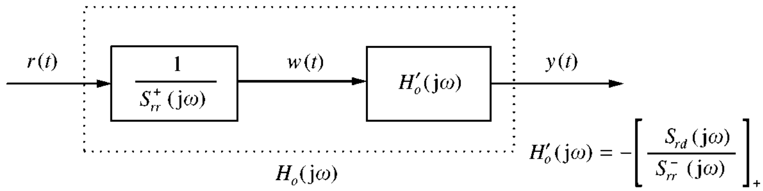

Figure 2. Wiener filter comprising a whitening filter and a sub-Wiener filter.

given by [2, 3]

$$H_o(\mathrm{j}\omega) = -\frac{1}{S_{rr}^{+}(\mathrm{j}\omega)}\left[\frac{S_{rd}(\omega)}{S_{rr}^{-}(\mathrm{j}\omega)}\right]_{+}. \tag{6}$$

The functions $S_{rr}^{+}(\mathrm{j}\omega)$ and $S_{rr}^{-}(\mathrm{j}\omega)$ are the minimum causal stable and minimum anti-causal stable parts of the auto-spectrum of the received signal, respectively, so that $S_{rr}(\omega) = S_{rr}^{+}(\mathrm{j}\omega)S_{rr}^{-}(\mathrm{j}\omega)$, and $[\bullet]_{+}$ denotes the causal part. Figure 2 shows a block diagram of the optimal filter, which can be formed by a cascade of a whitening filter, $1/(S_{rr}^{+}(\mathrm{j}\omega))$ and the sub-Wiener filter $H'_o(\mathrm{j}\omega) = -[S_{rd}(\mathrm{j}\omega)/S_{rr}^{-}(\mathrm{j}\omega)]_{+}$. Discrete time domain analysis is necessary when the filter is implemented by a digital control system or when computer simulations are carried out, and this is described in Appendix A.

3. ACTIVE CONTROL OF SISO VIBRATING AND ACOUSTICAL SYSTEMS

Feedforward control of a vibrating system is shown schematically in Figure 3(a). The system is excited by a primary force $f_p(t)$, and the system response, $e(t)$, is minimized in the mean square sense by the secondary force actuator $f_s(t)$ using the optimal controller $H_o(\mathrm{j}\omega)$. For convenience, a pure time delay τ is used to model the electrical control system, and is represented by $\mathrm{e}^{-\mathrm{j}\omega\tau}$. The control system is drawn as a block diagram in Figure 3(b) where the upper signal flow is the primary path and the lower is the secondary path. The frequency response functions of the mechanical plants in the primary and secondary paths are represented by $G_p(\mathrm{j}\omega)$ and $G_s(\mathrm{j}\omega)$ respectively. Since the systems are linear time invariant the positions of $H_o(\mathrm{j}\omega)$ and $G_s(\mathrm{j}\omega)$ have been exchanged to simplify the analysis, and the time delay is moved into the primary path as a time advance term, $\mathrm{e}^{\mathrm{j}\omega\tau}$.

To find the causal stable optimal filter $H_o(\mathrm{j}\omega)$ the theoretical framework described in section 2 can be used. Since the excitation is white noise, $f_p(t)$ in Figure 3(b) can be equivalently replaced by an impulse $\delta(t)$. The problem is now transformed to *the active control of an impulse response* and a deterministic approach can be applied to the stochastic optimal control problem. As discussed in section 2, the Wiener filter is represented as a cascade of a whitening filter and a sub-Wiener filter as shown in Figure 4. The signals for the Wiener filter are $r(t) = g_s(t)$ and $d(t) = g_p(t+\tau)$ where $g_s(t)$ and $g_p(t)$ are the impulse responses of $G_s(\mathrm{j}\omega)$ and $G_p(\mathrm{j}\omega)$ respectively. Thus, the problem is to find a causal stable filter which predicts the time-advanced impulse response of the primary path as closely as possible in the mean square sense. Following the procedure described in section 2, the optimal controller is written as $H_o(\mathrm{j}\omega) = H'_o(\mathrm{j}\omega)/G_s^{+}(\mathrm{j}\omega)$. If the secondary plant is minimum phase, i.e., $G_s(\mathrm{j}\omega) = G_s^{+}(\mathrm{j}\omega)$, then the impulse $\delta(t)$ passing through both $G_s(\mathrm{j}\omega)$ and $1/G_s^{+}(\mathrm{j}\omega)$ remains the same; i.e., $r'(t) = \delta(t)$ in Figure 4. If the plant is non-minimum phase, the received signal $r'(t)$ becomes the impulse response of an all-pass filter [13, 14].

