Journal of Sound and Vibration (2001) **246**(2), 379–386
doi:10.1006/jsvi.2000.3582, available online at http://www.idealibrary.com on IDEAL®

LIMIT CYCLE BEHAVIOR IN THREE- OR HIGHER-DIMENSIONAL NON-LINEAR SYSTEMS: THE LOTKA–VOLTERRA EXAMPLE

S. M. SHAHRUZ AND D. A. KALKIN

Berkeley Engineering Research Institute, P.O. Box 9984, *Berkeley, CA* 94709, *U.S.A.*
E-mail: *shahruz@robotics.eecs.berkeley.edu*

(*Received* 18 *December* 2000, *and in final form* 29 *December* 2000)

1. INTRODUCTION

Limit cycle (periodic or oscillatory) behavior is observed in many physical and biological systems. The problem of determining when a non-linear dynamical system exhibits limit cycle behavior has been of great interest for more than a century. Researchers have been devising techniques by which the existence of limit cycle behavior in non-linear systems can be established (see, e.g., references [1–4]). The Poincare'–Bendixon theorem provides a rather easily applicable technique by which the existence of limit cycle behavior in planar (two-dimensional) non-linear systems can be established (see, e.g., references [1–4]). This theorem, however, is applicable only to planar systems since it is based on the Jordan curve theorem (see, e.g., references [5, 6]), which holds for simply connected curves in a plane. There are generalizations of the Poincare'–Bendixon theorem for three- or higher-dimensional non-linear systems (see, e.g., references [7, 8]). These generalizations, however, are inapplicable to most systems of interest. Another technique by which the existence of limit cycle behavior can be established is the Poincare'–Andronov–Hopf bifurcation technique (see, e.g., references [9–14]). This technique can be applied to planar or higher-dimensional non-linear systems, however, it is computationally demanding. There is yet another technique, known as the describing function method, by which the existence of limit cycle behavior in non-linear systems can be predicted (see, e.g., references [15, 16]). The describing function method is mostly used for non-linear control systems and presently is seldom used. This technique can lead to erroneous results: (1) it can fail to predict existing limit cycles (see, e.g., reference [17]), and (2) it can spuriously predict non-existing limit cycles (see, e.g., reference [18]).

Considering the difficulties of existing techniques by which the existence of limit cycle behavior in three- or higher-dimensional non-linear systems is established, it is desirable to devise techniques which predict such behavior conveniently. In references [19, 20], a systematic procedure is presented by which the existence of limit cycle behavior (self-pulsation) in lasers can be established. This procedure is applicable to planar or higher-dimensional non-linear systems and is computationally straightforward. In order to apply this procedure, it is necessary to have the state-space representation of the system. Having such a representation, (1) the boundedness of the system states is established, and (2) the system parameters are chosen so that all equilibrium points of the system are destabilized. The boundedness of the system states and the instability of its equilibrium points imply that the system can have periodic, quasi-periodic, or chaotic behavior. When the system states are periodic, the system has limit cycle behavior.

0022-460X/01/370379 + 08 $35.00/0

In this note, the procedure in references [19, 20] is applied to an important three-dimensional Lotka–Volterra system to determine the conditions under which this system has limit cycle behavior (periodic solution). Lotka–Volterra systems have been of interest as they have been studied extensively by researchers (see, e.g., references [21–29] and the references therein).

2. A LOTKA–VOLTERRA SYSTEM

Lotka–Volterra systems model the dynamics of interacting species. These systems have been studied by many researchers (see, e.g., references [21–29] and the references therein). The Lotka–Volterra system considered in this note is the following three-dimensional system (see reference [26]):

$$\dot{N}_1(t) = r_1[1 + a - N_1(t) - aN_3(t)]\,N_1(t), \quad N_1(0) =: N_{10} > 0, \tag{1a}$$

$$\dot{N}_2(t) = r_2[N_1(t) - N_2(t)]\,N_2(t), \quad N_2(0) =: N_{20} > 0, \tag{1b}$$

$$\dot{N}_3(t) = r_3[N_2(t) - N_3(t)]\,N_3(t), \quad N_3(0) =: N_{30} > 0, \tag{1c}$$

for all $t \geqslant 0$, where the system state $N_i(t) \in \mathbb{R}$ for all $i = 1, 2, 3$, and the parameters r_1, r_2, r_3, and a are constant positive numbers.

