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SENSOR CONFIGURATION EFFICIENCY AND ROBUSTNESS AGAINST SPATIAL ERROR IN THE PRIMARY FIELD FOR ACTIVE SOUND CONTROL

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The efficiency and the robustness of active noise control against fluctuations depend, amongst other things, on the secondary source and control microphone configurations. Within the context of harmonic sound attenuation, the performance (efficiency and robustness) according to the source configurations has recently been dealt with. The present study is focused on the performance according to microphone, or sensor, configurations. It is demonstrated that all sensor configurations may become equally efficient due to the fact that they may be made to control equally well a domain defined by a certain number of observation points. The paper also gives a definition of robustness starting from the result of previous work on source configurations. Robustness is understood here in relation to the spatial fluctuations of the primary sound field. Given this definition, indications are provided for choosing the most robust sensor configuration or how to improve the robustness of any one of the sensor configurations.

1. INTRODUCTION

Active noise control is today a well-known procedure for reducing noise, essentially in the low audio-frequency range. It consists in voluntarily adding a sound field to the disturbing or 'primary field' in such a way that they cancel each other out. The ideal voluntarily radiated noise, also referred to as the 'secondary field', ought to replicate the disturbing one, with a sign change at all times and everywhere within the area where sound cancellation is sought. Achieving perfect synchronicity everywhere is generally an impossible task and active noise control attenuates the primary field as much as is possible. The well-known means of reaching the goal progressively adapts over time the coefficients of filters upstream from the secondary sources in order for the source driving signals to radiate the appropriate secondary field to minimize the total sound pressure at microphones located in the area of interest. In the frequency domain, this technique of autoadaptive active noise control is equivalent to minimizing a function which gives the global sound level from complex amplitudes of primary pressures, transfer functions and source driving signals. In these conditions, the attenuation depends on three inputs: the geometrical configuration of the secondary sources and the control microphones, the acoustic transfer functions between these sources and the microphones, the primary field at the control microphones (the latter input is not explicit from the technical point of view but it is from the physical and mathematical points of view). In the field of active sound reduction, predictive calculations must provide not only the achievable attenuation but also the attenuation that can be guaranteed when fluctuations affect the inputs. Indeed, in the real world, transfer functions as well as primary fields are subjected to fluctuations around what can be called a reference situation. Only the spatial fluctuation or error of primary fields is considered in this paper.

Efficiency and robustness according to source configurations have been dealt with recently, either by analyzing only the case of the error in the primary field [1] or by describing the case of all types of errors [2]. Here the concern is with efficiency *and* robustness according to microphone or sensor configurations. The problem is worth studying due to the fact that the number of control microphones is limited by the number of channels in controllers and, moreover, they often cannot be located where sound attenuation is sought. For example, in cars or airplanes, control microphones at the passenger's head will have to be displaced towards the ceiling or fuselage for practical reasons. Only a few works relevant to the present subject have been published. One of them mentions that the sensors should be located at the nodes of the residual field in the reference situation (without any error) [3] and another shows the possibility of placing the sensors in the vicinity of the domain to the controlled rather than inside it [4]. Both works on sensor locations speak of efficiency and not of robustness against error. In its contribution to the problem of efficiency *and* robustness of sensor configurations in the presence of errors in the spatial distribution of the primary harmonic field, the present paper will show that all sensor configurations may become equally efficient but that, at first sight, they are not equally robust. However, it will be shown that robustness can be improved.

Efficiency is obtained by means of filters associated with the sensors. Two situations are investigated here. When one filter is dedicated to each sensor, the weighting achieved is said to be diagonal because a diagonal matrix describes it; when the filters couple the sensor outputs, the weighting is said to be full because it is represented by a full matrix. Only one diagonal weighting matrix can be associated with each sensor configuration while an infinity of full weighting matrices exists for each configuration. Here the analytical method leads to the filters. Robustness does not have as classical a definition as efficiency and a large part of the paper is devoted to this. First a qualitative definition is sought, essentially owing to the vocabulary of sets. Then a quantitative definition is obtained through the minimum guaranteed attenuation, derived from previous work on the robustness of source configurations [1]. At this stage, robustness measurements are sought. On the one hand, an indicator is proposed arising from sufficient conditions (and unfortunately not necessary conditions) described geometrically, and on the other hand, a distance between two matrices which will make it possible to give access to or measure the robustness.

The main practical conclusion concerns how robustness measurements are used. In the presence of a few possible sensor configurations, each can become efficient in the reference situation by means of a single diagonal weighting, *a priori* easily implemented. In this case, the configuration with the largest indicator of robustness will be chosen; if the indicator is really large, the configuration could be kept (more precisely because the chances are that it will be good); if not, it is best to improve the robustness of a particular configuration—chosen for practical reasons for example—the considered weighting of which is now full. Indeed, the existence of an infinity of full weighting matrices likely to keep the configuration efficient makes it possible to obtain robustness by optimizing this weighting by minimizing the distance quoted above. On the more speculative side, the paper could reveal that all control microphone configurations may become not only equally efficient but also almost equally robust. Some physical insights will be given in the text.

2. SENSOR CONFIGURATION EFFICIENCY

It has already been mentioned that active noise control consists in deliberately making secondary (or anti-) sources radiate a field that adds to the disturbing or primary field, so that they neutralize each other. In fact, the primary sound level can rarely be totally cancelled out and so optimal attenuation is sought.

2.1. EFFICIENCY OVER A WHOLE AREA WITH AUTOADAPTIVE ACTIVE CONTROL

One can define the sound level, optimal control and optimal attenuation as well as the measure of the optimal attenuation within the context of harmonic sound fields, the equivalent of which in the time-domain is implemented with the now classical X-LMS algorithm that adapts the coefficients of filters upstream from the secondary sources.

The harmonic primary field $p_n(x, \omega)$ is spatially discretized by N points where sensors are located. The pressure at each sensor constitutes one component of the primary field vector $\mathbf{p}_n$. The acoustic function, associated with the primary sound level, is $J = J(\mathbf{p}_n) = \mathbf{p}_n^* . \mathbf{p}_n = \|\mathbf{p}_n\|_{L^2}^2$: i.e., the sum of the square modulus of the primary sound pressures at the sensors. The transfer functions associated with the N_s secondary sources will also be discretized by the fact that the pressure responses are seen from the number N of sensors, and they will be written in the matrix form $\mathbf{G}$. Vector $\boldsymbol{\phi}$ is made up of the driving signals of the secondary sources, or actuators. The addition of the secondary field to the primary field leads to what is called the residual level

$$J_{res}(\boldsymbol{\phi}) = \|\mathbf{G}.\boldsymbol{\phi} + \mathbf{p}_n\|_{L^2}^2 = \boldsymbol{\phi}^*.\mathbf{H}.\boldsymbol{\phi} + 2\Re e(\boldsymbol{\phi}^*.\boldsymbol{\Phi}) + J, \tag{1}$$

with $\mathbf{a}^*$ being the transpose conjugate of $\mathbf{a}$, $\mathbf{H} = \mathbf{G}^*.\mathbf{G}$, Hermitian positive definite matrix, of dimensions (N_s, N_s), and $\boldsymbol{\Phi} = \mathbf{G}^*.\mathbf{p}_n$ of dimensions $(N_s, 1)$.