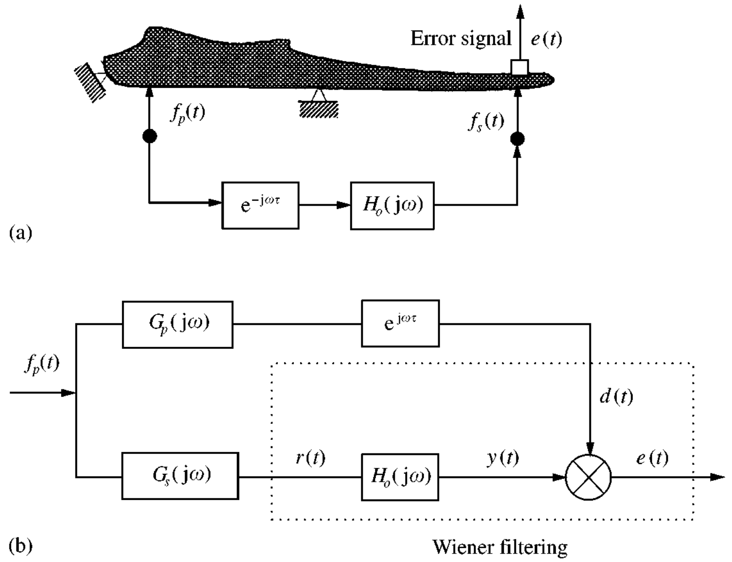

Figure 3. Feedforward control of random vibration: (a) feedforward control of a SISO system, (b) block diagram representation.

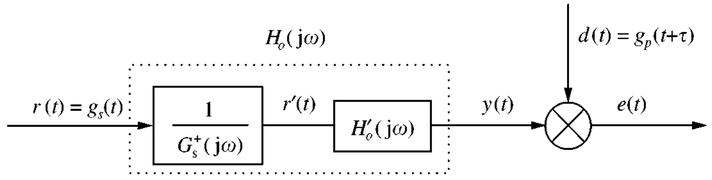

Figure 4. Feedforward optimal controller using the inverse model.

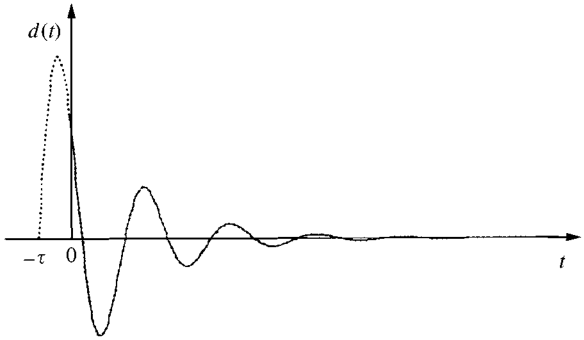

Figure 5. An arbitrary signal $d(t) = g_p(t + \tau)$. Only the causal part when $t \geqslant 0$ is used for the Fourier transform.

For minimum phase secondary plant, the received signal of the sub-Wiener filter $H'_o(\mathrm{j}\omega)$ is an impulse $\delta(t)$ and using equation (4) $H'_o(\mathrm{j}\omega) = -[\mathbf{F}(g_p(t + \tau))]_+$, where $\mathbf{F}$ denotes the Fourier transform and $[\bullet]_+$ denotes the causal part as shown in Figure 5. Thus, the optimum controller for a minimum phase secondary plant is given by

$$H_o(\mathrm{j}\omega) = -\frac{[\mathbf{F}(g_p(t + \tau))]_+}{G_s(\mathrm{j}\omega)}. \tag{7}$$

Since it is guaranteed to be causal, its impulse response is obtained from the inverse Fourier transform of equation (7). Since $r'(t) = \delta(t)$, the minimum mean square error J_o can be obtained from equation (5) by using $h'_o(\tau_1)$ instead of $h_o(\tau_1)$ to give

$$J_o = R_{dd}(0) - \int_0^{\infty} h_o'^2(\tau_1)\,\mathrm{d}\tau_1. \tag{8}$$

This can be normalized by the initial mean square error $J_i = R_{dd}(0)$ due to the primary force only to give

$$J' = \frac{J_o}{J_i} = 1 - \frac{\int_\tau^{\infty} g_p^2(t)\,\mathrm{d}t}{\int_0^{\infty} g_p^2(t)\,\mathrm{d}t}, \tag{9}$$

where $R_{dd}(0) = \int_0^\infty g_p^2(t)\,\mathrm{d}t$. The uncontrollable signal is the non-causal part of the time-advanced primary impulse response $d(t) = g_p(t + \tau)$. As τ increases, J' increases and it will finally become unity. For a minimum phase secondary plant, the performance is solely determined by the time delay τ. If $\tau = 0$, the optimal controller given in equation (7) is simply $H_o(\mathrm{j}\omega) = -G_p(\mathrm{j}\omega)/G_s(\mathrm{j}\omega)$. Thus, provided the secondary plant is minimum phase and there is no control time delay, the optimal random disturbance controller is simply an unconstrained Wiener filter.

