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AN INPUT/OUTPUT-BASED PROCEDURE FOR FULLY EVALUATING AND MONITORING DYNAMIC PROPERTIES OF STRUCTURAL SYSTEMS VIA A SUBSPACE IDENTIFICATION METHOD

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Dynamic behaviour of complex structural systems may be modelled by a system of second order linear ordinary differential equations, i.e., $\mathbf{M}\ddot{\mathbf{w}}(t) + \mathbf{D}\dot{\mathbf{w}}(t) + \mathbf{S}\mathbf{w}(t) = \mathbf{f}(t)$, by means of either structural analysis for finite degree-of-freedom systems or discretization procedures (e.g., FE methods) for continuous systems. Here, $\mathbf{w}(t)$ and $\mathbf{f}(t)$ are the displacement vector and the force vector. Owing to erosion, friction, and internal damage and cracks, etc., a working process of a system always accompanies gradual degradation of the performance of this system: the stiffness of the system weakens, whereas the damping of the system strengthens. To evaluate such degradation, the usual way is to model the evolution of property of a system, obtain system property parameters, trace the history of motion and loading, carry out complicated analysis and computation under prescribed initial and boundary value conditions, and finally derive the degraded property and responses of the system. This traditional way, however, might be cumbersome and unsatisfactory in some cases due to the lack of adequate experimental data and well-founded theoretical basis, etc. Another way is to apply "inverse" methods, such as modal analysis methods with FFT and a subspace identification method, etc., developed in the theory of system identification, which extracts information about system properties directly from experimental input/output measurement data and hence do not involve the foregoing traditional analysis. The latter method, however, could not supply full information about system properties due to the assumption of the "black box" viewpoint. In this work, with suitable experimental input/output measurement data, a simple, effective procedure is described by which the stiffness matrix **S** and the damping matrix **D** may be determined in a complete, unique manner using a subspace identification method. The possibility of such a procedure arises from the observation of the self-evident fact: the conservation of mass of any part of a structural system implies that the mass matrix **M** of this system is constant and hence is given by its initial value. The stiffness and damping matrices **S** and **D** determined by the proposed procedure may be used to evaluate and monitor, in a full sense, the degradation of dynamic properties of structural systems. Further, with the information about the stiffness distribution of constituent elements of a structural system it is shown that it may be possible to estimate the locations of the damaged or faulty elements in this system. An example is given to illustrate the application of the proposed procedure.

1. INTRODUCTION AND MOTIVATION

By means of either structural analysis for finite-degree-of-freedom systems or discretization through spatial displacement interpolation to a finite number of variables (e.g., FE methods) for continuous systems, dynamic behaviour of complex structural systems, such as frame and truss structures, machine systems, plate and shell structures, etc., may be modelled by a system of second order linear ordinary differential equations (see e.g., references [1, 2]), i.e.,

$$\mathbf{M}\ddot{\mathbf{w}}(t) + \mathbf{D}\dot{\mathbf{w}}(t) + \mathbf{S}\mathbf{w}(t) = \mathbf{f}(t). \tag{1}$$

Here, $\mathbf{w}(t)$ and $\mathbf{f}(t)$ are the displacement vector and the force vector, and $\mathbf{M}$, $\mathbf{S}$ and $\mathbf{D}$ are, respectively, the mass, stiffness and damping matrices of the system under consideration. Let n be the number of degrees of freedom of the system. Then

$$\mathbf{w}(t), \qquad \mathbf{f}(t) \in R^{n\times 1}, \quad \mathbf{M}, \mathbf{S}, \mathbf{D} \in R^{n\times n}.$$

Here and henceforth, $R^{r\times s}$ is used to signify the set of all real matrices of r rows and s columns.

Owing to various complicated dissipative mechanisms, such as erosion, friction, and internal damage and cracks, etc., a working process of a system always accompanies gradual degradation of the performance of this system: the stiffness of the system weakens, whereas the damping of the system strengthens. Thus, after every service period, the stiffness and damping matrices $\mathbf{S}$ and $\mathbf{D}$ of a system will change and hence differ from their initial values $\mathbf{S}_0$ and $\mathbf{D}_0$. To evaluate and monitor the performance of a system during its course of service, it is important to have a practical and robust procedure to evaluate the current values of $\mathbf{S}$ and $\mathbf{D}$, etc.

As is done in damage mechanics and other relevant fields, the usual way is to model the evolution of the property of a system, obtain system property parameters, trace the history of motion and loading, carry out complicated analysis and computation under prescribed initial and boundary value conditions, and finally derive the degraded property and responses of the system of interest. This traditional way, however, might be cumbersome and unsatisfactory in some cases due to the lack of adequate experimental data and well-founded theoretical basis.

Another way is to use "inverse" methods developed in the theory of system identification, which extracts information about system properties from experimental input/output measurement data and hence do not involve the foregoing traditional analysis (see e.g., references [3–6]). This way is based on the following procedures.

(i) Arrange m actuator(s) at some location(s) in the structural system to generate excitation(s) to the system. In doing so, the force vector $\mathbf{f}(t)$ in equation (1) will be replaced by an *input* $\mathbf{G}\mathbf{u}(t)$, i.e.,

$$\mathbf{f}(t) = \mathbf{G}\mathbf{u}(t), \tag{2}$$

where $\mathbf{u}(t) \in R^{m\times 1}$ is an *m-input force vector*, and $\mathbf{G} \in R^{n\times m}$ the corresponding *input location influence matrix*.

(ii) At the same time, arrange l sensors at some locations in the structural system to measure the output or the response of the system under the foregoing excitation or input generated by the arranged actuator(s). The output measured by the arranged sensors is expressible in the form

$$\mathbf{y}(t) = \mathbf{C}_d\mathbf{w}(t) + \mathbf{C}_v\dot{\mathbf{w}}(t) + \mathbf{C}_a\ddot{\mathbf{w}}(t) + \bar{\mathbf{D}}'\mathbf{u}(t). \tag{3}$$

In the above equation, $\mathbf{y}(t) \in R^{l\times 1}$ is an *l-sensor output vector*; $\mathbf{C}_d \in R^{l\times n}$, $\mathbf{C}_v \in R^{l\times n}$ and $\mathbf{C}_a \in R^{l\times n}$ are the corresponding *output displacement*, *velocity and acceleration location* *influence matrices*, respectively; and $\bar{\mathbf{D}}' \in R^{l\times m}$ is the *direct transmission matrix* corresponding to direct input/output feedthrough. If the sensors supply displacement (velocity, acceleration) data, there will be *displacement* (resp. *velocity*, *acceleration*) *sensing*. In these three cases, two of the matrices $\mathbf{C}_v$, $\mathbf{C}_d$ and $\mathbf{C}_a$ vanish respectively.

(iii) Find the triplet $\{\mathbf{M}, \mathbf{S}, \mathbf{D}\}$ such that the dynamic system modelled by equation (1) exactly supplies the output data (3) measured with the input data (2).

The above procedures constitute a problem of identifying dynamic characteristics of a second order structural system. Traditionally, methods called experimental modal analysis, which is based on FFT (fast Fourier transformation), are used to determine mode and mode shapes in the time-domain and frequency-domain (see e.g., references [7–10]). In spite of their many advantages and successes, limitations and deficiencies of these traditional modal analysis methods in some cases have now been recognized, such as difficulties due to systematic errors, expensive high resolution, etc. An alternative, simple, and effective method, called the subspace identification method for identification of state space models, was introduced in 1966 by Ho and Kalman [11], which results in a minimal order realization of a general form of state space model (cf. equations (4) and (5) below) with a high degree of accuracy. A comprehensive account of this method can be found in, e.g., reference [12]. In recent years, this method has been successfully and fruitfully used and developed to deal with structural system identification, see e.g., references [13–23], as well as the related references therein.

However, the structural model equations identified either by the modal analyses based on FFT or by the subspace identification method for state space models or by other known identification techniques, are not really the second order dynamic differential equations (1). The former, while useful for some purposes of interest, are a form of equation (1) under an unknown co-ordinate transformation and therefore could not supply the complete information about the stiffness and damping matrices **S** and **D**, etc. Generally, it is difficult to transform, in a complete and unique sense, the former into the second order structural dynamic equations (1).

The major concern is in a full sense the evaluation of the system property matrices **S** and **D** at any stage, which may be degraded by internal dissipative mechanisms in the structural system in the course of its service. To identify the second order structural dynamic model (1) in a full sense, some information about the structural system should be provided. Moreover, the input/output measurement should be arranged in an appropriate manner such that the obtained input/output data facilitate the present purpose. Of the two respects, the latter is adjustable or controllable, while the former implies that the system is no longer regarded to be fully "black". Although the viewpoint of "black boxes" in a system theory has universal applicability for all systems, it could not lead to the identification in a complete sense within a given class of systems. The main reason is that, within a class of systems, many different systems may be chosen to match the given input/output data, as will be seen in the next section.

Indeed, for a specific class of systems, certain characteristics that are common to this class may be known. For a structural system, whether discrete or continuous in space variables, the construction of its property matrices **M**, **S** and **D** in the dynamic equations (1) may be derived from structural analysis of the system or from FE discretization procedure of the system, etc. For structural systems, a simple yet crucial observation is that the mass of every part of a mechanical system is conservative in the course of any dynamic process experienced by this system.† This commonly known principle of conservation of mass for

†Of course, collapses and failures of structural systems are not included, which result in the loss of structural integrity.

mechanical systems implies[‡] that *the mass matrix of a system in model* (1) *is constant and hence given by its initial value* $\mathbf{M}_0$. Accordingly, of the three structural property matrices **M**, **S** and **D**, the former remains unchanged and only the latter two may change due to the emergence of damage and cracks etc.