A problem of interest is to determine when system (1) exhibits limit cycle behavior (see, e.g., references [26, 27]). Thus, the problem to be solved is:

Problem P. Determine the positive parameters r_1, r_2, r_3, and a for which system (1) exhibits limit cycle behavior.

In this note, Problem P is solved by applying the procedure in references [19, 20]. A useful property of system (1), which is used in solving Problem P, is proved next.

The state space of system (1) is $\mathbb{R}^3$. Let the state vector of system (1) be denoted by $X(t) := [N_1(t)\ N_2(t)\ N_3(t)]^T \in \mathbb{R}^3$ for all $t \geqslant 0$. Let the solution of the system starting at $t = 0$ from the initial vector $X_0 := X(0) = [N_{10}\ N_{20}\ N_{30}]^T$ be denoted by the vector $X(t, 0, X_0)$ for all $t \geqslant 0$.

The useful property to be proved is that the non-negative orthant $\mathbb{R}^3_+$ is an invariant set of system (1). That is, if the system starts from any initial vector $X_0 \in \mathbb{R}^3_+$, then the solution vector $X(t, 0, X_0) \in \mathbb{R}^3_+$ for all $t \geqslant 0$.

Lemma 2.1. *The non-negative orthant* $\mathbb{R}^3_+$ *is an invariant set of system* (1).

Proof. Define the subspace S_{N_i} for all $i = 1, 2, 3$ as

$$S_{N_i} := \{(N_1, N_2, N_3) \in \mathbb{R}^3 \,|\, N_i = 0\}. \tag{2}$$

From equation (1), it is concluded that for any point in S_{N_i}, $\dot{N}_i(t) = 0$ for all $t \geqslant 0$ and $i = 1, 2, 3$. Thus, any trajectory of system (1) starting in $\mathbb{R}^3_+$ cannot traverse into regions of $\mathbb{R}^3$ for which $N_1 < 0$ or $N_2 < 0$ or $N_3 < 0$. $\square$

The invariance of $\mathbb{R}^3_+$, i.e., the non-negativeness of state $N_i(\cdot)$ for all $i = 1, 2, 3$, will be used repeatedly in the note.

3. LIMIT CYCLE BEHAVIOR

In this section, Problem P is solved by applying the procedure in references [19, 20], according to which two steps should be taken: (1) the boundedness of the system states should be established, and (2) all equilibrium points of the system should be destabilized. The solution of Problem P is a set of parameters r_1, r_2, r_3, and a for which system (1) exhibits limit cycle behavior. Having this set obtained, an example is given by which the existence of limit cycles is corroborated.

3.1. BOUNDEDNESS OF SYSTEM STATES

In order to show that the state $N_i(\cdot)$ is bounded for all $i = 1, 2, 3$, let a scalar function of time be defined as

$$V(t) := N_1(t) + N_2(t) + N_3(t), \tag{3}$$

for all $t \geqslant 0$. By Lemma 2.1, the function $V(\cdot)$ is non-negative. It is next proved that $V(\cdot)$ is a bounded function of time by which the boundedness of the states follows.

Theorem 3.1. *Let $0 < \gamma < 1$ be an arbitrary constant number. If*

$$r_2 > \frac{r_3}{4}, \quad r_1 > \frac{r_2^2}{(1-\gamma)(4r_2 - r_3)}, \tag{4a, b}$$

then $V(\cdot)$ is a bounded function of time, and so is the state $N_i(\cdot)$ for all $i = 1, 2, 3$.