The goal of active noise control is to reduce the primary level: i.e., to obtain $J_{res}(\boldsymbol{\phi}) < J$. The optimal driving signals, also called optimal control, which minimizes the acoustic function $J_{res}(\boldsymbol{\phi})$ is $\boldsymbol{\phi}_n = -\mathbf{H}^{-1}.\boldsymbol{\Phi}$ from which is derived the minimum residual level

$$J_{res}^{min}(\boldsymbol{\phi}_n) = J - \mathbf{p}_n^*.\mathbf{G}.\mathbf{H}^{-1}.\mathbf{G}^*.\mathbf{p}_n = J - \mathbf{p}_n^*.\mathscr{A}.\mathbf{p}_n = J - \boldsymbol{\phi}_n^*.\mathbf{H}.\boldsymbol{\phi}_n. \tag{2}$$

The matrix $\mathscr{A}$, of dimensions (N, N), is Hermitian and also is a projection matrix $(\mathscr{A}^2 = \mathscr{A})$; its eigenvalues are 1 or 0.

The quantity $J_a = \mathbf{p}_n^*.\mathscr{A}.\mathbf{p}_n$ is interpreted as the greatest part of the primary level J that has been cancelled out by active control, leading to J_{res}^{min} as the minimal remaining part. The definition of the optimal attenuation is $J_{res}^{min}/J = 1 - J_a/J = 1 - R_n$ where R_n represents the ratio of the part of J cancelled out to the primary level J. In some cases it is possible to have $J_a = J$, but usually only $J_a < J$ is reachable.

In dB, the optimal sound attenuation is the difference between the primary and the minimum residual sound levels:

$$A^{opt}(\mathbf{p}_n) = -10\log_{10}\left(\frac{J_{res}^{min}}{J}\right) = -10\log_{10}\left(1 - \frac{\mathbf{p}_n^*.\mathscr{A}.\mathbf{p}_n}{J}\right) = -10\log_{10}\left(\frac{\mathbf{p}_n^*.(\mathscr{T} - \mathscr{A}).\mathbf{p}_n}{J}\right), \tag{3}$$

where $\mathscr{T}$ is the identity matrix. Let $R_n = \mathbf{p}_n^*.\mathscr{A}.\mathbf{p}_n/J = \boldsymbol{\phi}_n^*.\mathbf{H}.\boldsymbol{\phi}_n/\mathbf{p}_n^*.\mathbf{p}_n = \|\boldsymbol{\phi}_n\|_H^2/\|\mathbf{p}_n\|_{L^2}^2$.

In the present context, the norms of the pressure-like quantities are always in the L^2 sense while the control-like quantities are in the H sense. This information will not be repeated in the paper.

2.2. HOW TO MAINTAIN EFFICIENCY WITH A SMALL NUMBER OF MICROPHONES

It was already mentioned in the introduction that the question of microphone positioning arises for two practical reasons at least. First, the number of control microphones or error sensors is limited by the number of inputs available in current controllers, generally between 4 and 16—the number of calculations carried out by digital signal processors during a sample period increases with the number of microphones, resulting in this limitation. Second, the control microphones can rarely be located where sound attenuation is sought. The first reason leads one to look at the possibility of keeping the efficiency obtained with N sensors when only N_c, less than N, are available. In this paper, the N_c sensors are chosen from among the N but it just so happens that the development presented is sufficiently general to deal also with the problem of positioning the sensors outside the area of interest. It will be shown that any configuration of any number N_c of control microphones ($N_c < N$) may become as efficient as the initial configuration with N microphones. The N points where attenuation is wanted will be referred to as observation points while the N_c points leading to the control of the secondary sources will be referred to as control points.

Consider the reference situation where the primary field is $\mathbf{p}_n$. The optimal control $\boldsymbol{\phi}_n$ is the solution of programming $\min_{\boldsymbol{\phi}} \|\mathbf{G}.\boldsymbol{\phi} + \mathbf{p}_n\|^2$. With J_{res} of equation (1), the minimum value of J_{res} is J_{res}^{min} of equation (2). In the very particular case where $N_s = N$ and $\mathbf{G}$ is non-singular, J_{res} is of zero minimum value. With a number N_c of control sensors, less than the initial number N, let $\mathbf{G}_c$ be the transfer matrix, the dimensions of which are (N_c, N_s), which is filled with the transfer functions from the N_s secondary sources towards the N_c control sensors, and write $\mathbf{p}_n^c$ as the primary field of reference at the N_c sensors. How is it possible, in the situation of reference, to obtain the optimal attenuation at the N sensors with the help of a control working with the reduced number N_c of control sensors? If $\mathbf{v}_n$ is the optimal control originating from the N_c error sensors, the goal would be reached by satisfying $\mathbf{v}_n = \boldsymbol{\phi}_n$.

With N_c sensors, in the situation of reference, the same programming as before will lead to the optimal control $\mathbf{v}_n$, the solution of $\min_{\mathbf{v}} J_{res}^c$ where $J_{res}^c = \|\mathbf{G}_c.\mathbf{v} + \mathbf{p}_n^c\|^2$ and $J_{res}^{c,min} = \|\mathbf{G}_c.\mathbf{v}_n + \mathbf{p}_n^c\|^2$. As there is no reason for $\mathbf{v}_n$ to be identical to $\boldsymbol{\phi}_n$, the minimization process will be applied no longer to J_{res}^c but to another sound level $\tilde{J}_{res}^c$ that has to be found. The functional $\tilde{J}_{res}^c$ found has the form $\tilde{J}_{res}^c = \|\mathbf{G}_c.\mathbf{v} + \mathscr{D}.\mathbf{p}_n^c\|^2$ where the matrix $\mathscr{D}$, of dimensions (N_c, N_c), is such that $\mathbf{v}_n = \boldsymbol{\phi}_n$.