With the above observation and suitable input–output measurement data, it shall be shown that the structural dynamic model (1) may be identified in a full sense by utilizing Ho–Kalman subspace identification method. This makes it possible to establish an effective, practical procedure for evaluating and monitoring the degraded dynamic properties of a structural system at any stage. Moreover, with information about the construction of the stiffness matrix **S**, namely, information about the stiffness distribution of constituent elements of a structural system, it is possible to estimate the locations of the damaged or faulty elements, if any, in the structural system.

The main context of this paper is arranged as follows. In section 2, a recapitulation of subspace identification method is given for future use. In section 3, with the principle of conservation of mass a procedure is proposed by which the stiffness and damping matrices **S** and **D**, etc., may be determined from a complete set of input–output data. For purposes of practical applications, usually it is not easy to measure a complete set of output data for a model system (1) with a large number of degrees of freedom. To resolve this issue, section 4 shows how to utilize the proposed procedure to achieve the goal in the case of incomplete output data. Section 5, with the identified stiffness matrix **S** and information about the stiffness distribution of constituent elements of a structural system, shows how to estimate the locations of the damaged or faulty elements, if any, inside this system. An example is given to illustrate the application of the procedure. Finally, some further relevant aspects are pointed out in section 6.

Throughout this article, upper-case boldface letters are used to designate matrices. In particular, boldface lower-case letters are used to represent column matrices or vectors. In addition, $\mathbf{I}_k$ is used to denote the $k \times k$ unit matrix and $\mathbf{0}$ zero matrices.

2. SUBSPACE IDENTIFICATION METHOD FOR STATE SPACE MODELS

Consider linear multi-input multi-output systems modelled by state space descriptions in the form

$$\dot{\mathbf{x}}(t) = \bar{\mathbf{A}}\mathbf{x}(t) + \bar{\mathbf{B}}\mathbf{u}(t), \tag{4}$$

$$\mathbf{y}(t) = \bar{\mathbf{C}}\mathbf{x}(t) + \bar{\mathbf{D}}\mathbf{u}(t). \tag{5}$$

In accordance with the terms used in system theory, $\mathbf{x}(t) \in R^{2n\times 1}$ is a $2n$-dimensional state vector; $\mathbf{u}(t) \in R^{m\times 1}$ is an m-dimensional input vector; $\mathbf{y}(t) \in R^{l\times 1}$ is an l-dimensional output vector; $\bar{\mathbf{A}} \in R^{2n\times 2n}$ is a $2n \times 2n$ system matrix; $\bar{\mathbf{B}} \in R^{2n\times m}$ is a $2n \times m$ control matrix; $\bar{\mathbf{C}} \in R^{l\times 2n}$ is an $l \times 2n$ observer matrix; and $\bar{\mathbf{D}} \in R^{l\times m}$ is an $l \times m$ direct transmission matrix.

Equations (4) and (5) are known as a *state space model* with multi-inputs and multi-outputs. Linear and quasi-linear ordinary differential equations of any given order with input/output, including the second order differential equations (1) with equations (2) and (3), may equivalently be expressed in a form of state-space model by virtue of a standard procedure.

[‡]A detailed account will be given in section 3.

In theory of system identification and realization, a linear dynamic system is regarded to be a "black box", whose internal properties are totally unknown and hence in a fully "black" state. This means that there is total ignorance of any information about the system matrix $\bar{\mathbf{A}}$, including its dimension. As a result, to find out the internal properties of a "black" system, excitations can be input to the system and then the output or response of the system measured. Then, from the available input/output measurement data an attempt to extract information about the internal property of the system can be made. Mathematically, given the experimental input/output data, an attempt can be made to find a state space model (4) and (5) of a minimal dimension which exactly matches the measured input/output relation.

The latter is just one of the central problems in theory of identification of linear systems. Many methods and techniques have been proposed and developed to deal with this problem. Among them, a simple, effective method, called the subspace identification method and introduced by Ho and Kalman [11] (see, e.g., reference [12] for detail), has attracted much attention and has been widely used in the recent years. Since this method is fundamental to the attainment of the goal of this paper some notions and results of this method are outlined as follows.

Integrating equation (4) over $(0, t)$ produces the state vector

$$\mathbf{x}(t) = \mathrm{e}^{\bar{\mathbf{A}}t}\mathbf{x}(0) + \int_0^t \mathrm{e}^{\bar{\mathbf{A}}(t-\tau)}\bar{\mathbf{B}}\mathbf{u}(\tau)\,\mathrm{d}\tau, \quad t \geqslant 0. \tag{6}$$

Then, equation (5) yields the output vector

$$\mathbf{y}(t) = \bar{\mathbf{C}}\mathrm{e}^{\bar{\mathbf{A}}t}\mathbf{x}(0) + \bar{\mathbf{D}}\mathbf{u}(t) + \int_0^t \bar{\mathbf{C}}\mathrm{e}^{\bar{\mathbf{A}}(t-\tau)}\bar{\mathbf{B}}\mathbf{u}(\tau)\,\mathrm{d}\tau, \quad t \geqslant 0. \tag{7}$$

Now, consider the discrete-time series

$$t_i = i\Delta t, \quad i = 0, 1, 2, 3, \ldots, \tag{8}$$

where Δt is a constant time increment. Let

$$\mathbf{A} = \mathrm{e}^{\bar{\mathbf{A}}\Delta t}, \qquad \mathbf{B} = \left(\int_0^{\Delta t} \mathrm{e}^{-\bar{\mathbf{A}}\tau}\,\mathrm{d}\tau\right)\bar{\mathbf{B}}. \tag{9}$$

Observing that the value of the input $\mathbf{u}(\tau)$ over each small interval $\tau \in (t_i, t_{i+1})$ may be regarded to be constant and hence given by $\mathbf{u}(t_i)$ gives

$$\mathbf{y}(t_k) = \bar{\mathbf{C}}\mathrm{e}^{\bar{\mathbf{A}}t_k}\mathbf{x}(0) + \bar{\mathbf{D}}\mathbf{u}(t_k) + \sum_{i=0}^{k-1} \mathbf{Y}(k-i)\mathbf{u}(t_i), \tag{10}$$

where

$$\mathbf{Y}(k-i) = \bar{\mathbf{C}}\mathbf{A}^{(k-i)}\mathbf{B} \in R^{l \times m}. \tag{11}$$

Compared with the closed-form solution (7), expression (10) is accurate up to a small quantity of at least the same order of magnitude as the time increment Δt. In particular, it is accurate for an impulse input. In fact, it may be shown that expression (10) with (9) supplies the solution of the output vector $\mathbf{u}(t_k)$ governed by the discrete state space model

$$\mathbf{x}(t_{k+1}) = \mathbf{A}\mathbf{x}(t_k) + \mathbf{B}\mathbf{u}(t_k), \tag{12}$$

$$\mathbf{y}(t_k) = \bar{\mathbf{C}}\mathbf{x}(t_k) + \bar{\mathbf{D}}\mathbf{u}(t_k). \tag{13}$$

The above-mentioned equations are just a discretized state space model of the continuous model (4) and (5) with reference to the discrete-time series $\{t_k\}$ given by equation (8).

Often, $\{\mathbf{Y}(s)\}$ are called *Markov parameters* of the system (12) and (13). Let the input $\mathbf{u}(t)$ be a Dirac impulse function and let the system be initially relaxed, i.e., $\lim_{t\to 0} - \mathbf{x}(t) = \mathbf{0}$. Then, the output $\mathbf{y}(t_k)$ is exactly $\mathbf{Y}(k)$, and, moreover,

$$\bar{\mathbf{D}} = \mathbf{Y}(0). \tag{14}$$

It turns out (see reference [24]) that, if the system is initially relaxed, the Markov parameters $\{\mathbf{Y}(s)\}$ of the system are just the impulse response of the system corresponding to the discrete-time series (8).

Thus, the system realization problem stated before may be reformulated as follows: given the impulse response functions of the system, i.e., a set of the Markov parameters, $\{\mathbf{Y}(s)\}$, of the system, find a triplet $\{\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}\}$, called a realization of the state space model (4)–(5), such that

$$\mathbf{Y}(s) = \bar{\mathbf{C}}\mathbf{A}^s\mathbf{B}, \quad s = 0, 1, 2, 3, \ldots \tag{15}$$

with equation (9). Evidently, $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ are obtainable from (9), whenever $\mathbf{A}$ and $\mathbf{B}$ are available.

The subspace identification method provides a systematic approach to model order determination for a given accuracy, and the derivation of the discrete state space model. Now a standard algorithm based on this method is available, known as *Eigensystem Realization Algorithm* (*ERA*) (see e.g., references [13–15, 18, 17]. The key procedure of this technique is in using the discrete-time shift of the Markov parameters to form the following *Hankel matrix*:

$$\mathbf{H}_{pq}(s) = \begin{bmatrix} \mathbf{Y}(s+1) & \mathbf{Y}(s+2) & \cdots & \mathbf{Y}(s+q) \\ \mathbf{Y}(s+2) & \mathbf{Y}(s+3) & \cdots & \mathbf{Y}(s+q+1) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \mathbf{Y}(s+p) & \mathbf{Y}(s+p+1) & \cdots & \mathbf{Y}(s+p+q-1) \end{bmatrix} \tag{16}$$

Then, computing a singular value decomposition of $\mathbf{H}_{pq}(0)$ and truncating the series following the $2n$ largest singular values gives

$$\mathbf{H}_{pq}(0) \approx \mathbf{P}\boldsymbol{\Sigma}\mathbf{Q}^{\mathrm{T}}, \tag{17}$$

where $\boldsymbol{\Sigma} \in R^{2n\times 2n}$ is the diagonal matrix whose diagonal elements are the first $2n$ singular values of $\mathbf{H}_{pq}(0)$ in the descending order; and $\mathbf{P} \in R^{pl\times 2n}$ and $\mathbf{Q} \in R^{qm\times 2n}$.