Proof. From equations (3) and (1), it is concluded that

$$\dot{V}(t) = r_1(1+a)N_1(t) - r_1\gamma N_1^2(t) - r_1 a N_3(t)N_1(t)$$
$$- r_1(1-\gamma)N_1^2(t) + r_2 N_1(t)N_2(t) - r_2 N_2^2(t) + r_3 N_2(t)N_3(t) - r_3 N_3^2(t), \tag{5}$$

for all $t \geqslant 0$. It can be easily verified that the maximum value of the quadratic function $r_1(1+a)N_1 - r_1\gamma N_1^2$, which is attained at $N_1 = (1+a)/(2\gamma)$, is equal to $r_1(1+a)^2/(4\gamma)$. Using this fact and the non-negativeness of $N_1(\cdot)$ and $N_3(\cdot)$ established in Lemma 2.1, equation (5) can be written as

$$\dot{V}(t) \leqslant -X^{\mathrm{T}}(t)Q\,X(t) + \frac{r_1(1+a)^2}{4\gamma}, \tag{6}$$

for all $t \geqslant 0$, where the symmetric matrix $Q \in \mathbb{R}^{3\times 3}$ is

$$Q = \begin{bmatrix} r_1(1-\gamma) & -\frac{r_2}{2} & 0 \\ -\frac{r_2}{2} & r_2 & -\frac{r_3}{2} \\ 0 & -\frac{r_3}{2} & r_3 \end{bmatrix}. \tag{7}$$

The principal minors of Q are

$$\Delta_1 = r_3, \qquad \Delta_2 = r_3\left(r_2 - \frac{r_3}{4}\right), \tag{8a, b}$$

$$\Delta_3 = r_1 r_3 (1-\gamma)\left(r_2 - \frac{r_3}{4}\right) - \frac{r_2^2 r_3}{4}. \tag{8c}$$

Clearly, $\Delta_1 > 0$. By inequalities (4), it is clear that $\Delta_2 > 0$ and $\Delta_3 > 0$. Since $\Delta_i > 0$ for all $i = 1, 2, 3$, the matrix Q is positive definite, hence all its eigenvalues are positive. Having all eigenvalues of Q positive, inequality (6) can be written as

$$\dot{V}(t) \leqslant -\lambda_{min}(Q)\,\|X(t)\|_E^2 + \frac{r_1(1+a)^2}{4\gamma}, \tag{9}$$

for all $t \geqslant 0$, where $\lambda_{min}(Q)$ denotes the smallest eigenvalue of the matrix Q and

$$\|X(t)\|_E := [N_1^2(t) + N_2^2(t) + N_3^2(t)]^{1/2}. \tag{10}$$

Therefore, for all $t \geqslant 0$ at which

$$\|X(t)\|_E^2 > \frac{r_1^{1/2}(1+a)}{2(\gamma\lambda_{min}(Q))^{1/2}}, \tag{11}$$

$\dot{V}(t) < 0$. That is, $V(\cdot)$ is a decaying function of time when the trajectory corresponding to the solution of system (1) in $\mathbb{R}_+^3$ is outside of the ball of radius $r_1^{1/2}(1+a)/(2(\gamma\lambda_{min}(Q))^{1/2})$ centered at the origin. Therefore, $V(\cdot)$ is a bounded function of time.

Finally, having $V(\cdot)$ a bounded function and the state $N_i(\cdot)$ non-negative for all $i = 1, 2, 3$, it follows from equation (3) that each $N_i(\cdot)$ is a bounded function. □

3.2. DESTABILIZATION OF EQUILIBRIUM POINTS

In order to destabilize the equilibrium points of system (1), they should be located first, in particular, those that are in $\mathbb{R}_+^3$, because by Lemma 2.1, the solution of system (1) is in $\mathbb{R}_+^3$.

Lemma 3.2. *The equilibrium points of system* (1) *in* $\mathbb{R}_+^3$ *are at*

$$X_{000} = (0, 0, 0), \qquad X_{+00} = (1+a, 0, 0), \tag{12a, b}$$

$$X_{++0} = (1+a, 1+a, 0), \qquad X_{+++} = (1, 1, 1). \tag{12c, d}$$

Proof. Set the right-hand side of system (1) equal to zero to obtain the equilibrium points in equations (12). □

Next, it is determined for what values of the parameters r_1, r_2, r_3, and a the equilibrium points in equations (12) are unstable.