To identify $\mathscr{D}$, it is required that $\tilde{J}_{res}^c(\mathbf{v}) = 0$ when $\mathbf{v} = \mathbf{v}_n = \boldsymbol{\phi}_n$. $\tilde{J}_{res}^c(\mathbf{v})$ being a quadratic functional of $\mathbf{v}$ and $\mathbf{H}_c = \mathbf{G}_c^*.\mathbf{G}_c$ being positive definite, its zero value is its minimum value

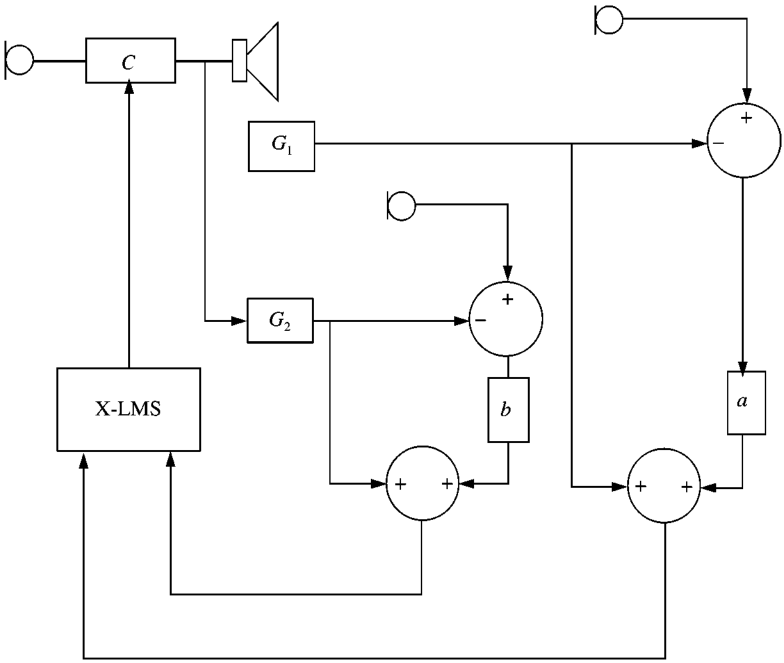

Figure 1. Diagram of the electroacoustic channels for a control using one secondary source and two control microphones weighted by a diagonal matrix $\mathscr{D}$, the terms of which are a and b.

resulting in $\tilde{J}_{res}^{c,min}(\boldsymbol{\phi}_n) = 0$: i.e., $\| - \mathbf{G}_c.\mathbf{H}^{-1}.\mathbf{G}^*.\mathbf{p}_n + \mathscr{D}.\mathbf{p}_n^c\|^2 = 0$ from which

$$\mathscr{D}.\mathbf{p}_n^c = \mathbf{G}_c.\mathbf{H}^{-1}.\mathbf{G}^*.\mathbf{p}_n, \tag{4}$$

where the vector on the right-hand side is well determined. With equation (4) it appears that the weighting matrix $\mathscr{D}$ is, for each sensor configuration, unique when diagonal. Indeed there are as many equations (N_c) as unknowns. On the contrary, for a non-diagonal weighting matrix, there are fewer equations than unknowns and an infinity of matrices exists which allow one to reach the optimal attenuation at the N sensors: when $\mathscr{D}$ is a full matrix, the system has N_c equations for $N_c \times N_c$ unknowns. Thus equation (4) shows that it is always possible to obtain efficient sensor configurations, the price to be paid being the introduction of a weighting matrix, the role of which is to modify the acoustic pressure at the N_c microphones. Figures 1 and 2 show schematically the insertion of the filter $\mathscr{D}$, respectively, diagonal and full, in the diagram of electroacoustic channels for adaptive control.

The trend consisting in locating the error sensors at the nodes of the residual field [3] is a particular case of equation (4). Indeed, at the nodes of the residual field, one has $\mathbf{p}_{sec}^c = -\mathbf{p}_n^c$ ($\mathbf{p}_{sec}^c$ is the pressure radiated at the N_c microphones by the secondary sources: i.e., $-\mathbf{G}_c.\mathbf{H}^{-1}.\mathbf{G}^*.\mathbf{p}_n$) from which it can be seen that $\mathscr{D}$ is the identity matrix $\mathscr{T}$. The control is thus $\mathbf{v}_n = \boldsymbol{\phi}_n$. Recall that the necessary condition to find $\boldsymbol{\phi}_n$ is $N_c \geqslant N_s$.

Here one can digress to answer the following question: can the error sensors be located at any nodal position: i.e., where the residual pressure is perfectly zero? The response is yes according to the reasoning that follows. With $\mathscr{D} = \mathscr{T}$, one has $\tilde{J}_{res}^c(\mathbf{v}) = J_{res}^c(\mathbf{v}) = \sum_{i=1}^{N_c} |p_{res}(\mathbf{x}_i)|^2$ where $p_{res}(\mathbf{x}_i)$ is the residual pressure at point $\mathbf{x}_i$ made up of the primary and the secondary pressures. When $\mathbf{v}$ has the optimal value, the function reaches its minimum

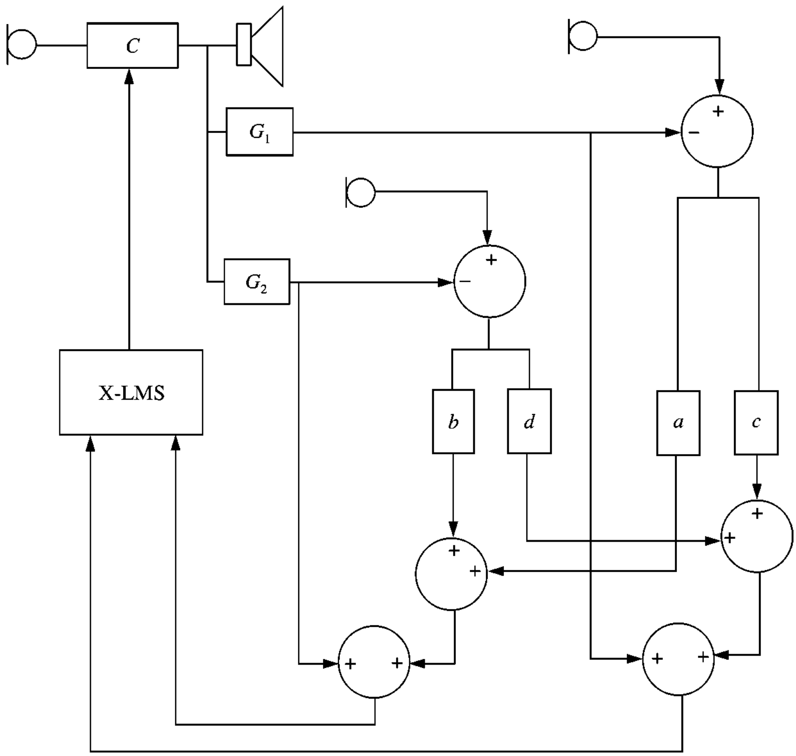

Figure 2. Diagram of the electroacoustic channels for a control using one secondary source and two control microphones weighted by a full matrix $\mathscr{D}$, the terms of which are a and b on the first row, c and d on the second row.

value $J_{res}^{c,min}$. Suppose now that there exists one location $\mathbf{x}_j$ where $|p_{res}(\mathbf{x}_j)|^2 = 0$ for $\mathbf{v} = \mathbf{v}_n$. As $p_{res}(\mathbf{x}_j) = p_n(\mathbf{x}_j) + \sum_{s=1}^{N_s} G(\mathbf{x}_j, \mathbf{x}_s)\, v_s = p_n(\mathbf{x}_j) + \mathbf{g}_j^t.\ \mathbf{v}$, it appears that $|p_{res}(\mathbf{x}_j)|^2$ is also a quadratic function of $\mathbf{v}$. If this quadratic function is zero at $\mathbf{v}_n$ and since $\mathbf{g}_j^*.\mathbf{g}_j$ is positive definite, this zero is a minimum for $|p_{res}(\mathbf{x}_j|^2$: i.e., $\mathbf{v}_n$ is the solution of the programming $\min_\mathbf{v} |p_{res}(\mathbf{x}_j)|^2$. The same reasoning applies when considering more than one nodal location.