Thus, a discrete-time realization of the continuous state space model (4) and (5) is given by

$$\mathbf{A} = \sqrt{\boldsymbol{\Sigma}^{-1}}\mathbf{P}^{\mathrm{T}}\mathbf{H}_{pq}(1)\mathbf{Q}\sqrt{\boldsymbol{\Sigma}^{-1}}, \tag{18}$$

$$\mathbf{B} = \sqrt{\boldsymbol{\Sigma}}\mathbf{Q}^{\mathrm{T}}\begin{bmatrix}\mathbf{I}_m\\ \mathbf{0}\end{bmatrix}, \tag{19}$$

$$\bar{\mathbf{C}} = [\mathbf{I}_l\ \mathbf{0}]\mathbf{P}\sqrt{\boldsymbol{\Sigma}}, \tag{20}$$

$$\bar{\mathbf{D}} = \mathbf{Y}(0). \tag{21}$$

In the above-mentioned equations, the first $\mathbf{0}$ is the $(qm - m) \times m$ zero matrix, while the second $\mathbf{0}$ is the $l \times (pl - l)$ zero matrix. Since these facts are easily derived from the matrix multiplications involved here and in similar cases later on, there is no distinction between such zero matrices and they are simply denoted by the same symbol $\mathbf{0}$.

However, the realization given by equations (18)–(21) is not unique. It is merely one among an infinite number of equivalent realizations for the given data. In fact, for any given non-singular matrix $\mathbf{R} \in R^{2n\times 2n}$, the triplet $\{\mathbf{R}^{-1}\mathbf{A}\mathbf{R}, \mathbf{R}^{-1}\mathbf{B}, \bar{\mathbf{C}}\mathbf{R}\}$ is also an *equivalent realization*. This implies that from the input–output data, the ERA based on the Ho–Kalman subspace identification method can identify a system up to only within a non-singular co-ordinate transformation and hence cannot extract full information about the system matrix $\bar{\mathbf{A}}$. Further developments will be considered in the succeeding sections.

3. DETERMINATION OF STIFFNESS AND DAMPING MATRICES

Now the second order linear dynamic model (1) for structural systems is of concern. Introducing the state vector $\mathbf{x}(t)$ and the phase velocity vector $\dot{\mathbf{x}}(t)$ by

$$\mathbf{x}(t) = \begin{bmatrix}\mathbf{w}(t)\\ \dot{\mathbf{w}}(t)\end{bmatrix}, \qquad \dot{\mathbf{x}}(t) = \begin{bmatrix}\dot{\mathbf{w}}(t)\\ \ddot{\mathbf{w}}(t)\end{bmatrix}, \tag{22}$$

Equation (1) is transformed with equations (2) and (3) to the equivalent state space form (4) and (5), with

$$\bar{\mathbf{A}} = \begin{bmatrix}\mathbf{0} & \mathbf{I}_n\\ -\mathbf{M}^{-1}\mathbf{S} & -\mathbf{M}^{-1}\mathbf{D}\end{bmatrix}, \tag{23}$$

$$\bar{\mathbf{B}} = \begin{bmatrix}\mathbf{0}\\ \mathbf{M}^{-1}\mathbf{G}\end{bmatrix}, \tag{24}$$

$$\bar{\mathbf{C}} = [\mathbf{C}_d \quad \mathbf{C}_v] - \mathbf{C}_a\mathbf{M}^{-1}[\mathbf{S} \quad \mathbf{D}]. \tag{25}$$

Let the actuator arranged in the structural system input an impulse excitation to the structural system. Then the output data recorded by the sensors arranged in the structural system supply the Markov parameters $\{\mathbf{Y}(s)\}$ of the system with reference to a discrete-time series $\{t_s\}$ given by equation (8). Thus, the ERA based upon Ho–Kalman subspace identification method furnishes a realization of the structural system through equations (16)–(21).

As has been indicated at the end of the last section, such a realization is not unique and hence could not determine the system matrix $\bar{\mathbf{A}}$ in a full sense. In fact, only the eigenvalues of $\bar{\mathbf{A}}$ and mode shapes are identified through the ERA. Although the former contain important information about the dynamic properties of the structural system, comprehensive information about the stiffness and damping matrices could not be determined, which is crucial to a complete understanding of the system properties. It seems that this issue is inextricably related with the black box viewpoint. With the assumption of the latter, any further information about the internal structure of the system is not used, which may be necessary for a proper identification of the system property.

As far as all systems in a general sense are concerned, it seems to be meaningless to prescribe in advance any information about internal structures of the systems, since generality means being unconditional. For some specific class of systems, however, useful information could be known. That is the case for structural systems modelled by equation (1). As has been mentioned earlier, the principle of conservation of mass implies that during every dynamic process the mass matrix $\mathbf{M}$ of a structural system is constant and hence is given by its initial value $\mathbf{M}_0$. In fact, a discrete finite-degree-of-freedom structural system is formed by certain constituent elements. On the other hand, in a sense of approximation, a continuous structural system may be modelled by a discrete system which is also formed by certain constituent elements, viz., finite elements, via an FE discretization programme (see e.g., references [1, 2]). For any given discrete structural system in either of the latter two senses, the mass matrix of each constituent element, denoted by $\mathbf{M}^{(e)}$, is determined by the mass of this element as well as certain characteristic length(s) (e.g., the length of a bar or a rod or a beam, the diameter of a disc or a sphere, the length and width of a plate, etc.) of this element, the latter being associated with the mass moment of inertia of this element and choices of generalized co-ordinates, etc. (see e.g., references [10, 25]). Within the scope of the applicability of the model equation (1), i.e., linearization and small deformation, the changes of the characteristic lengths of a constituent element are negligible and they may be treated as being constant. Moreover, the mass matrix of the model equation (1) is derived by assembling the mass matrices of all the constituent elements, i.e., by the operation of adding the coefficients of the element mass matrices into the proper locations in the aggregate mass matrix $\mathbf{M}$. Let each element mass matrix $\mathbf{M}^{(e)}$ be expressed in terms of the aggregate nodal displacement vector. Then

$$\mathbf{M} = \sum_e \mathbf{M}^{(e)}.$$

Here, $\sum_e$ means the summation for the constituent elements. Consequently, from the above facts and the conservation of mass it is deduced that the foregoing fact concerning the mass matrix $\mathbf{M}$ of equation (1) is true.

With the above-mentioned self-evident fact and suitable output data, in what follows, it will be shown that the stiffness and damping matrices may be uniquely determined.

First, transform the structural dynamic system (1) with input (2) and output (3) to the multi-input multi-output state space model (4) and (5), where the matrices $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$ and $\bar{\mathbf{C}}$ are of the forms given by equations (23)–(25). Then, given the Markov parameters (15) with (9), a realization $\bar{\mathbf{A}}' \in R^{2n\times 2n}$, $\bar{\mathbf{B}}' \in R^{2n\times m}$ and $\bar{\mathbf{C}}' \in R^{l\times 2n}$ can be determined by means of the ERA based on Ho–Kalman subspace identification method as shown above.

Since the foregoing realization merely provides a form of the state space model (4) and (5) relative to an unknown transformed co-ordinate system, the structures of the matrices $\bar{\mathbf{A}}'$

and $\bar{\mathbf{B}}'$ and $\bar{\mathbf{C}}'$, unless by chance, cannot be the same as those of their respective counterparts given by equations (23)–(25). The main idea of the subsequent development is the attempt to find a non-singular transformation $\mathbf{R} \in R^{2n\times 2n}$ such that the transformed matrices $\mathbf{R}\bar{\mathbf{A}}\mathbf{R}^{-1}$ and $\mathbf{R}\bar{\mathbf{B}}$ and $\bar{\mathbf{C}}\mathbf{R}^{-1}$ are exactly those given by equations (23)–(25), i.e.,

$$\mathbf{R}\bar{\mathbf{A}}'\mathbf{R}^{-1} = \bar{\mathbf{A}} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_n \\ -\mathbf{M}^{-1}\mathbf{S} & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix}, \tag{26}$$

$$\mathbf{R}\bar{\mathbf{B}}' = \bar{\mathbf{B}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{G} \end{bmatrix}, \tag{27}$$

$$\bar{\mathbf{C}}'\mathbf{R}^{-1} = \bar{\mathbf{C}} = [\mathbf{C}_d \quad \mathbf{C}_v] - \mathbf{C}_a\mathbf{M}^{-1}[\mathbf{S} \quad \mathbf{D}]. \tag{28}$$

Let

$$\bar{\mathbf{A}}' = \begin{bmatrix} \bar{\mathbf{A}}_1' & \bar{\mathbf{A}}_3' \\ \bar{\mathbf{A}}_2' & \bar{\mathbf{A}}_4' \end{bmatrix}, \qquad \bar{\mathbf{B}}' = \begin{bmatrix} \bar{\mathbf{B}}_1' \\ \bar{\mathbf{B}}_2' \end{bmatrix}, \qquad \bar{\mathbf{C}}' = [\bar{\mathbf{C}}_1' \quad \bar{\mathbf{C}}_2'] \tag{29}$$

and

$$\mathbf{R} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_3 \\ \mathbf{X}_2 & \mathbf{X}_4 \end{bmatrix}. \tag{30}$$

In the above, the block matrices are of the properties $\bar{\mathbf{A}}_i', \mathbf{X}_i \in R^{n\times n}$, $\bar{\mathbf{B}}_j' \in R^{n\times m}$ and $\bar{\mathbf{C}}_j' \in R^{l\times n}$, with $i = 1, 2, 3, 4$ and $j = 1, 2$.