Figure 6(a) shows a feedback configuration where the vibrating system is excited by a single actuator $f_p(t)$ and the secondary actuator is driven by the error signal via a feedback controller $-C(\mathrm{j}\omega)$. It can be represented by a block diagram as shown in Figure 6(b), and the error signal can be written in the frequency domain as

$$E(\omega) = \frac{D(\omega)}{[1 + G_s(\mathrm{j}\omega)C(\mathrm{j}\omega)]}, \tag{10}$$

where $E(\omega)$ and $D(\omega)$ are the Fourier transforms of signals $e(t)$ and $d(t)$ respectively. The variables ω and $\mathrm{j}\omega$ are used for denoting signals and systems respectively. To change the feedback structure to an open-loop feedforward structure, the internal model control (IMC) technique is used. This facilitates analysis using Wiener filter theory as discussed by Elliott and Sutton [15]. Following their work, the feedback controller $-C(\mathrm{j}\omega)$ is set to be

$$-C(\mathrm{j}\omega) = \frac{H_o(\mathrm{j}\omega)}{[1 + H_o(\mathrm{j}\omega)\hat{G}_s(\mathrm{j}\omega)]}, \tag{11}$$

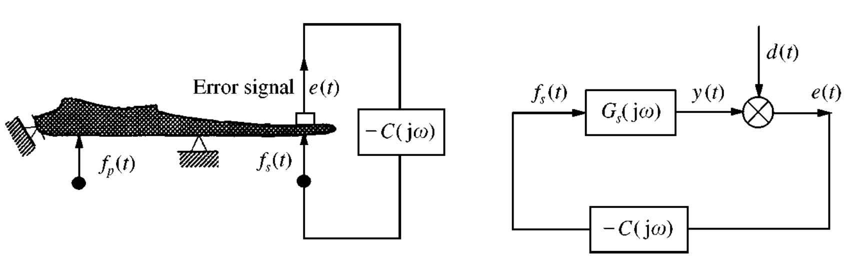

Figure 6. Feedback control of random vibration. (a) Feed back Control; (b) Block diagram representation.

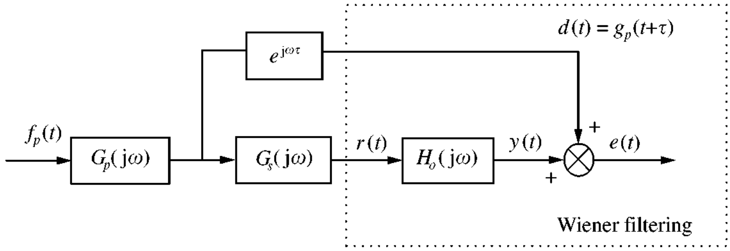

Figure 7. Internal model control of an impulse response with a pure time advance on the primary path.

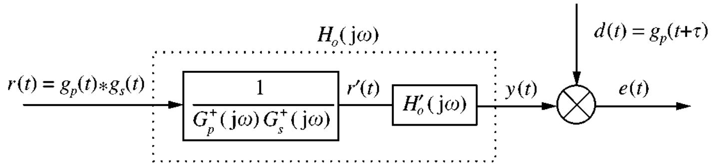

Figure 8. Wiener filter using the inverse model. The operator $*$ denotes the convolution integral.

where $H_o(j\omega)$ is the optimal control filter and $\hat{G}_s(j\omega)$ is an estimate of the secondary plant. Substituting equation (11) into equation (10) with $\hat{G}_s(j\omega) = G_s(j\omega)$, the control system is transformed into a feedforward system and is written as

$$E(\omega) = [1 + G_s(j\omega)H_o(j\omega)]D(\omega). \tag{12}$$

The corresponding block diagram is shown in Figure 7. A pure time delay is again assumed to model the electrical control process, and is transferred to the primary path as the time advance term $e^{j\omega\tau}$. Again, this is a Wiener filter problem, and the optimal filter is obtained by using the inverse model as shown in Figure 8. Unlike feedforward control, in feedback control both primary and secondary plants are required to design the inverse model. Thus, the Wiener filter is written as $H_o(j\omega) = H_o'(j\omega)/G_p^+(j\omega)G_s^+(j\omega)$. If both primary and secondary plants are minimum phase, i.e., $G_p(j\omega) = G_p^+(j\omega)$ and $G_s(j\omega) = G_s^+(j\omega)$, the received signal to the sub-Wiener filter is $r'(t) = \delta(t)$ and the optimum filter is

$$H_o(j\omega) = -\frac{[\mathbf{F}(g_p(t+\tau))]_+}{G_p(j\omega)G_s(j\omega)}. \tag{13}$$

Comparing Figures 4 and 8, it can be seen that both sub-Wiener filters $H_o'(j\omega)$ have the same received signal $\delta(t)$ and that they have to estimate the future signal of the impulse response of the primary plant. If both primary and secondary plants are minimum phase, the mean square error after control is the same as that for feedforward control given in equation (9). If, in addition, the control time delay is zero, the Wiener filter in equation (13) becomes $H_o(j\omega) = -1/G_s(j\omega)$, and the mean square error becomes zero. However, such perfect control is not achievable because an instability occurs in the physical controller $-C(j\omega)$ in equation (11) [10]. IMC that is robust to model and input uncertainty is generally required for practical implementation [16]. Nevertheless, knowledge of the best achievable performance can be helpful in assessing different control schemes.