The remote microphone technique [4] with the microphones located outside the control area can also be written in the same general way with $\mathscr{D} \neq \mathscr{T}$.

A posteriori the role played by the weighting matrix $\mathscr{D}$ can be understood in the following manner. At first, the goal is for the secondary sources to cancel out the residual level at N microphones, in other words to reach $\mathbf{p}_{res} \equiv \mathbf{0}$. As this goal is impossible to reach perfectly one must minimize the acoustic level in order to obtain the optimal attenuation and residual field. The latter is thus accessible thanks to the secondary sources. Now the objective consists in reaching this residual pressure $\mathbf{p}_{res}$ perfectly i.e., with zero distance from the objective. So $\mathbf{G}.\boldsymbol{\phi}$ was asked to reach $-(\mathbf{p}_{res} - \mathbf{0})$ while $\mathbf{G}.\mathbf{v}$ must reach $-(\mathbf{p}_n^c - \mathbf{p}_{res}^c)$. This last quantity, written $-\mathscr{D}.\mathbf{p}_n^c$, gives the role of $\mathscr{D}$ in equation (4).

It must be emphasized that the effort made to find a weighting matrix capable of leading to the optimal attenuation at the N microphones does not bring any new information concerning the results of the optimal control in the reference situation, as these results were necessary to obtain the weighting matrix. However, in the following study of sensor configuration robustness, the weighting matrix plays a fundamental role by making it possible to tackle the problem, resulting in new information outside the reference situation.

2.3. NOTIONS OF CONCEPT, DEFINITION AND MEASUREMENT IN THE CASE OF EFFICIENCY

The definition of efficiency for an active control system is nowadays so classical that there is no point in giving its roots before using it at the beginning of section 2. Concerning robustness, analyzed from section 3 to section 5, no tradition exists and the whole approach, going from the concept to its definition right through to its measurement, has to be written explicitly. Before using these three notions in the robustness analysis, the well-known example of efficiency is taken to clarify them.

Active control of sound fields is designed to reduce disturbing sounds; its goal is sound attenuation. A control system is efficient if it greatly attenuates the disturbing field. Attenuation is the concept associated with efficiency and can take various forms like $||\mathbf{p}_{res}| - |\mathbf{p}_n||$ or $|\mathbf{p}_{res}|/|\mathbf{p}_n|$, without yet giving the adequate norms.

The choice of the definition has been the ratio $\|\mathbf{p}_{res}\|/\|\mathbf{p}_n\|$ where the norms are L^2-norms. It so happens that this definition is a quantity easily reached, but it has to be remembered that other definitions exist [5].

The measurement of the attenuation is $-20\log_{10}(\|\mathbf{p}_{res}\|/\|\mathbf{p}_n\|)$. This quantity is also easily obtainable. It makes it possible to compare various control systems because the measurement is a monotonic function of $\|\mathbf{p}_{res}\|/\|\mathbf{p}_n\|$: i.e., the measurement always increases or decreases when the definition increases; presently, the measurement increases when the definition decreases. A system is all the more efficient the smaller the definition and the higher the value of the attenuation measured.

Concerning efficiency, the previous notions are so simple that their explanation brings nothing new. However, in general cases these notions may not be so obvious because concept and definition are not universally accepted, because the definition chosen for its accuracy or objectivity is not a quantity easily obtainable, because easily reachable quantities, the role of which could be to measure, are not rigorously monotonic functions of the definition. These difficulties seem to belong to the new field of research concerned by the robustness of active control systems against fluctuations of data.

3. CONCEPTS AND QUALITATIVE ASPECTS FOR SENSOR CONFIGURATION ROBUSTNESS

Contrary to efficiency, robustness has not received a universal meaning in the field of active noise control. That is why the first reflection consists in imagining what the concept of robustness could be.

Consider whatever primary field denoted as $\mathbf{p}$ at the N sensors. When the number N_c of control sensors is less than the number N of observation points, the optimal driving signals at the secondary sources, ordered in vector $\mathbf{v}$, are the solution of $\min_{\mathbf{v}} \|\mathbf{G}_c.\mathbf{v} + \mathscr{D}.\mathbf{p}^c\|^2$ i.e., $\mathbf{v} = -\mathbf{H}_c^{-1}.\mathbf{G}_c^*.\mathscr{D}.\mathbf{p}^c$, and $\mathbf{v}$ depends on $\mathbf{p}$ since $\mathbf{p}^c = \mathscr{P}.\mathbf{p}$ where $\mathscr{P}$ is a rectangular projection matrix representing the choice of N_c sensors from among N. The optimal control resulting from the N_c error sensors have thus the form

$$\mathbf{v} = -\mathbf{H}_c^{-1}.\mathbf{G}_c^*.\mathscr{D}.\mathscr{P}.\mathbf{p}. \tag{5}$$

For the sensor configuration under study, it has been seen that it is possible to find one diagonal matrix or an infinite number of full matrices $\mathscr{D}$ such that $\mathbf{v}_n = \boldsymbol{\phi}_n$, when $\mathbf{p} = \mathbf{p}_n$, thus leading to the optimal attenuation in the reference situation. But when $\mathbf{p}$ differs from $\mathbf{p}_n$ then $\mathbf{v}$, the optimal control to attenuate $\mathbf{p}^c$ at the N_c sensors, differs from $\boldsymbol{\phi}$, the optimal control to attenuate $\mathbf{p}$ at the N sensors. In those conditions, what can be expected from a control system where the number of control sensors is less than the number of observation points? For the time being, a very vague definition of the robustness of a sensor configuration is its capacity to attenuate a primary field $\mathbf{p}$ different from the primary field of reference $\mathbf{p}_n$. The objective of this third section is to reach a more precise idea. Three possible ways of presenting the concept of robustness, that is to say of finding what could be called a robust sensor configuration against errors in, or fluctuations of, the spatial distribution of the primary field, are proposed.