Theorem 3.3. *The equilibrium points* X_{000}, X_{+00}, and X_{++0} *of system* (1) *are unstable for any positive value of the parameters* r_1, r_2, r_3, *and* a.

Proof. The instability of an equilibrium point of a system is decided upon by the eigenvalues of the coefficient matrix of the linearized model of the system around that point, namely, the Jacobian matrix. The Jacobian matrices corresponding to the equilibrium points X_{000}, X_{+00}, and X_{++0} are, respectively,

$$J_{000} = \begin{bmatrix} r_1(1+a) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{13a}$$

$$J_{+00} = \begin{bmatrix} -r_1(1+a) & 0 & -r_1 a(1+a) \\ 0 & r_2(1+a) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{13b}$$

$$J_{++0} = \begin{bmatrix} -r_1(1+a) & 0 & -r_1 a(1+a) \\ r_2(1+a) & -r_2(1+a) & 0 \\ 0 & 0 & r_3(1+a) \end{bmatrix}. \tag{13c}$$

It is clear that every Jacobian matrix in equations (13) has one positive eigenvalue. Thus, the equilibrium points X_{000}, X_{+00}, and X_{++0} are unstable. □

Next, it is determined when the equilibrium point X_{+++} in equation (12d) is unstable.

Theorem 3.4. *The equilibrium point X_{+++} of system* (1) *is unstable if and only if*

$$a > \frac{r_2 + r_3}{r_1} + \frac{r_3 + r_1}{r_2} + \frac{r_1 + r_2}{r_3} + 2. \tag{14}$$

Proof. The Jacobian matrix corresponding to the equilibrium point X_{+++} is

$$J_{+++} = \begin{bmatrix} -r_1 & 0 & -r_1 a \\ r_2 & -r_2 & 0 \\ 0 & r_3 & -r_3 \end{bmatrix}. \tag{15}$$

It can be easily verified that the eigenvalues of the matrix J_{+++}, denoted by λ, are the solutions of the following cubic equation:

$$\lambda^3 + (r_1 + r_2 + r_3)\lambda^2 + (r_1 r_2 + r_2 r_3 + r_3 r_1)\lambda + r_1 r_2 r_3 (1+a) = 0. \tag{16}$$

By the Routh–Hurwitz test (see, e.g., references [30, 31]), it is concluded that two solutions of equation (16) are in the open right-half complex plane, i.e., the equilibrium point X_{+++} is unstable, if and only if inequality (14) holds. □

Theorems 3.1, 3.3 and 3.4 provide a solution for Problem P. According to these theorems, if the positive parameters r_1, r_2, r_3 and a satisfy inequalities (4) and (14), then the solution of system (1) will not settle at a constant vector. The solution will be a time-varying function of time which will wander in $\mathbb{R}^3_+$ and when it is a periodic vector, the system has limit cycle behavior.