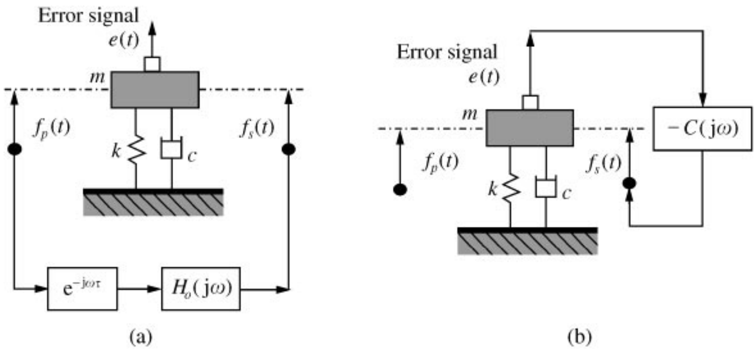

Figure 9. Active control of a single-degree-of-freedom vibrating system: (a) feedforward control; (b) feedback control.

4. ACTIVE CONTROL OF A MINIMUM PHASE VIBRATING SYSTEM

The active control of a single-degree-of-freedom minimum phase system is considered. Both feedforward and feedback techniques are considered and are shown in Figure 9(a) and 9(b), where m, k, and c are the mass, spring constant, and damping of the system respectively. The systems are excited by the white-noise primary force $f_p(t)$, and the velocity response $e(t)$ is minimized using the secondary force actuator $f_s(t)$. They are solved both analytically and numerically using the theoretically framework described in the previous section. The plant frequency response is given by [17]

$$G(\mathrm{j}\omega) = \frac{1}{m}\frac{\mathrm{j}\omega}{(\omega_n^2 - \omega^2 + \mathrm{j}2\varsigma\omega_n\omega)} \tag{14}$$

and the corresponding impulse response is given by

$$g(t) = -\frac{\omega_n}{m\omega_d}\,\mathrm{e}^{-\varsigma\omega_n t}\sin(\omega_d t - \varphi), \quad t \geqslant 0, \tag{15}$$

where the damped natural frequency $\omega_d = \omega_n\sqrt{1-\varsigma^2}$, the natural frequency $\omega_n = \sqrt{k/m}$, the damping coefficient $\varsigma = c/2\sqrt{mk}$, and the phase angle $\varphi = \tan^{-1}\sqrt{1-\varsigma^2}/\varsigma$. Feedforward control is considered first. By replacing white noise by an impulse, the approach described in section 3 is used to determine optimal controller. The Wiener filter is $H_o(\mathrm{j}\omega) = H_o'(\mathrm{j}\omega)/G(\mathrm{j}\omega)$ and $h_o'(t)$ is given by

$$h_o'(t) = \frac{\omega_n}{m\omega_d}\,\mathrm{e}^{-\varsigma\omega_n\tau}\,\mathrm{e}^{-\varsigma\omega_n t}\sin(\omega_d t + \omega_d\tau - \varphi), \quad t \geqslant 0. \tag{16}$$

Using the identity $\sin(A+B) = \sin A\cos B + \cos A\sin B$, equation (16) becomes

$$h_o'(t) = \frac{\omega_n}{m\omega_d}\,\mathrm{e}^{-\varsigma\omega_n\tau}\,\mathrm{e}^{-\varsigma\omega_n t}\{\sin(\omega_d t)\cos(\omega_d\tau - \varphi) + \cos(\omega_d t)\sin(\omega_d\tau - \varphi)\}, \quad t \geqslant 0. \tag{17}$$

Now the one-sided Fourier transforms give the relationships [18]

$$\int_0^\infty e^{-at}\sin(bt)e^{-j\omega t}\,dt = \frac{b}{((j\omega + a)^2 + b^2)}\quad \int_0^\infty e^{-at}\cos(bt)e^{-j\omega t}\,dt = \frac{(j\omega + a)}{((j\omega + a)^2 + b^2)}, \tag{18}$$

where a and b are arbitrary constants, so the Fourier transform of $h'_o(t)$ can be written as

$$H'_o(j\omega) = \frac{j\omega}{((j\omega + \varsigma\omega_n)^2 + \omega_d^2)}\left(-g(\tau) + \frac{1}{j\omega}\,\omega_n^2 g_d(\tau)\right), \tag{19}$$

where $g(\tau)$ and $g_d(\tau)$ are the velocity and displacement impulse responses at time τ, respectively, and are given by $g(\tau) = -(\omega_n/m\omega_d)e^{-\varsigma\omega_n\tau}\sin(\omega_d\tau - \varphi)$ and $g_d(\tau) = (1/m\omega_d)e^{-\varsigma\omega_n t}\sin(\omega_d\tau)$ [17]. The optimal controller is thus given by

$$H_o(j\omega) = -mg(\tau) + m\omega_n^2 g_d(\tau)\,\frac{1}{j\omega}, \tag{20}$$

which contains both proportional and integral operators. Its inverse Fourier transform gives

$$h_o(t) = -mg(\tau)\delta(t) + m\omega_n^2 g_d(\tau). \tag{21}$$

As expected, when $\tau = 0$, $H_o(j\omega) = -1$ and $h_o(t) = -\delta(t)$ which means the secondary force is equal but opposite to the primary signal, and the error is zero. From equation (15) the square of the impulse response function can be written as

$$g^2(t) = \left(\frac{\omega_n}{m\omega_d}\right)^2 e^{-2\varsigma\omega_n^t}\sin^2(\omega_d t - \varphi). \tag{22}$$