3.1. SEARCH FOR THE CONFIGURATION GUARANTEEING THE HIGHEST MINIMUM ATTENUATION

One way of classifying the sensor configurations according to their capacity to attenuate a non-reference acoustic field, consists in extending what has recently been done in the case where all observation points are control points [1]. For each primary field $\mathbf{p}$, and for each sensor configuration called configuration "c", there exists an optimal attenuation $A_c(\mathbf{p})$ resulting from the optimal control $\mathbf{v}(\mathbf{p})$. Indeed $A_c(\mathbf{p}) = -10\log_{10}(J_{res}^{min}/J)$ where J concerns the primary level associated with $\mathbf{p}$: i.e., $J = \|\mathbf{p}\|^2$ and where J_{res}^{min} is obtained with the optimal control $\mathbf{v}$ resulting from the N_c control sensors to reduce $\mathbf{p}$: i.e., $J_{res}^{min} = \|\mathbf{G}.\mathbf{v} + \mathbf{p}\|^2$. The control $\mathbf{v}$ minimizes $\tilde{J}_{res}^c = \|\mathbf{G}_c.\mathbf{v} + \mathscr{D}.\mathscr{P}.\mathbf{p}\|^2$ and has the form $\mathbf{v} = -\mathbf{H}_c^{-1}.\mathbf{G}_c^*.\,\mathscr{D}.\mathscr{P}.\mathbf{p}$ (equation (5)). Upon introducing

$\mathscr{A}_c = (\mathbf{H}_c^{-1}.\mathbf{G}_c^*.\mathscr{D}.\mathscr{P} - \mathbf{H}^{-1}.\mathbf{G}^*)^*.\mathbf{H}.(\mathbf{H}_c^{-1}.\mathbf{G}_c^*.\mathscr{D}.\mathscr{P} - \mathbf{H}^{-1}.\mathbf{G}^*)$ and $\mathscr{A} = \mathbf{G}.\mathbf{H}^{-1}.\mathbf{G}^*$, (equation 2) results in

$$\mathbf{A}_c(\mathbf{p}) = -10\log_{10}\left(\frac{\mathbf{p}^*.(\mathscr{T} - \mathscr{A} + \mathscr{A}_c).\mathbf{p}}{\|\mathbf{p}\|^2}\right). \tag{6}$$

Now define the set E_e by $E_e = \{\mathbf{p} = \mathbf{p}_n + \delta\mathbf{p}$ such that $\|\delta\mathbf{p}\|/\|\mathbf{p}_n\| \leqslant e\}$. By following the same method as was used to obtain the minimum attenuation guaranteed by a secondary source system in the presence of errors in the spatial distribution of the primary field, one could hope to obtain the minimum attenuation guaranteed by a secondary source system and a sensors configuration made up of N_c sensors instead of N. The formulation could be

$$A_c^{min}(e) = \min_{p \in E_e} A_c(\mathbf{p}). \tag{7}$$

Unfortunately, contrary to the case where $N_c = N$, it will be seen that $A_c^{min}(e)$ cannot be reached by analytical means but can be achieved by an exhaustive way or a genetic algorithm.

For a given value of 'e', the sensor configuration could be said to be all the more robust as the value of $A_c^{min}(e)$ is high. The approach above requires $\mathscr{D}$ to be known for each sensor configuration. It is thus suitable for the case where there is a single diagonal weighting matrix $\mathscr{D}$ for each configuration. For a full weighting matrix, the search for $\mathscr{D}$ that maximizes $A_c^{min}(e)$, seems impossible, apparently more for practical than for conceptual reasons. The method can thus be adopted only after having chosen one full weighting matrix from among the infinity, and more particularly after having optimized it to improve the robustness of the configuration thanks to another method that has to be devised.

3.2. SEARCH FOR THE CONFIGURATION ATTENUATING THE LARGEST SET OF PRIMARY FIELDS

A second way of dealing with the problem of sensor configuration robustness has already been published [6, 7]. However, it will be presented here in a slightly different manner without changing the result. In this paper, the new information presented concerning this approach lies in the numerical and experimental tests described in sections 5.1.2. and 5.1.3.

For the primary field of reference, written in the form of a vector $\mathbf{p}_n$ filled with the sound pressures at the N observation points, the vector of the driving signals $\mathbf{v}(\mathbf{p}_n)$ applied to the secondary sources, obtained from a number N_c of control sensors, leads to the optimal attenuation in the whole area Ω where sound attenuation is sought, defined here by the number of observation points N larger than N_c. For non-reference primary fields $\mathbf{p}$, the driving signals $\mathbf{v}(\mathbf{p})$ no longer result in the optimal attenuation in Ω, and even amplification is possible. In a very vague manner again, a sensor configuration can be said to be robust if it is capable of reducing a non-reference primary field; in other words if $\mathbf{v}(\mathbf{p})$ attenuates $\mathbf{p}$. In this approach to the problem, another consideration has to be taken into account: attempting to reduce the primary level $J(\mathbf{p}_n)$ expressed in dB, for the situation of reference, shows that this level is high, say 100 dB. When the primary field departs from that of reference $\mathbf{p}_n$ and becomes $\mathbf{p}$, the primary sound level $J(\mathbf{p})$ can be still higher, say 120 dB. So the control $\mathbf{v}(\mathbf{p})$ has to attenuate not only $J(\mathbf{p})$ but also $J(\mathbf{p}_n)$ as this latter level was already considered to be high. More generally, $\mathbf{v}(\mathbf{p})$ must reduce the weakest level of both. With what will be called the sensor configuration c, let us associate the set E_c of fields $\mathbf{p}$ such that the driving signals $\mathbf{v}(\mathbf{p})$ satisfy what is needed:

$$E_c = \{\mathbf{p} = \mathbf{p}_n + \delta\mathbf{p} \text{ such that } \mathbf{v}(\mathbf{p}) \text{ attenuates } J(\mathbf{p}) \text{ and } \mathbf{v}(\mathbf{p}) \text{ attenuates } J(\mathbf{p}_n)\}.$$

It will be shown that the set E_c could be written as $E_c = \{\mathbf{p} = \mathbf{p}_n + \delta\mathbf{p}$ such that $\|\delta\mathbf{p}\|/\|\mathbf{p}\| \leqslant e_{\mathscr{AC}}\}$ where the value of $e_{\mathscr{AC}}$ is totally dependent on the configuration and easily calculated. The largest set is the one with the largest indicator $e_{\mathscr{AC}}$. The condition under which it represents the most robust configuration will be given.

Knowledge of $\mathscr{D}$ for each configuration is needed for the above approach which is well suited to the case of a single diagonal weighting matrix. It might also have been useful to optimize a full weighting matrix in order to maximize the indicator $e_{\mathscr{AC}}$ and it will be explained later why this path was abandoned.

3.3. SEARCH FOR THE CONFIGURATION MOST CLOSELY "RESEMBLING" THE IDEAL CASE

It has been mentioned that an indicator other than $A_c^{min}(e)$ and $e_{\mathscr{AC}}$ had to be found. The one imagined is similarity, defined as a norm of the difference between two matrices. The first matrix is the matrix $\mathscr{A}$ related to the situation where all observation points are control sensors. The second, written $\mathbf{G}.\mathbf{H}_c^{-1}.\mathscr{D}.\mathscr{P}$, is partly related to the situation where the number of control sensors N_c is less than the number of observation points N where attenuation is sought. The similarity so defined depends on $\mathscr{D}$. When comparing various configurations, each with its single diagonal weighting matrix $\mathscr{D}$, it will turn out that the best similarity, i.e., the smallest value of the difference, is associated with the configuration of greatest robustness obtained by the indicator $e_{\mathscr{AC}}$. When working with one configuration and a full matrix $\mathscr{D}$ filtering the control sensor outputs, the choice of $\mathscr{D}$ exists and there could be a possibility of minimizing the distance to improve the robustness. From this point of view, similarity has proved to be very successful.