Then, using the above partitioned forms and the fact that

$$\mathbf{M} = \mathbf{M}_0, \tag{31}$$

and noting that equations (26) and (28) may be recast in the forms

$$\mathbf{R}\bar{\mathbf{A}}' = \bar{\mathbf{A}}\mathbf{R}, \qquad \bar{\mathbf{C}}' = ([\mathbf{C}_d \quad \mathbf{C}_v] - \mathbf{C}_a\mathbf{M}_0^{-1}[\mathbf{S} \quad \mathbf{D}])\mathbf{R},$$

the following equations are derived from equations (26)–(30):

$$\left.\begin{aligned} \mathbf{X}_2 &= \mathbf{X}_1\bar{\mathbf{A}}_1' + \mathbf{X}_3\bar{\mathbf{A}}_2', \\ \mathbf{X}_4 &= \mathbf{X}_1\bar{\mathbf{A}}_3' + \mathbf{X}_3\bar{\mathbf{A}}_4', \end{aligned}\right\} \quad \text{i.e. } [\mathbf{X}_2 \quad \mathbf{X}_4] = [\mathbf{X}_1 \quad \mathbf{X}_3]\bar{\mathbf{A}}', \tag{32}$$

$$\left.\begin{aligned} \mathbf{X}_2\bar{\mathbf{A}}_1' + \mathbf{X}_4\bar{\mathbf{A}}_2' &= -\mathbf{M}_0^{-1}\mathbf{S}\mathbf{X}_1 - \mathbf{M}_0^{-1}\mathbf{D}\mathbf{X}_2, \\ \mathbf{X}_2\bar{\mathbf{A}}_3' + \mathbf{X}_4\bar{\mathbf{A}}_4' &= -\mathbf{M}_0^{-1}\mathbf{S}\mathbf{X}_3 - \mathbf{M}_0^{-1}\mathbf{D}\mathbf{X}_4, \end{aligned}\right\} \quad \text{i.e. } [\mathbf{X}_2 \quad \mathbf{X}_4] = -\mathbf{M}_0^{-1}[\mathbf{S} \quad \mathbf{D}]\mathbf{R}, \tag{33}$$

$$\left.\begin{aligned} \mathbf{X}_1\bar{\mathbf{B}}_1' + \mathbf{X}_3\bar{\mathbf{B}}_2' &= \mathbf{0}, \\ \mathbf{X}_2\bar{\mathbf{B}}_1' + \mathbf{X}_4\bar{\mathbf{B}}_2' &= \mathbf{M}_0^{-1}\mathbf{G}. \end{aligned}\right\} \quad \text{i.e.} \quad \left\{\begin{aligned} &[\mathbf{X}_1 \quad \mathbf{X}_3]\bar{\mathbf{B}}' = \mathbf{0}, \\ &[\mathbf{X}_2 \quad \mathbf{X}_4]\bar{\mathbf{B}}' = \mathbf{M}_0^{-1}\mathbf{G}, \end{aligned}\right. \tag{34}$$

$$\begin{aligned} (\mathbf{C}_d - \mathbf{C}_a\mathbf{M}_0^{-1}\mathbf{S})\mathbf{X}_1 + (\mathbf{C}_v - \mathbf{C}_a\mathbf{M}_0^{-1}\mathbf{D})\mathbf{X}_2 &= \bar{\mathbf{C}}_1', \\ (\mathbf{C}_d - \mathbf{C}_a\mathbf{M}_0^{-1}\mathbf{S})\mathbf{X}_3 + (\mathbf{C}_v - \mathbf{C}_a\mathbf{M}_0^{-1}\mathbf{D})\mathbf{X}_4 &= \bar{\mathbf{C}}_2'. \end{aligned} \tag{35}$$

The main result is as follows.

Theorem 1. *Let the output displacement location influence matrix* $\mathbf{C}_d$ *for displacement sensing or the output velocity location influence matrix* $\mathbf{C}_v$ *for velocity sensing or the output acceleration location influence matrix* $\mathbf{C}_a$ *for acceleration sensing be a non-singular* $n \times n$ *matrix, and let the triplet* $\{\bar{\mathbf{A}}', \bar{\mathbf{B}}', \bar{\mathbf{C}}'\}$ *be a realization of the structural dynamic system* (4)–(5) *with equations* (23)–(25). *Then the stiffness, damping and input location influence matrices of this system may be completely and uniquely determined as follows*:

$$\mathbf{R} = \begin{bmatrix} \mathbf{C}_d^{-1}\bar{\mathbf{C}}' \\ \mathbf{C}_d^{-1}\bar{\mathbf{C}}'\bar{\mathbf{A}}' \end{bmatrix}, \tag{36}$$

$$[\mathbf{S} \quad \mathbf{D}] = -\mathbf{M}_0\mathbf{C}_d^{-1}\bar{\mathbf{C}}'\bar{\mathbf{A}}'^2\mathbf{R}^{-1}, \tag{37}$$

$$\mathbf{G} = \mathbf{M}_0\mathbf{C}_d^{-1}\bar{\mathbf{C}}'\bar{\mathbf{A}}'\bar{\mathbf{B}}' \tag{38}$$

for displacement sensing,

$$\mathbf{R} = \begin{bmatrix} \mathbf{C}_v^{-1}\bar{\mathbf{C}}'\bar{\mathbf{A}}'^{-1} \\ \mathbf{C}_v^{-1}\bar{\mathbf{C}}' \end{bmatrix}, \tag{39}$$

$$[\mathbf{S} \quad \mathbf{D}] = -\mathbf{M}_0\mathbf{C}_v^{-1}\bar{\mathbf{C}}'\bar{\mathbf{A}}'\mathbf{R}^{-1}, \tag{40}$$

$$\mathbf{G} = \mathbf{M}_0\mathbf{C}_v^{-1}\bar{\mathbf{C}}'\bar{\mathbf{B}}' \tag{41}$$

for velocity sensing, *and*

$$\mathbf{R} = \begin{bmatrix} \mathbf{C}_a^{-1}\bar{\mathbf{C}}'\bar{\mathbf{A}}'^{-2} \\ \mathbf{C}_a^{-1}\bar{\mathbf{C}}'\bar{\mathbf{A}}'^{-1} \end{bmatrix}, \tag{42}$$

$$[\mathbf{S} \quad \mathbf{D}] = -\mathbf{M}_0\mathbf{C}_a^{-1}\bar{\mathbf{C}}'\mathbf{R}^{-1}, \tag{43}$$

$$\mathbf{G} = \mathbf{M}_0\mathbf{C}_a^{-1}\bar{\mathbf{C}}'\bar{\mathbf{A}}'^{-1}\bar{\mathbf{B}}' \tag{44}$$

for acceleration sensing.

Proof. First, for displacement sensing

$$\mathbf{C}_v = \mathbf{C}_a = \mathbf{0}.$$

Then, equation (35) gives

$$\mathbf{C}_d\mathbf{X}_1 = \bar{\mathbf{C}}'_1, \qquad \mathbf{C}_d\mathbf{X}_3 = \bar{\mathbf{C}}'_2,$$

The latter and $(29)_3$ lead to

$$\mathbf{C}_d[\mathbf{X}_1 \quad \mathbf{X}_3] = \bar{\mathbf{C}}'. \tag{45}$$

Hence, if $\mathbf{C}_d$ is a non-singular $n \times n$ matrix, then from equation (45)

$$[\mathbf{X}_1 \quad \mathbf{X}_3] = \mathbf{C}_d^{-1}\bar{\mathbf{C}}'. \tag{46}$$

With (32)

$$[\mathbf{X}_2 \quad \mathbf{X}_4] = \mathbf{C}_d^{-1}\bar{\mathbf{C}}'\bar{\mathbf{A}}'. \tag{47}$$

Thus, equations (46) and (47) yield equation (36). From equations (33), $(29)_1$ and (30) it can be inferred that

$$[\mathbf{S} \quad \mathbf{D}] = -\mathbf{M}_0[\mathbf{X}_2 \quad \mathbf{X}_4]\bar{\mathbf{A}}'\mathbf{R}^{-1}. \tag{48}$$

Hence, equations (47) and (48) gives equation (37). Finally, equation (38) follows from equations $(34)_2$ and (47).

Second, for velocity sensing, $\mathbf{C}_d = \mathbf{C}_a = \mathbf{0}$. Then, equations (35) and $(29)_3$ result in

$$\mathbf{C}_v[\mathbf{X}_2 \quad \mathbf{X}_4] = \bar{\mathbf{C}}'. \tag{49}$$

Hence, if $\mathbf{C}_v$ is a non-singular $n \times n$ matrix, then from equation (49)

$$[\mathbf{X}_2 \quad \mathbf{X}_4] = \mathbf{C}_v^{-1}\bar{\mathbf{C}}'. \tag{50}$$

Equations (32) and (50) give

$$[\mathbf{X}_1 \quad \mathbf{X}_3] = \mathbf{C}_v^{-1}\bar{\mathbf{C}}'\bar{\mathbf{A}}'^{-1}. \tag{51}$$

Thus, equations (50) and (51) yield equation (39).

Third, for acceleration sensing, $\mathbf{C}_d = \mathbf{C}_v = \mathbf{0}$. Equations (35), $(29)_2$ and (30) give

$$\mathbf{C}_a\mathbf{M}_0^{-1}[\mathbf{S} \quad \mathbf{D}]\mathbf{R} = -\bar{\mathbf{C}}'. \tag{52}$$

Hence, if $\mathbf{C}_a$ is a non-singular $n \times n$ matrix, then from the latter equation (43) is deduced. Then, equation (33) gives

$$[\mathbf{X}_2 \quad \mathbf{X}_4] = \mathbf{C}_a^{-1}\bar{\mathbf{C}}'\bar{\mathbf{A}}'^{-1}, \tag{53}$$

and from equation (32)

$$[\mathbf{X}_1 \quad \mathbf{X}_3] = \mathbf{C}_a^{-1}\bar{\mathbf{C}}'\bar{\mathbf{A}}'^{-2}. \tag{54}$$

Thus, equations (53)–(54) yield equation (42). Finally, equation (44) may be derived from equations $(34)_2$ and (53).