Setting $(t - \varphi/\omega_d) = \tau_1$, equation (22) can be written as $g^2(\tau_1) = Ae^{a\tau_1}\sin^2(b\tau_1)$ where $A = (\omega_n/m\omega_d)^2 e^{2\varsigma\omega_n\varphi/\omega_d}$, $a = -2\varsigma\omega_n$, and $b = \omega_d$. Thus, equation (9) can be written as

$$J' = 1 - \frac{\int_{\tau-\varphi/\omega_d}^{\infty} g^2(\tau_1)\,d\tau_1}{\int_{-\varphi/\omega_d}^{\infty} g^2(\tau_1)\,d\tau_1}. \tag{23}$$

Using the relationship [18]

$$\int Ae^{at}\sin^2 bt\,dt = \frac{A}{(a^2 + 4b^2)}\left(e^{at}\sin bt(a\sin bt - 2b\cos bt) + \frac{2b^2}{a}e^{at}\right) \tag{24}$$

one obtains

$$J' = 1 - e^{-2\varsigma\omega_n\tau}\left(1 + \frac{2\varsigma\omega_n^2}{\omega_d^2}\sin(\omega_d\tau)\sin(\omega_d\tau - \varphi)\right), \tag{25}$$

which can be written as

$$J' = 1 - e^{-4\pi\varsigma\tau'}\left(1 + \frac{2\varsigma}{1-\varsigma^2}\sin(2\pi\sqrt{1-\varsigma^2}\tau')\sin(2\pi\sqrt{1-\zeta^2}\tau' - \varphi)\right), \tag{26}$$

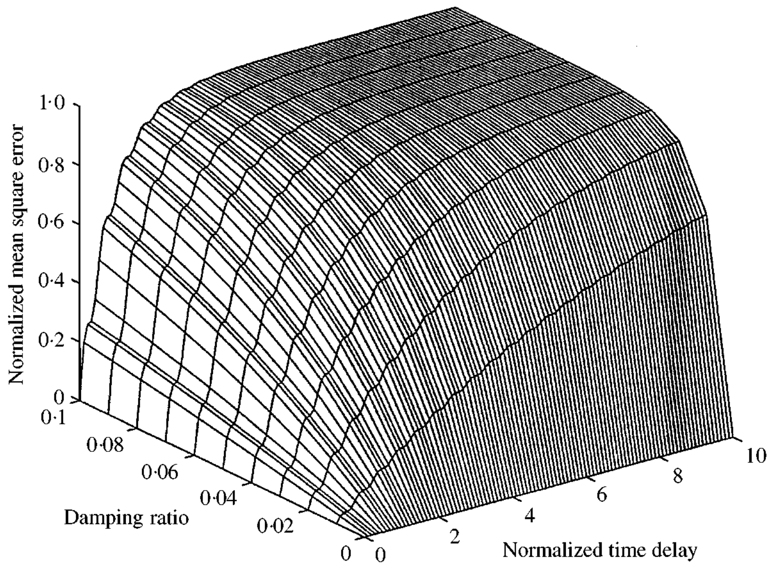

Figure 10. The normalized mean square error J′ as a function of the normalized time delay $\tau' = \tau/T_n$ and damping ratio.

where $\tau' = \tau/T_n$ is the normalized delay to the natural period where the period of the impulse response $T_n = 2\pi/\omega_n$. The normalized mean square error J' as a function of the normalized time delay τ' and damping ratio ζ is shown in Figure 10. It can be seen that perfect control is possible for the system with no damping, regardless of the delay. This is because the impulse response of this system is a sinusoidal signal. If the oscillatory term inside the brackets is neglected, equation (26) simplifies to

$$J' \cong 1 - \mathrm{e}^{-4\pi\varsigma\tau'}. \tag{27}$$

Thus, the mean square error is a function of $\varsigma\tau'$. The smaller the value of $\varsigma\tau'$, the better the control performance, and perfect control is achieved when it is zero. This result is similar to that obtained by Nelson *et al.* [7], who used white-noise excitation for a similar system.

Feedback control using IMC is now considered for the system shown in Figure 9(b). Following the procedure in section 3 equation (13) can be rewritten as

$$H_o(\mathrm{j}\omega) = \frac{1}{G^2(\mathrm{j}\omega)} H'_o(\mathrm{j}\omega), \tag{28}$$

where $H'_o(\mathrm{j}\omega)$ is given by equation (19). The resulting optimal controller is given by

$$H_o(\mathrm{j}\omega) = A_1\mathrm{j}\omega + A_0 + A_{-1}\frac{1}{\mathrm{j}\omega} + A_{-2}\frac{1}{(\mathrm{j}\omega)^2} \tag{29}$$

where $A_1 = -m^2 g(\tau)$, $A_0 = m^2(\omega_n^2 g_d(\tau) - 2\varsigma\omega_n g(\tau))$, $A_{-1} = m^2(2\varsigma\omega_n^3 g_d(\tau) - \omega_n^2 g(\tau))$, and $A_{-2} = m^2\omega_n^4 g_d(\tau)$. Its impulse response is given by

$$h_o(t) = A_1\delta'(t) + A_0\delta(t) + A_{-1} + A_{-2}t. \tag{30}$$

where $\delta'(t)$ is the unit doublet function.