4. CHOICE OF A QUANTITATIVE DEFINITION OF SENSOR CONFIGURATION ROBUSTNESS

From the three possible approaches regarding robustness, the indicator $e_{\mathscr{AC}}$ as well as the similarity will result from relatively easy calculations but they are not immediately interpretable from the quantitative point of view. The interpretation is clearer in the curves $A_c^{min}(e)$ and, besides, they appear to be less constrained by hypotheses. In these conditions, $A_c^{min}(e)$ will play the role of robustness definition while the two others, studied in section 5, will be tested as measurements of robustness.

4.1. DETERMINATION OF $A_c^{min}(e)$ BY AN EXHAUSTIVE WAY

Given a configuration of N_c control microphones, the problem, introduced in section 3.1, consists in searching for the minimum attenuation A_c^{min}, seen at the total number of observation points which constitute the whole acoustic area under study. The search is carried out for sound pressure fields around a field of reference. The relative errors of the fields considered are less than a predetermined value 'e'. The aim is to determine the most robust sensor configuration that, for a given value of 'e', is associated with the highest curve $A_c^{min}(e)$.

The starting equations are

$$A_c^{min}(e) = \min_{p \in E_e} A_c(\mathbf{p}), \qquad E_e = \{\mathbf{p} = \mathbf{p}_n + \delta\mathbf{p} \text{ such that } \|\delta\mathbf{p}\|/\|\mathbf{p}_n\| \leqslant e\}, \tag{8}$$

where $A_c(\mathbf{p})$, defined in equation (6), is the optimal attenuation of the primary pressure $\mathbf{p}$, obtained with the configuration made up of the N_c sensors.

The attenuation $A_c(\mathbf{p})$ depends only on the direction of the primary field vector $\mathbf{p}$. It turns out that, thanks to this property, the quantity $\varepsilon(\mathbf{p}) = \|\delta\mathbf{p}\|/\|\mathbf{p}_n\| = \|\mathbf{p}_n - \mathbf{p}\|/\|\mathbf{p}_n\|$ may be minimized without changing the solution of problem (8). This leads to

$$\varepsilon_{min}(\mathbf{p}) = \left(\frac{\|\mathbf{p}_n - \mathbf{p}\|}{\|\mathbf{p}_n\|}\right)_{min} = \frac{\left\|\mathbf{p}_n - \dfrac{\mathbf{p}^*.\mathbf{p}_n}{\|\mathbf{p}\|^2}\,\mathbf{p}\right\|}{\|\mathbf{p}_n\|},$$

the value of which must now be less than or equal to the quantity denoted as e_{min} [1].

The search of the curve $A_c^{min}(e_{min})$ by an exhaustive way consists in working with a large number of arbitrary primary fields $\mathbf{p}$. For each $\mathbf{p}$, e_{min} and the optimal attenuation A_c are calculated. A cloud of points is thus obtained in the co-ordinate system (e_{min}, A_c). The curve $A_c^{min}(e_{min})$ is the inferior limit of the cloud of points.

In an attempt to obtain more directly this curve, an effort has been made to find the problem of which $A_c^{min}(e_{min})$ is the solution.

4.2. PROBLEM OF WHICH $A_c^{min}(e_{min})$ IS THE SOLUTION

Since the attenuation $A_c(\mathbf{p})$ depends only on the direction of the primary field vector, for each pressure $\mathbf{p}$, one can take among all the fields of the same direction the field $\mathbf{p}''$ such that $\mathbf{p}''^*.\mathbf{p}_n = \|\mathbf{p}''\|^2$. The constraint

$$\frac{\left\|\mathbf{p}_n - \dfrac{\mathbf{p}''.\mathbf{p}_n}{\|\mathbf{p}''\|^2}\,\mathbf{p}''\right\|}{\|\mathbf{p}_n\|} \leqslant e_{min}$$

becomes $\|\mathbf{p}_n - \mathbf{p}''\|/\|\mathbf{p}_n\| \leqslant e_{min}$ or $\|\mathbf{p}''\|^2 \geqslant (1 - e_{min}^2)\|\mathbf{p}_n\|^2$ via $\|\mathbf{p}_n - \mathbf{p}''\|^2 = (\mathbf{p}_n - \mathbf{p}'')^*(\mathbf{p}_n - \mathbf{p}'')$. The problem to be solved is now

$$A_c^{min}(e_{min}) = \min_{\mathbf{p}''} A_c(\mathbf{p}''), \qquad \mathbf{p}''^*.\mathbf{p}_n = \|\mathbf{p}''\|^2, \qquad \|\mathbf{p}''\|^2 \geqslant (1 - e_{min}^2)\|\mathbf{p}_n\|^2. \tag{9}$$

From now on, $\mathbf{p}''$ will be denoted $\mathbf{p}$.

Before being solved, problem (9) has still to be modified. In the development of $A_c(\mathbf{p})$ in equation (6), denoted $\mathscr{M} = \mathscr{T} - \mathscr{A} + \mathscr{A}_c$. If $\mathscr{S}$ and $\mathscr{U}$ are, respectively, the diagonal and full matrices filled with the eigenvalues and eigenvectors of $\mathscr{M}$, then $\mathscr{M} = \mathscr{U}.\mathscr{S}.\mathscr{U}^*$. By changing $\mathbf{p}$ into $\tilde{\mathbf{p}}$ such that $\tilde{\mathbf{p}} = \mathscr{U}^*.\mathbf{p}$, $A_c(\mathbf{p})$ has the form

$$A_c(\mathbf{p}) = -10\log_{10}\left(\frac{\mathbf{p}^*.(\mathscr{T} - \mathscr{A} + \mathscr{A}_c).\mathbf{p}}{\|\mathbf{p}\|^2}\right) = -10\log_{10}\left(\frac{\tilde{\mathbf{p}}^*.\mathscr{S}.\tilde{\mathbf{p}}}{\|\tilde{\mathbf{p}}\|^2}\right) = -10\log_{10}\left(\frac{\sum_i s_i|\tilde{p}_i|^2}{\|\tilde{\mathbf{p}}\|^2}\right).$$

It will appear *a posteriori* that, in problem (9), $A_c^{min}(e_{min})$ decreases when e_{min} increases and thus the constraint $\|\mathbf{p}_n - \mathbf{p}''\|/\|\mathbf{p}_n\| \leqslant e_{min}$ can also be written as $\|\mathbf{p}_n - \mathbf{p}''\|/\|\mathbf{p}_n\| = e_{min}$ without any consequence on the solution of the problem. The information makes it possible to replace, in problem (9), the constraint $\|\mathbf{p}\|^2 \geqslant (1 - e_{min}^2)\|\mathbf{p}_n\|^2$ by $\|\mathbf{p}\|^2 = (1 - e_{min}^2)\|\mathbf{p}_n\|^2$. This results in the minimization of A_c being replaced by the following maximization: $\max_{\tilde{p}} \sum_i s_i|\tilde{p}_i|^2$ (upon remembering that $\|\tilde{\mathbf{p}}\|^2 = \|\mathbf{p}\|^2$).