From the above mentioned it is concluded that the expressions (36)–(38), (39)–(41) and (42)–(44) hold, respectively, for displacement, velocity and acceleration sensing. However, equation $(34)_1$ has not been used in the above process. That the three solutions given are, separately, consistent with this equation must be verified.

In fact, from equations (46), (32), (54) and $(34)_1$ the consistency conditions

$$\bar{\mathbf{C}}'\bar{\mathbf{B}}' = \mathbf{0} \tag{55}$$

are derived for displacement sensing,

$$\bar{\mathbf{C}}'\bar{\mathbf{A}}^{-1}\bar{\mathbf{B}}' = \mathbf{0} \tag{56}$$

for velocity sensing, and

$$\bar{\mathbf{C}}'\bar{\mathbf{A}}'^{-2}\bar{\mathbf{B}}' = \mathbf{0} \tag{57}$$

for acceleration sensing. The respective first equalities of equations (26)–(28) yield

$$\bar{\mathbf{A}}' = \mathbf{R}^{-1}\bar{\mathbf{A}}\mathbf{R}, \qquad \bar{\mathbf{B}}' = \mathbf{R}^{-1}\bar{\mathbf{B}}, \qquad \bar{\mathbf{C}}' = \bar{\mathbf{C}}\mathbf{R}.$$

Accordingly,

$$\bar{\mathbf{C}}'\bar{\mathbf{B}}' = (\bar{\mathbf{C}}\mathbf{R})(\mathbf{R}^{-1}\bar{\mathbf{B}}) = \bar{\mathbf{C}}\bar{\mathbf{B}}$$

for displacement sensing, and

$$\bar{\mathbf{C}}'\bar{\mathbf{A}}'^{-1}\bar{\mathbf{B}}' = (\bar{\mathbf{C}}\mathbf{R})(\mathbf{R}^{-1}\bar{\mathbf{A}}^{-1}\mathbf{R})(\mathbf{R}^{-1}\bar{\mathbf{B}}) = \bar{\mathbf{C}}\bar{\mathbf{A}}^{-1}\bar{\mathbf{B}}$$

for velocity sensing, and

$$\bar{\mathbf{C}}'\bar{\mathbf{A}}'^{-2}\bar{\mathbf{B}}' = (\bar{\mathbf{C}}\mathbf{R})(\mathbf{R}^{-1}\bar{\mathbf{A}}^{-2}\mathbf{R})(\mathbf{R}^{-1}\bar{\mathbf{B}}) = \bar{\mathbf{C}}\bar{\mathbf{A}}^{-2}\bar{\mathbf{B}}$$

for acceleration sensing.

From the last three expressions and the respective second equalities of equations (26)–(28), as well as the facts that

$$\bar{\mathbf{C}} = [\mathbf{C}_d \quad \mathbf{0}]$$

for displacement sensing, and

$$\bar{\mathbf{C}} = [\mathbf{0} \quad \mathbf{C}_v]$$

for velocity sensing, and

$$\bar{\mathbf{C}} = -\mathbf{C}_a\mathbf{M}_0^{-1}[\mathbf{S} \quad \mathbf{D}]$$

for acceleration sensing, it is inferred that the consistency conditions (55)–(57) may be satisfied for their respective cases of sensing. Indeed,

$$\bar{\mathbf{C}}'\bar{\mathbf{B}}' = \bar{\mathbf{C}}\bar{\mathbf{B}} = [\mathbf{C}_d \quad \mathbf{0}]\begin{bmatrix}\mathbf{0}\\ \mathbf{G}\end{bmatrix} = \mathbf{0}$$

for displacement sensing. Hence equation (55) is satisfied. Next, for velocity sensing,

$$\bar{\mathbf{C}}'\bar{\mathbf{A}}'\bar{\mathbf{B}}' = \bar{\mathbf{C}}\bar{\mathbf{A}}^{-1}\bar{\mathbf{B}} = [\mathbf{0} \quad \mathbf{C}_v]\begin{bmatrix}\mathbf{Z}_1\\ \mathbf{Z}_2\end{bmatrix} = \mathbf{C}_v\mathbf{Z}_2.$$

Here,

$$\begin{bmatrix}\mathbf{Z}_1\\ \mathbf{Z}_2\end{bmatrix} = \bar{\mathbf{A}}^{-1}\bar{\mathbf{B}}$$

with $\mathbf{Z}_1, \mathbf{Z}_2 \in R^{n \times m}$. Hence,

$$\bar{\mathbf{A}}\begin{bmatrix}\mathbf{Z}_1\\ \mathbf{Z}_2\end{bmatrix} = \bar{\mathbf{B}},$$

i.e.,

$$\begin{bmatrix}\mathbf{0} & \mathbf{I}_n\\ -\mathbf{M}_0^{-1}\mathbf{S} & -\mathbf{M}_0^{-1}\mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{Z}_1\\ \mathbf{Z}_2\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ \mathbf{G}\end{bmatrix}.$$

From the latter, it is deduced that $\mathbf{Z}_2 = \mathbf{0}$. Thus equation (56) is satisfied. Finally, for acceleration sensing,

$$\bar{\mathbf{C}}'\bar{\mathbf{A}}'^{-2}\bar{\mathbf{B}}' = \bar{\mathbf{C}}\bar{\mathbf{A}}^{-2}\bar{\mathbf{B}} = [\mathbf{W}_1 \quad \mathbf{W}_2]\begin{bmatrix}\mathbf{Z}_1\\ \mathbf{Z}_2\end{bmatrix} = \mathbf{W}_1\mathbf{Z}_1 + \mathbf{W}_2\mathbf{Z}_2.$$

Here $[\begin{smallmatrix}\mathbf{Z}_1\\ \mathbf{Z}_2\end{smallmatrix}]$, as is done above, is used to denote the matrix $\bar{\mathbf{A}}^{-1}\bar{\mathbf{B}}$, and $[\mathbf{W}_1 \;\; \mathbf{W}_2]$ to designate the matrix $\bar{\mathbf{C}}\bar{\mathbf{A}}^{-1}$. Hence, from the above, $\mathbf{Z}_2 = \mathbf{0}$ and, in addition,

$$[\mathbf{W}_1 \quad \mathbf{W}_2] = \mathbf{C}_a[-\mathbf{M}_0^{-1}\mathbf{S} \quad -\mathbf{M}_0^{-1}\mathbf{D}]\begin{bmatrix}\mathbf{0} & \mathbf{I}_n\\ -\mathbf{M}_0^{-1}\mathbf{S} & -\mathbf{M}_0^{-1}\mathbf{D}\end{bmatrix}^{-1}.$$

Evidently, $\mathbf{W}_1 = \mathbf{0}$. Thus, equation (57) is satisfied.

This completes the proof of Theorem 1. □

In Theorem 1, the output displacement (velocity, acceleration) location influence matrix $\mathbf{C}_d$ (resp., $\mathbf{C}_v$, $\mathbf{C}_a$) is assumed to be a non-singular $n \times n$ matrix. This means that the number of sensors, i.e. l, should be the same as that of degrees of freedom, i.e., n of the model equation (1). It has been shown that this condition is sufficient for the identification of the structural system in a full sense. This condition is also necessary. This can be deduced from equations (45), (32) and (48) for displacement sensing, from equations (49), (32) and (48) for velocity sensing, and from equations (52), (48) and (32) for acceleration sensing. In fact, for displacement sensing, equation (45) supplies infinite number of solutions for $[\mathbf{X}_1 \;\; \mathbf{X}_3]$, whenever the rank of the matrix $\mathbf{C}_d \in R^{l \times n}$ is smaller than the number of degrees of freedom, i.e., n. Accordingly, equation (32) yields infinite number of solutions for $[\mathbf{X}_2 \;\; \mathbf{X}_4]$, and then equation (48) produces infinite number of solutions for $[\mathbf{S} \;\; \mathbf{D}]$. In a similar way, it may be shown that the same is true for velocity sensing. For acceleration sensing, if the rank of the matrix $\mathbf{C}_a \in R^{l \times n}$ is smaller than n, then equation (52) supplies an infinite number of $\mathbf{M}_0^{-1}[\mathbf{S} \;\; \mathbf{D}]\mathbf{R}$. Accordingly, equation (48) provides an infinite number of $[\mathbf{X}_2 \;\; \mathbf{X}_4]$, and then equation (32) supplies an infinite number of $[\mathbf{X}_1 \;\; \mathbf{X}_3]$.

Since it is known from Theorem 1 that for the attainment of the goal it is adequate to set the number of sensors the same as that of degrees of freedom of the model equation (1), the case when the former exceeds the latter will be excluded. From this and the facts shown above, it is concluded that the condition $l = n$ is necessary for the uniqueness requirement. Thus, the following result is reached.

Theorem 2. *Do not let the number of sensors exceed that of degrees of freedom of the model equation* (1), *i.e.,* $l \leqslant n$. *Then, for displacement* (*resp., velocity, acceleration*) *sensing, the stiffness, damping and input location influence matrices of the structural system modelled by* *equation* (1) *with n degrees of freedom may be uniquely determined by its Markov parameters and its mass matrix* $\mathbf{M} = \mathbf{M}_0$, *if and only if the displacement* (*resp*., *velocity*, *acceleration*) *location influence matrix* $\mathbf{C}_d$ (*resp*., $\mathbf{C}_v$, $\mathbf{C}_a$) *is a non-singular* $n \times n$ *matrix. Accordingly, the number of sensors should be the same as that of degrees of freedom of the model equation* (1), *i.e.*, $l = n$, *and the locations of sensors should be suitably arranged.*

The expressions (36)–(38), (39)–(41) and (42)–(44) furnish the stiffness, damping and input location influence matrices **S**, **D** and **G**, separately, for displacement and velocity and acceleration sensing, whenever a realization $\{\bar{\mathbf{A}}', \bar{\mathbf{B}}', \mathbf{C}'\}$ is obtained from a set of Markov parameters. As has been mentioned earlier, there are an infinite number of equivalent realizations $\{\bar{\mathbf{A}}', \bar{\mathbf{B}}', \mathbf{C}'\}$, and any two of them are related to each other by a non-singular transformation matrix pertaining to $R^{2n\times 2n}$. Thus, it is necessary to clarify whether or not the expressions (37) and (38), (40) and (41) and (43) and (44) are "objective', i.e., invariant under every non-singular transformation $\mathbf{U} \in R^{2n\times 2n}$. This issue is treated as follows.