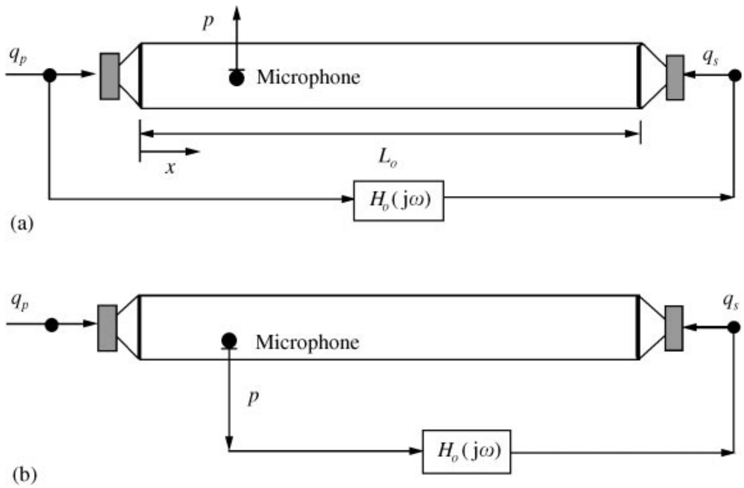

Figure 11. Feedforward and feedback control of sound in a duct: (a) feedforward control; (b) feedback control.

Since the plant is minimum phase, the normalized minimum mean square error for feedback control is the same as given in equation (26). As with feedforward control of this system, the signal processing delay is the most important factor in determining the performance, and should be minimized as far as possible.

5. ACTIVE CONTROL OF A NON-MINIMUM PHASE ACOUSTICAL SYSTEM

Regardless of whether feedforward or feedback control is used for either a minimum or non-minimum phase system, the Wiener filter is the optimal controller. In an acoustical system, the performance of the control system is determined by the wave propagation delay in the primary and secondary paths which may have non-linear phase characteristics [19]. To show the wave propagation time difference between the paths more precisely, the phase angle difference function may be introduced as

$$\Theta(\mathrm{e}^{\mathrm{j}\omega}) = \angle D(\mathrm{e}^{\mathrm{j}\omega}) - \angle R(\mathrm{e}^{\mathrm{j}\omega}), \tag{31}$$

where the variable $\mathrm{e}^{\mathrm{j}\omega}$ denotes frequency dependency of sampled signals and $\angle$ denotes the phase angle. The desired and received signals $D(\mathrm{e}^{\mathrm{j}\omega})$ and $R(\mathrm{e}^{\mathrm{j}\omega})$ can be obtained from Figure 3(b) for feedforward control, and from Figure 7 for feedback control. In this section, control mechanisms are investigated using the phase angle difference function. The one-dimensional acoustic system considered is shown in Figure 11. It is excited by white noise through the primary source $q_p(t)$ located at $x = 0$. Both feedforward and feedback techniques are considered to minimize the sound pressure response at the error sensor, as shown in Figure 11(a) and (b) respectively. The sound pressure at the error sensor located at x_e can be obtained by a summation of the first N modes given by [11]

$$p(x_e, t) = \sum_{n=1}^{N} \psi_n(x_e) a_n(t), \tag{32}$$

where $\psi_n(x_e)$ is the acoustic mode shape function at x_e and $a_n(t)$ is the complex amplitude of the nth mode. With the primary source at $x = 0$ and the secondary source at $x = L_o$, the

TABLE 1

The acoustic duct data

Length L_o (m)	Area S_o (m^2)	Density ρ_o (kg/m^3)	Sound speed c_o (m/s)	Sampling frequency (Hz)
3·4	0·1 × 0·1	1·21	340	1000

impulse responses of the primary and secondary plants are given by [11]

$$g_p(t) = \frac{\rho_o c_o^2}{V} \sum_{n=1}^{N} \psi_n(x_e)\psi_n(0)A_n(t), \tag{33a}$$

$$g_s(t) = \frac{\rho_o c_o^2}{V} \sum_{n=1}^{N} \psi_n(x_e)\psi_n(L_o)A_n(t), \tag{33b}$$

where the time-dependent term is given by

$$A_1(t) = \mathrm{e}^{t/T_a} \quad \text{when } n = 1,$$

$$A_n(t) = -\frac{\omega_n}{\omega_d}\mathrm{e}^{-\varsigma\omega_n t}\sin(\omega_d t - \varphi_n) \quad \text{when } n \neq 1.$$

T_a is the time constant for the first acoustic mode [20], $\varphi_n = \tan^{-1}\sqrt{1-\varsigma_n^2}/\varsigma_n$ is the phase angle, and ω_n and ω_d are the undamped and damped natural frequencies respectively. Regardless of which control technique is used, the acoustic pressure in equation (32) can be expressed as the error signal for the Wiener filter, and is written as $p(x_e, t) = d(t) + r(t)*h(t)$, where the operator $*$ denotes convolution. Since white noise is assumed as the excitation signal, $q_p(t) = \delta(t)$. Thus, the desired and received signals are $d(t) = g_p(t)$ and $r(t) = g_s(t)$ is feedforward control, and $d(t) = g_p(t)$, $r(t) = g_p(t)*g_s(t)$ in feedback control. No electrical control time delay is assumed so that the phase difference function is solely determined by the non-minimum phase characteristics of the primary and secondary mechanical plants.