Finally changing $\mathbf{p}$ into the variable $\mathbf{x}$, where the positive real components are $\mathbf{x}_i = |\tilde{p}_i|$, results in the last form of the problem, which is an extension of the formulation written when all the sensors are control sensors but with coefficients s_i, the values of which are no longer 0 or 1:

$$\max_{\mathbf{x}} f(\mathbf{x}) \quad \text{where} \quad f(\mathbf{x}) = \sum_i s_i \mathbf{x}_i^2,$$

$$g(\mathbf{x}) \leqslant 0 \quad \text{where} \quad g(\mathbf{x}) = (1 - e_{min}^2)\|\mathbf{x}_n\|^2 - \sum_i \mathbf{x}_i^2,$$

$$h(\mathbf{x}) = 0 \quad \text{where} \quad h(\mathbf{x}) = \sum_i \mathbf{x}_i \mathbf{x}_{n_i} - \sum_i \mathbf{x}_i^2. \tag{10}$$

Unfortunately, no analytical solution to problem (10) can be obtained. The solution may be found by a genetic algorithm.

It has to be noted here that there is no constraint on $\mathbf{p} = \mathbf{p}_n + \delta\mathbf{p}$ in the previous formulation of the problem and the method for finding its solution is given in the next paragraph. In fact in the real world, $\mathbf{p}$ and $\mathbf{p}_n$ both satisfy the same boundary conditions. Until now it has not been found how to formulate the problem that, taking into account these boundary conditions, has $A_c^{min}(e_{min})$ for solution. In such a case there is no choice but to determine $A_c^{min}(e_{min})$ by exhaustive means.

4.3. SOLUTION OBTAINED BY GENETIC ALGORITHM AND NUMERICAL SIMULATIONS

The procedure of the method, the main lines of which are given in Appendix A, is applied to the problem for each value of e_{min}.

The vector-individuals $\mathbf{x}$ belong to $\Re^+$ in the sense that each of their N (number of observation points) components x_i belongs to $\Re^+$. To satisfy the constraint $h(\mathbf{x}) = 0$, a component k depends on the $(N-1)$ others in each vector. Indeed the constraint is also written as

$$x_k^2 - x_k x_{n_k} - \left(\sum_{i \neq k} x_i x_{n_i} - x_i^2 \right) = 0 \quad \text{or} \quad x_k = \frac{x_{n_k} \pm \sqrt{x_{n_k}^2 + 4\sum_{i \neq k}(x_i x_{n_i} - x_i^2)}}{2}.$$

This component has to be real and positive, and the following inequalities must apply:

$$x_{n_k}^2 + 4 \sum_{i \neq k} (x_i x_{n_i} - x_i^2) \geqslant 0, \quad x_{n_k} \pm \sqrt{x_{n_k}^2 + 4 \sum_{i \neq k} (x_i x_{n_i} - x_i^2)} \geqslant 0.$$

Thus, a population with M individuals (around 40) is built by choosing randomly $(N-1)$ components and then deducing by calculation the remaining one in order to satisfy $h(\mathbf{x}) = 0$. The vector calculated in this way is kept only if it is real and positive. The vector must also satisfy $g(\mathbf{x}) \leqslant 0$. If not, it is rejected. Thus the first population is obtained (around 40 individuals).

For the members of the population, the values of $f(\mathbf{x})$ are calculated and the individuals $\mathbf{x}$ are classified in the order of the decreasing values of $f(\mathbf{x})$.

The parents are selected and their components are crossed and/or mutated. Only the products which satisfy the constraints are kept. The set of around 40 children constitutes the next generation. $f(\mathbf{x})$ is calculated and the individuals $\mathbf{x}$ are classified. If necessary the procedure carries on by returning to the selection of the individuals who are going to play the role of parents.

The algorithm stops when the maximum value of $f(\mathbf{x})$ no longer increases or is sufficiently high and the value of $A_c^{min}(e_{min})$ is finally deduced.

Numerical simulations will confirm that the minimum attenuation obtained by the method really is the lowest limit of all possible attenuations. Only one secondary source is present. Three sensor configurations are made up of two microphones chosen from among three that constitute the whole acoustic area of interest. The data are thus $N = 3$, $N_c = 2$ and $C_3^2 = 3$. Arbitrary transfer functions and primary field of reference are given.

First the exhaustive way is applied. Each of the three configurations is submitted to the control of a large number of primary fields (2000) which differ from the reference primary field by their spatial distribution. To this end, it is sufficient to add, at each point $\mathbf{x}$, a not too large random value of pressure to the pressure of reference. The error-attenuation couples form a cloud of points.

Then the genetic algorithm is implemented for each of the three configurations. If it has converged towards the curve of minimum attenuation against the minimized error e_{min}, the overall cloud must stay above this curve. Figures 3–5 present the results. The minimum attenuations calculated are very near those observed, the greatest difference being of 0·3 dB. On each of the three figures, there exists a value of e_{min} beyond which it is impossible to ensure no amplification; respectively beyond $e_{min}^{max} \approx 0{\cdot}7$, 0·6 and 0·47, amplifications are liable to occur. With a PC equipped with a clock at 300 MHz, the calculations of the minimum attenuation lasted almost 5 h! This is the practical reason why the definition of robustness is not easily determined, as long as this genetic algorithm is used. Moreover, as the duration is for a given $\mathscr{D}$, it is impossible with this technique to improve $A_c^{min}(e_{min})$ by optimizing $\mathscr{D}$.

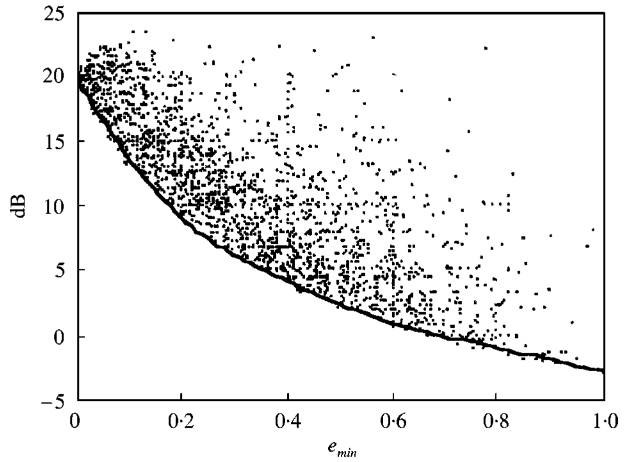

Figure 3. Minimum attenuation in dB against minimized relative error in the primary field. The curve results from a genetic algorithm applied to a configuration of two sensors, the role of which is to reduce the sound field at three sensors. The cloud made up of 2000 optimal attenuations associated with 2000 primary fields remains above the lowest limit.