Let $\mathbf{U} \in R^{2n\times 2n}$ be any given non-singular transformation. In equations (36)–(44), replacing the realization $\{\bar{\mathbf{A}}', \bar{\mathbf{B}}', \bar{\mathbf{C}}'\}$ with another equivalent realization $\{\bar{\mathbf{A}}'', \bar{\mathbf{B}}'', \bar{\mathbf{C}}''\}$ transformed by **U**, i.e.

$$(\mathbf{A}'', \mathbf{B}'', \bar{\mathbf{C}}'') = (\mathbf{U}\bar{\mathbf{A}}'\mathbf{U}^{-1}, \mathbf{U}\bar{\mathbf{B}}', \bar{\mathbf{C}}'\mathbf{U}^{-1}),$$

gives $\mathbf{R}''$, $\mathbf{S}''$, $\mathbf{D}''$ and $\mathbf{G}''$. The latter are given by

$$\mathbf{R}'' = \begin{bmatrix} \mathbf{C}_d^{-1}(\bar{\mathbf{C}}'\mathbf{U}^{-1}) \\ \mathbf{C}_d^{-1}(\bar{\mathbf{C}}'\mathbf{U}^{-1})(\mathbf{U}\bar{\mathbf{A}}'\mathbf{U}^{-1}) \end{bmatrix} = \mathbf{R}\mathbf{U}^{-1},$$

$$[\mathbf{S}'' \;\; \mathbf{D}''] = -\mathbf{M}_0\mathbf{C}_d^{-1}(\bar{\mathbf{C}}'\mathbf{U}^{-1})(\mathbf{U}\bar{\mathbf{A}}'^2\mathbf{U}^{-1})(\mathbf{U}\mathbf{R}^{-1}) = [\mathbf{S} \quad \mathbf{D}],$$

$$\mathbf{G}'' = \mathbf{M}_0\mathbf{C}_d^{-1}(\bar{\mathbf{C}}'\mathbf{U}^{-1})(\mathbf{U}\bar{\mathbf{A}}'\mathbf{U}^{-1})(\mathbf{U}\bar{\mathbf{B}}') = \mathbf{G},$$

for displacement sensing, and

$$\mathbf{R}'' = \begin{bmatrix} \mathbf{C}_v^{-1}(\bar{\mathbf{C}}'\mathbf{U}^{-1})(\mathbf{U}\bar{\mathbf{A}}'^{-1}\mathbf{U}^{-1}) \\ \mathbf{C}_v^{-1}(\bar{\mathbf{C}}'\mathbf{U}^{-1}) \end{bmatrix} = \mathbf{R}\mathbf{U}^{-1},$$

$$[\mathbf{S}'' \;\; \mathbf{D}''] = -\mathbf{M}_0\mathbf{C}_v^{-1}(\bar{\mathbf{C}}'\mathbf{U}^{-1})(\mathbf{U}\bar{\mathbf{A}}'\mathbf{U}^{-1})(\mathbf{U}\mathbf{R}^{-1}) = [\mathbf{S} \quad \mathbf{D}],$$

$$\mathbf{G}'' = \mathbf{M}_0\mathbf{C}_v^{-1}(\bar{\mathbf{C}}'\mathbf{U}^{-1})(\mathbf{U}\bar{\mathbf{B}}') = \mathbf{G},$$

for velocity sensing, and

$$\mathbf{R}'' = \begin{bmatrix} \mathbf{C}_a^{-1}(\bar{\mathbf{C}}'\mathbf{U}^{-1})(\mathbf{U}\bar{\mathbf{A}}'^{-2}\mathbf{U}^{-1}) \\ \mathbf{C}_a^{-1}(\bar{\mathbf{C}}'\mathbf{U}^{-1})(\mathbf{U}\bar{\mathbf{A}}'^{-1}\mathbf{U}^{-1}) \end{bmatrix} = \mathbf{R}\mathbf{U}^{-1},$$

$$[\mathbf{S}'' \;\; \mathbf{D}''] = -\mathbf{M}_0\mathbf{C}_a^{-1}(\bar{\mathbf{C}}'\mathbf{U}^{-1})(\mathbf{U}\mathbf{R}^{-1}) = [\mathbf{S} \quad \mathbf{D}],$$

$$\mathbf{G}'' = \mathbf{M}_0\mathbf{C}_a^{-1}(\bar{\mathbf{C}}'\mathbf{U}^{-1})(\mathbf{U}\bar{\mathbf{A}}'^{-1}\mathbf{U}^{-1})(\mathbf{U}\bar{\mathbf{B}}') = \mathbf{G},$$

for acceleration sensing.

Thus, the following result is reached.

Corollary 3. *The expressions* (37) *and* (38), (40) *and* (41) *and* (43) *and* (44) *furnish, separately, the same results for all possible equivalent realizations* $\{\bar{\mathbf{A}}', \bar{\mathbf{B}}', \mathbf{C}'\}$ *derived from any given set of Markov parameters.*

4. ITERATION ALGORITHM FOR THE CASE OF INCOMPLETE OUTPUT DATA

According to Theorem 2, the number of sensors should be the same as that of degrees of freedom of the model equation (1). As a result, when the latter is very large, a very large number of sensors are needed. A possible solution for this issue is to measure output data with different arrangements of locations of relatively fewer numbers of actuators and sensors. The output data required may be derived from a collection of output data measured several times. Moreover, appropriate arrangements of actuator and sensor locations may result in optimal measurement data. In modal analysis, this has been investigated in, e.g., references [26–29]. Use of sensors and actuators in sensing and control of structural dynamics has been discussed or reviewed in, e.g., references [29–37].

Usually, for a system with a large number of degrees of freedom, only an incomplete set of output data may be available, measured by a fewer number of sensors. In this case, the procedure proposed in the last section cannot be used in a straightforward manner. To circumvent this difficulty, in what follows, a further iteration algorithm for the case of incomplete output data is suggested.

Consider a structural system modelled by the discretized equation (1) with n degrees of freedom. In this structural system, l sensors are arranged at l locations. The output data $\mathbf{y}(t) \in R^{l\times 1}$, measured by the l arranged sensors for an excitation generated by the arranged actuator are described by equation (3), or, equivalently, by equations (5), (22) and (25) in terms of state space description. Let $l < n$. Then, according to Theorem 2, these output data are not sufficiently complete to determine the stiffness and damping matrices $\mathbf{S}$ and $\mathbf{D}$ in a unique manner. Then,

$$\tilde{\mathbf{y}}(t) = \tilde{\mathbf{C}}\mathbf{x}(t) + \tilde{\mathbf{D}}\mathbf{u}(t). \tag{58}$$

Once the response $\mathbf{x}(t)$ to the input $\mathbf{u}(t)$ is known, the last equation determines $\tilde{\mathbf{y}}(t)$.

It becomes clear that the data given by equation (58) join the output data $\mathbf{y}(t)$ measured by l sensors to supply a complete set of output data by which the system property matrix may be uniquely determined by the procedure described in the last section.

In a practical measurement, the data $\mathbf{y}(t) \in R^{l\times 1}$ are measured by the l arranged sensors and hence are available, whereas the data $\tilde{\mathbf{y}}(t) \in R^{(n-l)\times 1}$ are lacking due to the fact that it may not be possible to arrange enough number of sensors to measure the output. If the number of degrees of freedom, n, is very large, the number of arranged sensors, l, may be quite smaller than n, i.e., $l \ll n$. However, to apply the procedure established in the last section, it is necessary to have a complete set of output data, i.e., the missing data $\hat{\mathbf{y}}(t)$ have to be supplemented. Towards this end, an iteration algorithm will be used. The main idea of this algorithm is to evaluate the missing data $\tilde{\mathbf{y}}(t)$ and the system matrices $\mathbf{S}$ and $\mathbf{D}$ by means of a successive approximation procedure, which is elaborated below.

First, starting with the initial values of the mass, stiffness and damping matrices, i.e., $\mathbf{M}_0$, $\mathbf{S}_0$ and $\mathbf{D}_0$, as well as the initial input location influence matrix $\mathbf{G}_0$, equations (23)–(25) provide the initial system matrix $\bar{\mathbf{A}}_0$ and the initial observer matrix $\bar{\mathbf{C}}_0$, as well as the initial control matrix $\bar{\mathbf{B}}_0$. Then, the response $\mathbf{x}_0(t)$ is derived from the differential equation (4) with the replacement of $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ by $\bar{\mathbf{A}}_0$ and $\bar{\mathbf{B}}_0$. After that, the replacement of $\hat{\mathbf{C}}$ and $\tilde{\mathbf{C}}$ by $\hat{\mathbf{C}}_0$ and $\tilde{\mathbf{C}}_0$ supply the missing data $\tilde{\mathbf{y}}_0(t)$. The latter and the output data $\mathbf{y}(t)$ measured by the

l sensors together provides an approximation of a complete set of output data, i.e., $\mathbf{y}_0(t) = [\mathbf{y}(t)\ \tilde{\mathbf{y}}_0(t)]^{\mathrm{T}}$. Then, utilizing the latter and ERA described in section 2, a realization $(\bar{\mathbf{A}}'_1, \bar{\mathbf{B}}'_1, \bar{\mathbf{C}}'_1)$ may be found. Finally, by using equations (36)–(44) the new stiffness and damping matrices $\mathbf{S}_1$ and $\mathbf{D}_1$ may be obtained, as well as the input location influence matrix $\mathbf{G}_1$.