Physical data used for simulations are shown in Table 1. A time constant for the first mode of 0·2 s and a damping ratio of 0·05 for the other modes were assumed. The impulse responses given in equation (33a, b) were sampled at a frequency of 1000 Hz over 2 s and were modelled as 2000-length FIR filters. To avoid aliasing, only the first 10 acoustic modes whose last mode is at 450 Hz were assumed to be excited. Optimal controllers were calculated for 11 equidistant error microphone positions from $x = 0$ to L_o. Figure 12(a) shows the performance of feedforward control according to the position of the microphone plotted against the normalized duct length. At normalized locations less than 0·5, poor performance is observed since a pulse takes longer to travel from the secondary source to the error sensor than from the primary source (phase lead in the phase angle difference function). At the mid-position, both pulses arrive at the same time (zero phase) and thus perfect cancellation is achieved. At normalized locations greater than 0·5, where the error sensor is closer to the secondary source (phase lag in the phase angle difference function), a good performance is achieved, but not perfect minimization between positions 0·55 and 0·9. This is because the wave propagation inside the duct is dispersive due to damping [20].

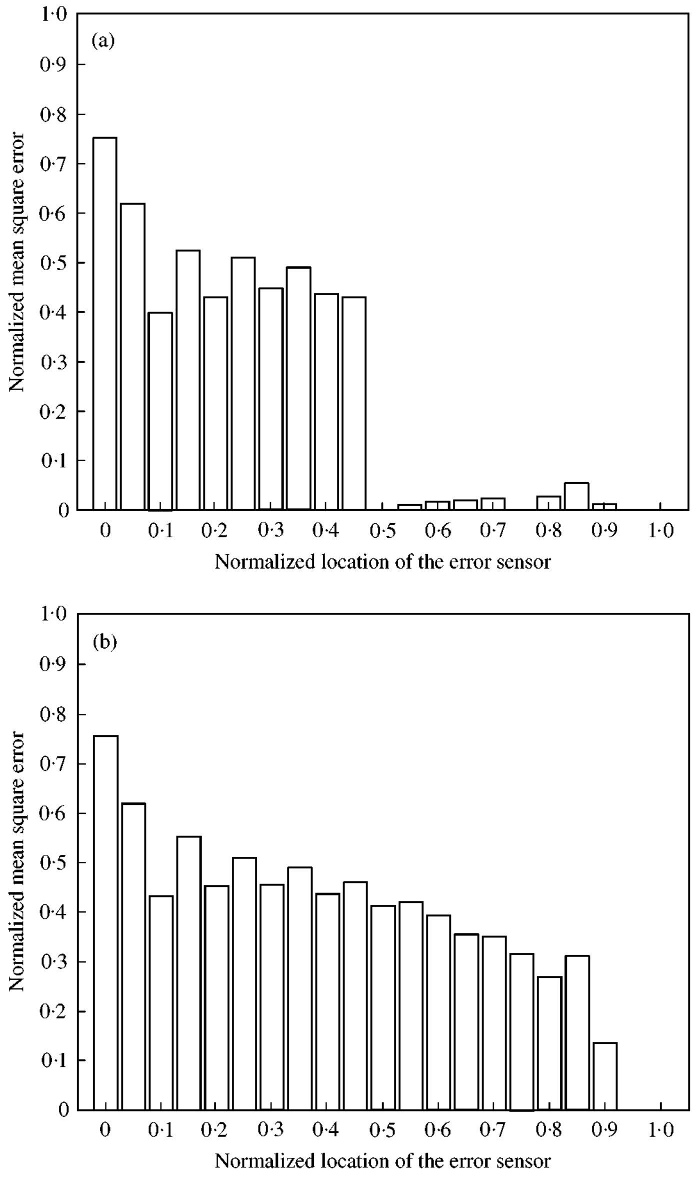

Figure 12. Performance of feedforward and feedback control according to the location of the error sensor which is normalized by the length of the duct: (a) feedforward control; (b) feedback control.

At positions 0·95 and 1, there is collocation control where the secondary plant is minimum phase.

The performance of feedback control according to the location of the microphone is shown in Figure 12(b). It can be seen that the performance improves as the sensor gets closer to the secondary source location at L_o. Note in this case that the phase angle difference function in equation (31) is entirely determined by the phase characteristics of the secondary plant. As the error sensor gets closer to the secondary source position, the time delay of the secondary path decreases. At the points 0·95 and 1, there is collocated control so that the error becomes zero.