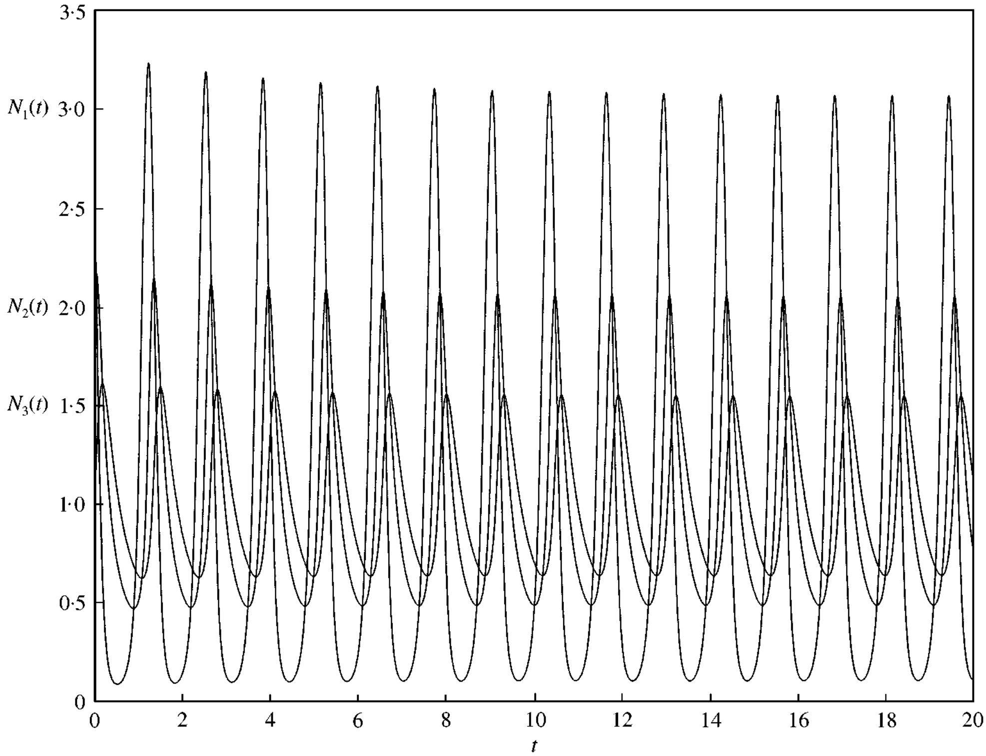

Figure 1. Time histories of $t \mapsto N_1(t)$ (largest amplitude), $t \mapsto N_2(t)$ (medium amplitude), and $t \mapsto N_3(t)$ (smallest amplitude). These time histories are periodic functions. That is, the system exhibits limit cycle behavior.

3.3. EXAMPLES

In this section, two examples are given to show the existence and non-existence of limit cycle behavior in system (1).

(1) In the first example, it is shown that if inequalities (4) and (14) are satisfied, then system (1) exhibits limit cycle behavior. Let

$$\gamma = 0{\cdot}25, \quad r_1 = 2, \quad r_2 = 3, \quad r_3 = 4, \quad a = 10, \tag{17}$$

in system (1). For these values of the parameters, it can be easily verified that inequalities (4) and (14) hold. Thus, system (1) can have limit cycle behavior. Indeed this is the case. Numerical simulations of system (1) attest the existence of limit cycle behavior; see Figure 1, where the time histories $t \mapsto N_1(t)$, $t \mapsto N_2(t)$, and $t \mapsto N_3(t)$ are shown. These time histories are periodic functions.

(2) In the second example, it is shown that if inequality (14) is not satisfied, i.e., if the equilibrium $X_{+++} = (1, 1, 1)$ is locally stable, then system (1) does not exhibit limit cycle behavior. Let the parameters γ, r_1, r_2, and r_3 be the same as those in equation (17), and let $a = 5$. For these values of the parameters, it can be easily verified that inequality (14) does not hold, and hence the equilibrium point X_{+++} is locally stable. Numerical simulations of system (1) show that the solution of the system converges to X_{+++}, no matter what the initial conditions N_{10}, N_{20}, and N_{30} are; see Figure 2. That is, system (1) does not exhibit limit cycle behavior.