The above procedures constitute the first step of the iteration algorithm. The second step is to use $\mathbf{S}_1$ and $\mathbf{D}_1$ and $\mathbf{G}_1$ as the initial values and to repeat the above procedures. As a result, $\mathbf{S}_2$ and $\mathbf{D}_2$ and $\mathbf{G}_2$ may be obtained. Then, follow the third step, the fourth step, etc., up to a step at which satisfactory values of the stiffness and damping matrices are available.

Below is the summation of the above iteration algorithm.

$$\bar{\mathbf{A}}_\alpha = \begin{bmatrix} \mathbf{0} & \mathbf{I}_n \\ -\mathbf{M}_0^{-1}\mathbf{S}_\alpha & -\mathbf{M}_0^{-1}\mathbf{D}_\alpha \end{bmatrix}, \tag{59}$$

$$\bar{\mathbf{B}}_\alpha = \begin{bmatrix} \mathbf{O} \\ \mathbf{M}_0^{-1}\mathbf{G}_\alpha \end{bmatrix}, \tag{60}$$

$$\tilde{\mathbf{C}}_\alpha = [\mathbf{0} \quad \mathbf{I}_{n-l}]([\hat{\mathbf{C}}_d \quad \hat{\mathbf{C}}_v] - \hat{\mathbf{C}}_a\mathbf{M}_0^{-1}[\mathbf{S}_\alpha \quad \mathbf{D}_\alpha]), \tag{61}$$

$$\dot{\mathbf{x}}_\alpha(t) = \bar{\mathbf{A}}_\alpha\mathbf{x}_\alpha(t) + \bar{\mathbf{B}}_\alpha\mathbf{u}(t), \tag{62}$$

$$\tilde{\mathbf{y}}_\alpha(t) = \tilde{\mathbf{C}}_\alpha\mathbf{x}_\alpha(t) + \tilde{\mathbf{D}}\mathbf{u}(t), \tag{63}$$

$$\hat{\mathbf{y}}_\alpha(t) = \begin{bmatrix} \mathbf{y}(t) \\ \tilde{\mathbf{y}}_\alpha(t) \end{bmatrix}, \tag{64}$$

$$(\bar{\mathbf{A}}'_{\alpha+1}, \bar{\mathbf{B}}'_{\alpha+1}, \bar{\mathbf{C}}'_{\alpha+1}) = \mathrm{ERA}(\hat{\mathbf{y}}_\alpha(t)), \tag{65}$$

$$(\mathbf{S}_{\alpha+1}, \mathbf{D}_{\alpha+1}, \mathbf{G}_{\alpha+1}) = \Phi(\bar{\mathbf{A}}'_{\alpha+1}, \bar{\mathbf{B}}'_{\alpha+1}, \bar{\mathbf{C}}'_{\alpha+1}). \tag{66}$$

In the above-mentioned equations, the symbol "ERA" is used to represent the fact that, given a complete set of output data, $\hat{\mathbf{y}}_\alpha(t)$, the eigensystem realization algorithm described in section 2 determines a realization $(\bar{\mathbf{A}}'_{\alpha+1}, \bar{\mathbf{B}}'_{\alpha+1}, \bar{\mathbf{C}}'_{\alpha+1})$ of the system (4) and (5). Moreover, the symbol Φ means that, with the above-mentioned realization, either equations (36)–(38) or equations (39)–(41) or equations (42)–(43) are used to determine the matrices $(\mathbf{S}_{\alpha+1}, \mathbf{D}_{\alpha+1}, \mathbf{G}_{\alpha+1})$.

The initial mass, stiffness and damping matrices $\mathbf{M}_0$, $\mathbf{S}_0$ and $\mathbf{D}_0$, which are needed in carrying out the above iteration algorithm, may be derived from the structural analysis of discretized systems or from discretization methods, such as finite element methods and finite difference methods, etc., of continuous systems. Usually, it is easier to construct the mass and stiffness matrices $\mathbf{M}_0$ and $\mathbf{S}_0$, while in some cases it may not be easy to obtain the damping matrix $\mathbf{D}_0$. The reason for the latter fact is mainly due to the complicated mechanisms of dynamic dissipation in a complex structural system. However, for structural systems with weak damping, the above iteration algorithm may be implemented by assuming $\mathbf{D}_0 = \mathbf{0}$. This final result will supply all the system property matrices, including the damping matrix $\mathbf{D}$.

It may be expected that the above iteration algorithm is convergent, provided that the current system property matrices are close to their initial values. That may be true for most of the engineering structures before their collapses and failures.

It is worthwhile to point out that the missing data $\tilde{\mathbf{y}}(t)$ may alternatively be evaluated by introducing the so-called *observers* for the state space model (4) and (5), see e.g., references [38, 39]. Extensive applications of observers in studying dynamic behaviour of the model system (1) have been made by Schmidt [40]. A suitable chosen observer may provide a good asymptotic estimation of the missing data $\tilde{\mathbf{y}}(t)$ using a linear dynamic system with lower dimension $n - l$. When the number l is close to the number n, this approach is attractive. A detailed account of this aspect is beyond our consideration. Reference may be made to reference [40] and the relevant literature therein.

5. APPLICATION AND AN ILLUSTRATIVE EXAMPLE

In recent years, analytical and experimental techniques have been proposed and developed to deal with the issue of damage or fault detection in structural systems, see, e.g., references [19, 20, 23, 40–51], as well as the references therein.

The procedure developed in the previous sections allows the evaluation and monitoring, in a full sense, of the degradation of dynamic properties of a structural system at each stage of its service. Specifically, if data for the stiffness matrix $\mathbf{S}$ of a structural system at a stage are available, it is possible to estimate whether or not the stiffness of this system has been appreciably weakened, and inside which elements or parts of the system the phenomenon of stiffness weakening occurs.

The basic consideration is schematically shown in Figure 1. Consider an initial state without damage and faults and a state with damaged and faulty constituent elements of a structural system modelled by equation (1). In sharp contrast to the conservation of its mass, a constituent element[§] in which appreciable damage or fault locates may suffer considerable loss of stiffness and strength as compared with its initial stiffness and strength. As a result, if the current and initial values of stiffness (e.g., axial, flexural, torsional rigidities, etc.) of every constituent element of a structural system modelled by equation (1) are known, then it is possible to judge inside which constituent elements an appreciable internal damage or fault, if any, has emerged. However, the matrix $\mathbf{S}$ itself merely characterizes the global stiffness property of the structural system as a whole. To achieve the foregoing goal, it is necessary to have information about the distribution of stiffnesses of constituent elements or parts of the structural system. This is indeed possible, since the global or aggregate stiffness matrix $\mathbf{S}$ of a structural system is obtained just by assembling the stiffnesses of its constituent elements or parts. The latter is achieved simply by the operation of adding the coefficients of the element stiffness matrices into the proper locations in the aggregate stiffness matrix. Symbolically,

$$\mathbf{S} = \sum_e \mathbf{S}^{(e)}.$$

Here, as has been done before, $\sum_e$ means the summation of all the constituent elements or parts of the structural system; $\mathbf{S}^{(e)}$ is used to represent an element stiffness matrix in terms of the aggregate displacement vector.

In order to attain the goal, it is necessary to consider an “inverse” process of the foregoing assembling operation: from the aggregate stiffness matrix, derive the stiffness of each constituent element of the structural system. To achieve this, a simple approach is described

[§]The constituent elements of a finite-degree-of-freedom structural system may be evident. However, for a continuous structural system, both the model equation (1) and the associated constituent elements depend on the discretization programme adopted.

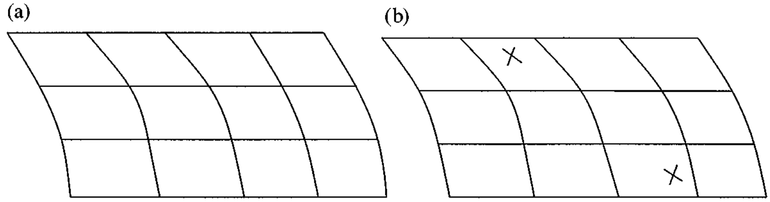

Figure 1. A structural system modelled by equation (1). (a) Initial state without damage and faults. (b) A state with damaged or faulty constituent elements.

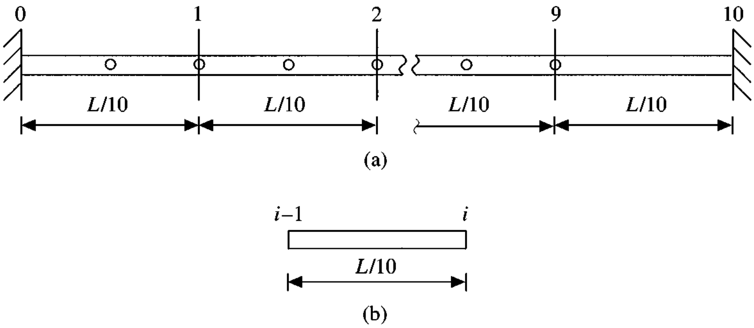

Figure 2. Beam with fixed ends. (a) Discretization with 10 elements. (b) Element i.

as follows. There are some entries of the aggregate stiffness matrix **S** which are precisely some entries $S_{ij}^{(p)}$ of some element stiffness matrices. Determine these entries $S_{ij}^{(p)}$ and obtain directly the stiffnesses (e.g., axial, flexural, and torsional rigidities, etc.) of the corresponding elements indexed by p. Then determine such entries of **S**, each of which is a sum of an unknown entry $S_{ij}^{(q)}$ and some known entries, and then obtain the stiffnesses of the corresponding elements indexed by q. Continue this process until the stiffnesses of all the constituent elements are derived.

