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LETTERS TO THE EDITOR

OSCILLATIONS IN AN $x^{4/3}$ POTENTIAL

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Much research has been done on the oscillations of non-linear, one-dimensional systems for which the linear terms correspond to the harmonic oscillator differential equation [1, 2]

$$\ddot{x} + x = 0. \tag{1}$$

However, dynamical systems for which this condition does not hold are also of interest. For example, Mickens and his collaborators have studied the equation [2]

$$\ddot{x} + x^3 = \varepsilon f(x, \dot{x}), \tag{2}$$

where $f(x, y)$ can be a rational function of x and y, and ε is a small positive parameter. (See Mickens [2] for a listing of other works on this problem.) This paper considers systems that can be modelled by equations of motion where the potential takes the form

$$V(x) = V_0 x^{(2n+2)/(2n+1)}, \tag{3}$$

where $n = 1, 2, 3, \ldots$, and V_0 is a positive constant. Note that the force derived from equation (3) is

$$f(x) = -\left(\frac{2n+2}{2n+1}\right) V_0 x^{1/(2n+1)}, \tag{4}$$

and that $V(x)$ and $f(x)$ are, respectively, even and odd functions of x, i.e.,

$$V(-x) = V(x), \qquad f(-x) = -f(x). \tag{5}$$

In the following calculations the value of n will be taken to be $n = 1$. However, the summary will present the results for arbitrary positive n. The central result is that a system under the influence of a force, such as equation (4), has only periodic solutions. Further, the method of harmonic balance will be used to calculate an analytical approximation to these periodic solutions.

A particle of mass, M, acted on by the force of equation (4), has the equation of motion

$$M \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \left(\frac{2n+2}{2n+1}\right) V_0 x^{1/(2n+1)} = 0. \tag{6}$$

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By a proper change of both the dependent and independent variables, this equation can be transformed to the dimensionless form

$$\frac{d^2\bar{x}}{d\bar{t}^2} + \bar{x}^{1/(2n+1)} = 0. \tag{7a}$$

In the work to follow, the "bars" will be dropped to give

$$\frac{d^2x}{dt^2} + x^{1/(2n+1)} = 0. \tag{7b}$$

For $n = 1$, equation (7b) becomes

$$\frac{d^2x}{dt^2} + x^{1/3} = 0. \tag{8}$$

The system equations for equation (8) are

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -x^{1/3}, \tag{9}$$

and the first order differential equation that the trajectories in the (x, y) phase-space satisfy is

$$\frac{dy}{dx} = -\left(\frac{x^{1/3}}{y}\right). \tag{10}$$

Since $x^{1/3}$ is an odd function of x, equation (10) is invariant under the three transformations

$$T_1 : x \to -x, \qquad y \to y, \tag{11a}$$

$$T_2 : \quad x \to x, \qquad y \to -y, \tag{11b}$$

$$T_3 : x \to -x, \qquad y \to -y. \tag{11c}$$

The corresponding null-clines [2], curves along which the slope dy/dx is either zero or bounded, are

$$\frac{dy}{dx} = 0 : x = 0 \quad \text{or along the } y\text{-axis}, \tag{12a}$$

$$\frac{dy}{dx} = \infty : y = 0 \quad \text{or along the } x\text{-axis}. \tag{12b}$$

The results given in equations (11) and (12) are exactly the same as those for the simple harmonic oscillator [1, 2]. Consequently, applying the standard phase-space qualitative methods [1–3], it can be concluded that all the trajectories in phase-space are closed [2]. This implies that all the solutions to equation (8) are periodic [1–3].

An approximate analytical solution can be calculated using the method of harmonic balance [2]. To proceed with this calculation, equation (8) is rewritten in the form

$$\left(\frac{d^2x}{dt^2}\right)^3 + x = 0, \qquad x(0) = x_0, \qquad \dot{x}(0) = 0, \tag{13}$$

where x_0 is given and the approximate solution is taken to be

$$x(t) \simeq A\cos\omega t. \tag{14}$$

The parameters A and ω are to be determined from the harmonic balance procedure [2]. Substitution of equation (14) into equation (13) gives

$$(-A\omega^2\cos\theta)^3 + A\cos\theta \simeq 0, \quad \theta = \omega t \tag{15}$$

or

$$A\left[1 - \left(\frac{3}{4}\right)A^2\omega^6\right]\cos\theta + (\text{higher order harmonics}) = 0. \tag{16}$$

Applying harmonic balance and using the first of the initial conditions in equation (13) gives

$$A = x_0, \qquad \omega = \left(\frac{4}{3x_0^2}\right)^{1/6}. \tag{17}$$

Thus, an approximation to the periodic solution to equation (8) is

$$x(t) \simeq x_0\cos\left[\left(\frac{4}{3x_0^2}\right)^{1/6} t\right]. \tag{18}$$

A similar, but more complicated calculation, can be done for the general case of equation (7b). Again, as above, this equation can be rewritten as

$$\left(\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right)^{2n+1} + x = 0. \tag{19}$$

Substitution of equation (14) into equation (19) gives

$$(-A\omega^2\cos\theta)^{2n+1} + A\cos\theta \simeq 0. \tag{20}$$

Using the relation [4]

$$(\cos\theta)^{2n+1} = \left(\frac{1}{2^{2n}}\right)\sum_{=0}^{n}\binom{2n+1}{k}\cos[2(n-k)+1]\theta \tag{21}$$

and keeping only the term in $\cos\theta$ allows the following result to be obtained:

$$A\left[1 - \binom{2n+1}{n}\left(\frac{A^{2n}\omega^{4n+2}}{2^{2n}}\right)\right]\cos\theta + (\text{higher order harmonics}) = 0. \tag{22}$$

This leads to the following approximate solution for equation (7b):

$$x(t) \simeq x_0\cos[\omega_n(x_0)t], \tag{23}$$

where

$$\omega_n(x_0) = \left[\frac{2^{2n}}{\binom{2n+1}{n}x_0^{2n}}\right]^{1/(4n+2)}. \tag{24}$$

In summary, the mathematical properties of a "truly" non-linear oscillator, given by equation (7b), has been studied. All the solutions are found to be periodic and the method of harmonic balance was used to construct an analytical approximation to these solutions. This continues our investigation of one-dimensional oscillators for which the linear terms do not correspond to a simple harmonic oscillator.

Finally, it should be indicated that equation (8) was also studied numerically using the non-standard finite difference schemes of Mickens [5]. The particular discrete model used was

$$\frac{x_{k+1} - x_k}{\sin(h)} = y_k, \qquad \frac{y_{k+1} - y_k}{\sin(h)} = -(x_{k+1})^{1/3}, \tag{25}$$

where $\phi(h) = \sin(h)$ is the so-called denominator function, with h being the time step-size. All numerical solutions were found to be periodic with closed trajectories in phase-space.

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