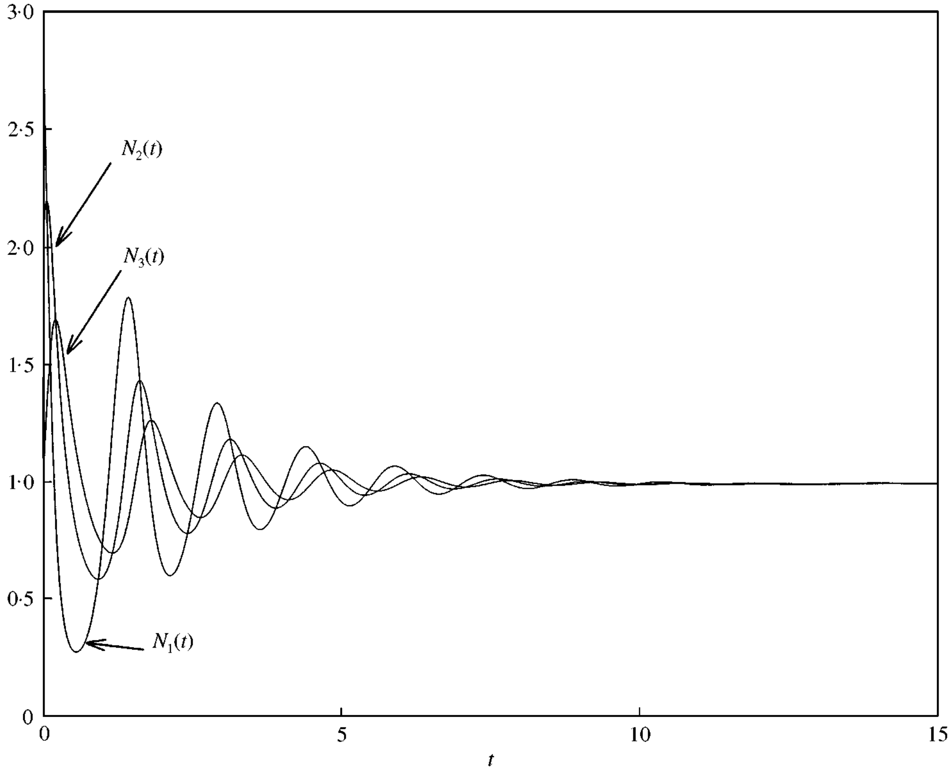

Figure 2. Time histories of $t \mapsto N_1(t)\, t \mapsto N_2(t)$, and $t \mapsto N_3(t)$. As $t \to \infty$, the function $N_i(t)$ tends to 1 for all $i = 1, 2, 3$. That is, the system does not exhibit limit cycle behavior.

4. CONCLUSIONS

In this note, a procedure is presented by which the existence of limit cycle behavior in three or higher-dimensional nonlinear systems can be established. The procedure has two steps: (1) the boundedness of the system states is established; and (2) all equilibrium points of the system are destabilized. These steps are applied to a three dimensional Lotka–Volterra system to determine a set of parameters for which this system exhibits limit cycle behavior.

The procedure by which the existence of limit cycle behavior in non-linear systems is established is conveniently applicable to planar or higher-dimensional systems. However, establishing the boundedness of the system states (step 1) may be difficult for some systems. Moreover, this procedure does not exclude the possibility of quasi-periodic or chaotic behaviors.

REFERENCES

1. A. A. ANDRONOV, E. A. LEONTOVICH, I. I. GORDON and A. G. MAIER 1973 *Qualitative Theory of Second-Order Dynamic Systems*. New York, NY: John Wiley.
2. M. FARKAS 1994 *Periodic Motions*. New York, NY: Springer-Verlag.
3. D. W. JORDAN and P. SMITH 1999 *Nonlinear Ordinary Differential Equations: An Introduction to Dynamical Systems*: Oxford, U.K.: Oxford University Press; third edition.
4. Y.-Q. YE 1986 *Theory of Limit Cycles*. Providence, RI: American Mathematical Society.