As is shown in Figure 2(a), a beam of length L with its two ends fixed is considered. The beam is discretized into $n_{el} = 10$ elements with the equilength L/n_{el} and the ends of these elements are denoted by $0, 1, \ldots, n_{el}$, as shown in Figure 2(a). The element $(i-1) - i$ (see Figure 2(b)) is also called element i.

Initially, it is assumed that the beam is uniform with reference to its section and material property; hence, its initial flexural rigidity is constant, denoted by K_0. However, after a process of deformation and loading, the flexural rigidity of the beam may no longer be constant due to the emergence of internal damage and cracks at some locations of the beam. According to the discretization programme, each element i is assumed to have constant flexural rigidity $K^{(i)}$. After a process of deformation and loading, if by virtue of a procedure it is found that $K^{(i)}$ is noticeably different from K_0, then it is clear that, inside the element i, noticeable degradation of rigidity has resulted due to accumulation of internal damage and cracks, etc. Evidently, the finer the discretization programme of the beam, the more accurate is the estimation of the locations of the damaged or faulty parts. Each element i (see Figure 2(b)) is regarded as a Euler–Bernoulli beam element.

Let the beam be made of carbon steel. Its length is $L = 100\,\text{in}$ and its section is a square of side $h = 2\,\text{in}$. The mass density and the elastic modulus of carbon steel are given by

$$\rho = 0{\cdot}28\,\text{lb/in}^3, \qquad E = 30 \times 10^6\,\text{psi}.$$

Then

$$\rho AL = \rho Lh^2 = 72\,\text{lb}, \qquad K_0 = \frac{Eh^4}{12} = 4 \times 10^7\,\text{psi}\,\text{in}^4.$$

Now put 18 sensors at the nodes $1, \ldots, 9$ as well as at the respective midpoints of the elements $1, \ldots, 9$, as is shown by the small circles in Figure 2(a). With this arrangement of sensors, the output vector $\mathbf{y}(t)$ provides the deflections or velocities or accelerations at the 18 points at which the sensors are. The relationship between the output vector $\mathbf{y}(t)$ and the aggregate nodal displacement vector $\mathbf{w} = \mathbf{w}(t)$ may be derived readily by using the value of the shape function at $\xi = 0$, and then the output location influence matrices $\mathbf{C}_d$ and $\mathbf{C}_v$ and $\mathbf{C}_a$ are available. The result is as follows:

$$\mathbf{C}_d = \mathbf{C}_v = \mathbf{C}_a = \begin{bmatrix} 0{\cdot}5 & -0{\cdot}25 & & & & & & \\ 1 & 0 & 0 & & & & & \\ 0{\cdot}5 & 0{\cdot}25 & 0{\cdot}5 & -0{\cdot}25 & & & & \\ & 0 & 1 & 0 & 0 & & & \\ & & \cdot & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \cdot & \\ & & & & & 0{\cdot}5 & -0{\cdot}25 & \\ & & & & & 1 & 0 & \end{bmatrix} \tag{67}$$

The above matrix is a non-singular 18×18 matrix and hence satisfies the condition given in Theorem 2.

Put an actuator at some location in the beam. Let the actuator generate an impulse excitation. Then, the arranged sensors record the discrete impulse responses at their respective locations with a sample time increment Δt. Hence, a set of Markov parameters is given by these measurement data. Find a discrete state space model realization modelled by equations (12) and (13) and convert the result to continuous-time, as described in section 2. Then, a realization for the state space model (4) and (5) can be obtained.

Assume that for the displacement sensing the above process supplies two realizations $\{\bar{\mathbf{A}}', \bar{\mathbf{B}}', \bar{\mathbf{C}}'\}$ and $\{\bar{\mathbf{A}}'', \bar{\mathbf{B}}'', \bar{\mathbf{C}}''\}$. The first realization results from a beam with original rigidities $K^{(i)} = K_0, (i = 1, \ldots, 10)$, while, in the second realization, the rigidity of beam element 2 was weakened by 0·1% and that of element 6 by 50%.

For the sake of demonstration the realization matrices are produced here by reverse transformation using an antidiagonal transformation matrix $\mathbf{R}$ and by taking the damping matrix $\mathbf{D} = \mathbf{0}$.

Now expressions (36)–(38) are utilized in sequence and $\mathbf{R}$, $\mathbf{G}$, $\mathbf{S}$ and $\mathbf{D}$ are evaluated for the first realization. The state space matrices of the beam with constant rigidity K_0 are found and it may therefore be concluded that no damage has occurred. Then equations (36)–(38) are applied to the matrices of the second realization. In particular, $\mathbf{S}$ is obtained from equation (37). The flexural rigidities of the different elements may be extracted from $\mathbf{S}$ by

$$\begin{aligned} K^{(i)} &= \frac{L^3}{12} S_{2i-3,2i}, \quad 2 < i < n-1, \\ K^{(1)} &= \frac{L^3}{12} S_{11} - K^{(2)}, \quad K^{(n_{el})} = \frac{L^3}{16} S_{2n_{el}-2,2n_{el}-2} - K^{(n_{el}-1)}. \end{aligned} \tag{68}$$

For the rigidities

$$\mathbf{K} = (40000000 \quad 39960000 \quad 40000000 \quad 40000000 \quad 40000000$$
$$20000000 \quad 40000000 \quad 40000000 \quad 40000000 \quad 40000000)^{\mathrm{T}},$$

the values are exact up to a relative error of 1E-14. It becomes clear that the flexural rigidity $K^{(6)}$ of element 6 dropped down by 50% from its initial value, i.e., appreciable damage, etc. has occurred inside element 6. Also, the rigidity of element 2 is slightly weakened compared to the initial state and element 2 should be inspected.

The relative accuracy of this calculation is 1E-14 and depends on the transformation matrix **R**. Indeed, with a randomly generated transformation **R** for the first realization

$$\mathbf{K} = (39999991 \quad 40000013 \quad 40000029 \quad 40000018 \quad 40000006$$
$$39999997 \quad 39999991 \quad 39999987 \quad 39999989 \quad 40000011)^{\mathrm{T}}$$

and for the second realization

$$\mathbf{K} = (39999999 \quad 39960001 \quad 40000001 \quad 39999994 \quad 40000000$$
$$19999998 \quad 40000000 \quad 40000001 \quad 40000006 \quad 39999994)^{\mathrm{T}},$$

i.e., the relative accuracy then was 3E-7. On the other hand, a randomly occupied damping matrix does not seem to significantly influence the accuracy of the results for **K**.

6. CONCLUDING REMARKS

With the observation of a self-evident fact, i.e., the conservation of mass of any part of a mechanical system, in the previous sections a procedure has been developed for fully evaluating and monitoring degradation of dynamic properties of structural systems via a subspace identification method. With the initial property matrices of a structural system as well as suitable experimental input/output measurement data, it has been shown that it may be possible to determine the stiffness and damping matrices of the system in a full and unique manner. The main results are presented by Theorems 1 and 2 and Corollary 3 in section 3 and by the iteration algorithm in section 4. Moreover, with the aggregate stiffness matrix of a structural system obtained as well as with information about the stiffness distribution of constituent elements of this system, it may be possible to estimate whether or not noticeable internal damage and crack, etc., has emerged inside the structural system and inside which elements or parts of this phenomenon occurred.

The proposed procedure has the same extent of applicability as that of the model equation (1). That means that the linear property is assumed. In some cases, non-linear properties may not be negligible and may even play a dominant role. This difficult aspect is beyond the present consideration. Within the context of linearization and small deformation, the proposed procedure may have wide applicability. Indeed, from analysis and computation of FE methods (see e.g., references [1, 2]), it is seen that dynamic behaviour of any structural system may be modelled by equation (1) either through structural analysis for finite-degree-of-freedom systems or through reasonable discretization and linearization for continuous systems.

For any continuous system in space variables, the model equation (1) is a finite-degree-of-freedom approximation to an infinite-degree-of-freedom case through discretization of space variables. Evidently, different discretization programmes will result in different forms of model equation (1). Choosing a suitable one from them may be essential for effectiveness and practicability of measurement and computation. A useful approach is to carry out an FE discretization process which refines the partition of constituent element step by step: start with a model equation (1) with fewer number of constituent elements. Then continue with a model equation (1) with a greater number of finer constituent elements. Evidently, the finer the discretization programme, the more accurate is the evaluation of the system property and the estimation of damage or fault locations. A finer discretization programme results in a model equation (1) with a larger number of degrees of freedom. As a result, the output data measured by arranged sensors may become incomplete. In this case, the iteration algorithm developed in section 4 is essential.

On the other hand, if the arranged actuator generates a general excitation to the system instead of an impulse input, then the output data recorded by the arranged sensors will not supply, in a straightforward manner, a set of Markov parameters. In this case, the latter is related to the former by expression (10) and hence may be derived from equation (10).

Input/output data for realistic measurements most often incorporate stochastic contribution to some extent due to the influence of noise, etc. This raises the issue as to whether or not the proposed procedure remains applicable and robust in the presence of noise. It is evident that this issue is dependent upon the applicability and robustness of the subspace identification method in the noise case. Whenever a realization $\{\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}\}$ is derived either in the noise-free case or in the noise case, the stiffness and damping matrices $\mathbf{S}$ and $\mathbf{D}$ are obtainable from expressions (36)–(44). It is expected that due to the sensitivity of the identified damping matrix to noisy data this will be a major point for examination.

It should be pointed out that output data may be measured by means of any other experimental technique except using sensors. This becomes necessary in the cases when it is not easy to arrange sensors to make measurement.

Applications of the proposed procedure to engineering structural systems with realistic input/output measurement data will be considered in the future development.

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