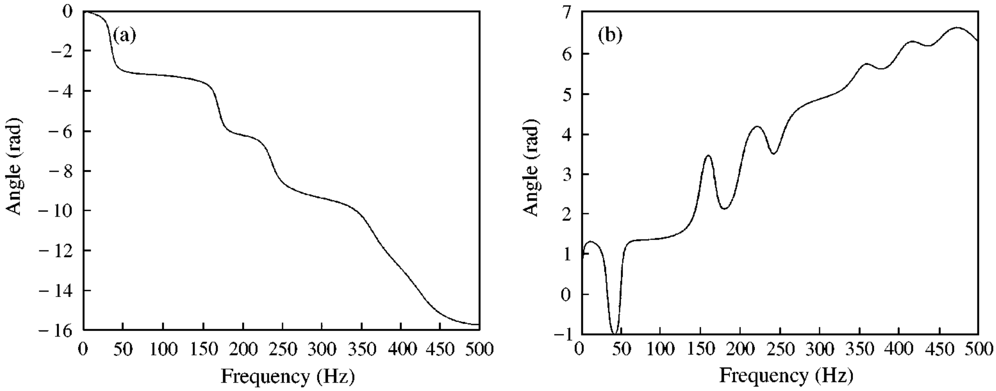

Figure 13. Phase angle difference functions at $x_1 = 0{\cdot}75\ L_o$: (a) feedforward control; (b) feedback control.

Figure 13 shows the phase angle difference functions for both feedforward and feedback control when the error sensor is located at the position 0·75. With feedforward control, the phase angle difference function shows that there is a phase lag so that nearly perfect minimization is achieved as shown in Figure 11(a). In feedback control, the phase difference function shows a phase lead, and results in a poorer performance as shown in Figure 13. This demonstrates that the phase angle difference function can be used as an indirect measure of predicting control performance, regardless of whether feedforward or feedback techniques are used.

6. CONCLUSIONS

The paper has considered the feedforward and feedback active control of stationary random disturbances in SISO vibrating and acoustical systems. A deterministic approach has been taken using Wiener filter theory in a unified framework to study optimal control with both strategies. The advantages of this approach are that it provides an easier understanding to the optimal control of random disturbances, and the calculation time for simulations is greatly reduced since no time-consuming averaging processes are required. Using this approach the active control of a single-degree-of-freedom vibrating system and an acoustic duct has been studied. The results from theoretical and numerical analyses demonstrate that the control performances with both feedforward and feedback control deteriorate as the electric time delay and system damping increase. The phase angle difference function was introduced to study the non-minimum phase characteristics of the primary and secondary paths in the acoustic system. This was used to investigate performance limitations imposed by the constraint of causality, where it was shown that a larger phase lead results in a poorer control performance.

ACKNOWLEDGMENTS

The authors would like to thank Professors P. A. Nelson and J. K. Hammond for their helpful discussions on the subject.

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APPENDIX A: DISCRETE TIME DOMAIN REPRESENTATION OF WIENER FILTER THEORY

The discrete form of the Wiener–Hopf equation is given by [21]

$$\sum_{k=0}^{\infty} R_{rr}[n-k]\,h_o[k] = -R_{rd}[n], \quad n \geqslant 0. \tag{A1}$$

Unlike the continuous time Wiener–Hopf equation, equation (A1) has a closed-form solution when the Wiener filter is modelled as a finite impulse response (FIR) filter. If the filter has I coefficients, then equation (A1) can be expressed in matrix form as

$$\mathbf{A}\mathbf{h}_{\mathbf{o}} = -\mathbf{b}, \tag{A2}$$

where the $(I \times I)$ auto-correlation matrix $\mathbf{A}$ is a Toeplitz matrix, and is given by

$$\mathbf{A} = \begin{bmatrix} R_{rr}(0) & R_{rr}(1) & \cdots & R_{rr}(I-1) \\ R_{rr}(-1) & R_{rr}(0) & \cdots & R_{rr}(I-2) \\ \vdots & \vdots & & \vdots \\ R_{rr}(-I+1) & R_{rr}(-I+2) & \cdots & R_{rr}(0) \end{bmatrix}$$

and the I length Wiener filter coefficient vector $\mathbf{h}_o$ and the I length cross-correlation vector $\mathbf{b}$ are given by

$$\mathbf{h_o} = \{h_o(0)\ h_o(1)\ \cdots\ h_o(I-1)\}^{\mathrm{T}},$$

$$\mathbf{b} = \{R_{rd}(0)\ R_{rd}(1) \cdots R_{rd}(I-1)\}^{\mathrm{T}}.$$

The matrix A is real-valued and symmetric since $R_{rr}(\tau) = R_{rr}(-\tau)$. When the received signal $r(t)$ is spectrally rich as in random processes, the matrix $\mathbf{A}$ is guaranteed to be positive definite [22]. The Wiener filter can be obtained from equation (A2) to give

$$\mathbf{h_o} = -\mathbf{A}^{-1}\mathbf{b} \qquad \text{(A3)}$$

and the minimum mean square error is

$$J_o = R_{dd}(0) - \mathbf{b}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{b}. \qquad \text{(A4)}$$

When the received signal is white noise, the correlation matrix $\mathbf{A}$ is the identity matrix. In this case, the Wiener filter is given by $\mathbf{h_o} = -\mathbf{b}$, and thus the minimum mean square error is

$$J_o = R_{dd}(0) - \mathbf{h_o^{\mathrm{T}}}\mathbf{h_o}. \qquad \text{(A5)}$$