5. S. Lefschetz 1949 *Introduction to Topology*. Princeton, NJ: Princeton University Press.
6. J. Dugundji 1966 *Topology*. Boston, MA: Allyn and Bacon.
7. G. A. Leonov, I. M. Burkin and A. I. Shepeljavyi 1996 *Frequency Methods in Oscillation Theory*. Dordrecht, The Netherlands: Kluwer Academic Publishers.
8. R. A. Smith 1980 *Proceedings of the Royal Society of Edinburgh* **85A**, 153–172. Existence of periodic orbits of autonomous ordinary differential equations.
9. E. Doedel and L. S. Tuckerman, editors 2000 *Numerical Methods for Bifurcation Problems and Large-Scale Dynamical Systems*. New York, NY: Springer-Verlag.
10. P. Glendinning 1994 *Stability, Instability, and Choas: An Introduction to the Theory of Nonlinear Differential Equations*. Cambridge, U.K.: Cambridge University Press.
11. B. D. Hassard, N. D. Kazarinoff and Y.-H. Wan 1981 *Theory and Applications of Hopf Bifurcation*. Cambridge, U.K.: Cambridge University Press.
12. Y. A. Kuznetsov 1998 *Elements of Applied Bifurcation Theory*. New York, NY: Springer-Verlag; second edition.
13. J. E. Marsden and M. McCracken 1976 *The Hopf Bifurcation and Its Applications*. New York, NY: Springer-Verlag.
14. R. Seydel 1994 *Practical Bifurcation and Stability Analysis: From Equilibrium to Chaos*. New York, NY: Springer-Verlag.
15. A. Gelb and W. E. Vander Velde 1968 *Multiple-Input Describing Functions and Nonlinear System Design*. New York, NY: McGraw-Hill.
16. M. Vidyasagar 1993 *Nonlinear Systems Analysis*. Englewood Cliffs, NJ: Prentice-Hall; second edition.
17. R. E. Fitts 1966 *IEEE Transactions on Automatic Control* **11**, 553–556. Two counterexamples to Aizerman's conjecture.
18. P. E. Rapp and A. I. Mees 1977 *International Journal of Control* **26**, 821–829. Spurious predictions of limit cycles in a non-linear feedback system by the describing function method.
19. S. M. Shahruz 2000 *Physica D* **142**, 291–305. Generation of self-pulsation in passively Q-switched lasers.
20. S. M. Shahruz 2000 *Advanced Semiconductor Lasers and Their Applications* (L. W. Hollberg and R. J. Lang, editors), *Trends in Optics and Photonics Series* (*TOPS*), Vol. 31, 203–213. Washington, DC: Optical Society of America. How to have narrow-stripe semiconductor lasers self-pulsate.
21. W. Ebeling and M. Peschel, editors 1985 *Lotka–Volterra-Approach to Cooperation and Competition in Dynamic Systems*. Berlin: Akademie-Verlag.
22. J. Hofbauer 1981 *Nonlinear Analysis, Theory, Methods & Applications* **5**, 1003–1007. On the occurrence of limit cycles in the Volterra–Lotka equation.
23. J. Hofbauer and K. Sigmund 1988 *The Theory of Evolution and Dynamical Systems: Mathematical Aspects of Selection*. Cambridge, U.K.: Cambridge University Press.
24. J. Hofbauer and K. Sigmund 1998 *Evolutionary Games and Population Dynamics*. Cambridge, U.K.: Cambridge University Press.
25. J. Hofbauer and J. W.-H. So 1994 *Applied Mathematics Letters* **7**, no. 6, 65–70. Multiple limit cycles for three dimensional Lotka–Volterra equations.
26. Y. S. Kolesov 2000 *Sbornik: Mathematics* **191**, 567–581. Non-classical relaxation cycle of three-dimensional system of Lotka–Volterra equations.
27. Y. Takeuchi 1996 *Global Dynamical Properties of Lotka–Volterra Systems*. Singapore: World Scientific.
28. D. Xiao and W. Li 2000 *Journal of Differential Equations* **164**, 1–15. Limit cycles for the competitive three dimensional Lotka–Volterra system.
29. M. L. Zeeman 1993 *Dynamics and Stability of Systems* **8**, 189–217. Hopf bifurcation in competitive three-dimensional Lotka–Volterra systems.
30. G. F. Franklin, J. D. Powell and A. Emami-Naeini 1994 *Feedback Control of Dynamic Systems*. Reading, MA: Addison-Wesley; third edition.
31. K. Ogata 1997 *Modern Control Engineering*. Upper Saddle River, NJ: Prentice-Hall; third edition.