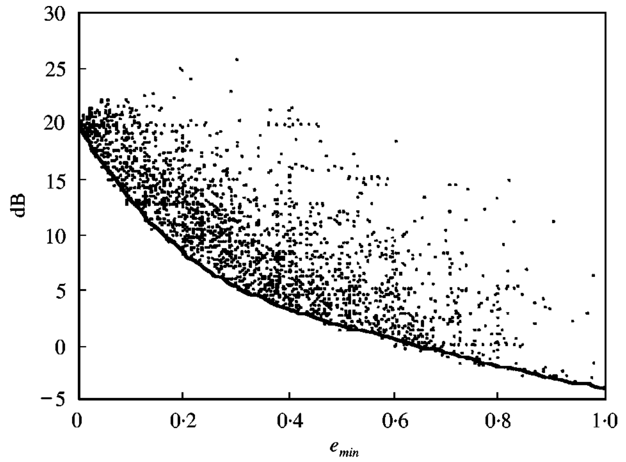

Figure 4. As Figure 3, but for the second configuration of two microphones from among three.

5. MEASUREMENT OF THE SENSOR CONFIGURATION ROBUSTNESS

From among the three concepts of robustness presented in section 3, $A_c^{min}(e_{min})$ has been chosen as the definition as it results from the smallest number of restrictive hypotheses. Unfortunately, as has been shown, it is not always easily accessible from the quantitative point of view. Are the other two concepts more easily accessible and are they able to give an insight into the robustness defined by $A_c^{min}(e_{min})$? The present section shows that the indicator $e_{\mathscr{AC}}$ arising from sufficient conditions only, is easily calculated for each sensor configuration but it reveals robustness mainly when its value is high. On the contrary, the similarity between two matrices, also easily calculated, reveals robustness in the sense that

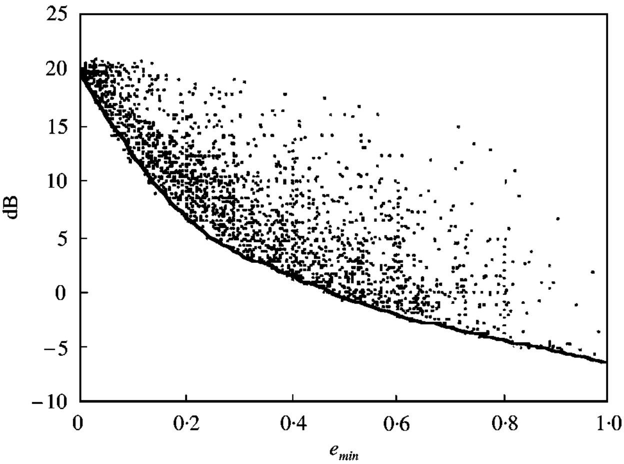

Figure 5. As Figure 3, but for the third configuration of two microphones from among three.

minimizing the distance between the two matrices maximizes the value of $A_c^{min}(e_{min})$. At this stage in the investigation, it could transpire that this similarity might be a promising indicator offering the characteristics of a measurement.

5.1. SUFFICIENT CONDITIONS AND VALUE OF $e_{\mathscr{AC}}$ AS AN INDICATOR OF ROBUSTNESS

At the end of section 4.3 where Figures 3–5 were presented, it was emphasized that for each curve a value e_{min}^{max} exists beyond which one cannot guarantee the absence of amplification of the primary level $J(\mathbf{p})$ associated with $\mathbf{p}$. In the same figures one can see that the higher the value of e_{min}^{max}, the better the robustness; i.e., the higher the curve $A_c^{min}(e_{min})$. Had it been easy to access the value of e_{min}^{max}, robustness would have been easily obtained. Until now the research in this direction has not given results. However, it has been possible to determine an indicator, of a simple form at the end of the demonstration, which can guarantee the absence of amplification of $J(\mathbf{p})$, among others. This coefficient is more restrictive than e_{min}^{max}. Is the restriction a handicap to find the robustness?

5.1.1. *Approach leading to the indicator* $e_{\mathscr{AC}}$

In the general presentation of section 3.2, the search for the set E_c has been justified: each sensor configuration called configuration "*c*" is associated with an optimal control $\mathbf{v}$ (obtained from the minimization of $\tilde{J}_{res}^c$) for each pressure field $\mathbf{p}$ as well as with a set E_c defined by

$$E_c = \{\mathbf{p} = \mathbf{p}_n + \delta\mathbf{p} \text{ such that } J_{res}(\mathbf{v}, \mathbf{p}) \leqslant \mathbf{J}(\mathbf{p}) \text{ and } J_{res}(\mathbf{v}, \mathbf{p}) \leqslant J(\mathbf{p}_n)\}.$$

To demonstrate that it is sufficient (and not necessary) for E_c to be defined by

$$E_c = \{\mathbf{p} = \mathbf{p}_n + \delta\mathbf{p} \text{ such that } \|\delta\mathbf{p}\|/\|\mathbf{p}_n\| \leqslant e_{\mathscr{AC}}\}$$

in order to satisfy the constraints $J_{res}(\mathbf{v}, \mathbf{p}) \leqslant J(\mathbf{p})$ and $J_{res}(\mathbf{v}, \mathbf{p}) \leqslant J(\mathbf{p}_n)$, a slightly different approach from those already presented [6, 7] is written here. The consideration are still of geometrical nature in the space of the driving signals or controls of the secondary sources.

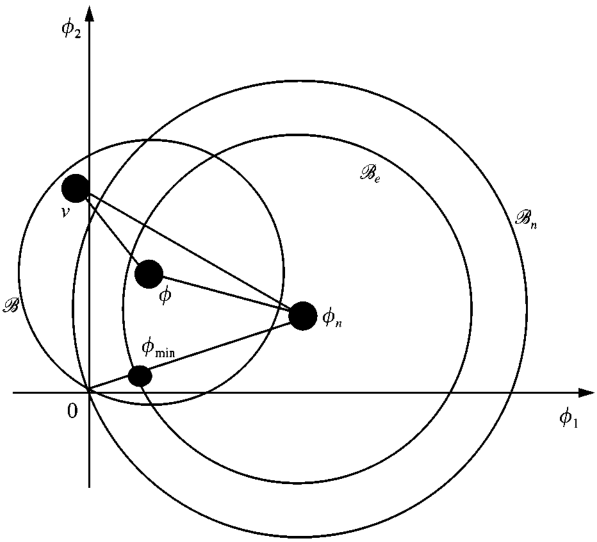

Figure 6. Geometrical representation to show that care must be taken not to amplify $J(\mathbf{p}_n)$; here two secondary sources are driven by real signals ϕ_1 and ϕ_2.

The approach is based on the following knowledge: in the reference situation, the optimal attenuation of $J(\mathbf{p}_n)$ through R_n obtained with $\boldsymbol{\phi}_n$ (see section 2.1); the relative error of the perturbed primary pressure field $\mathbf{p}$ to the primary pressure field of reference $\mathbf{p}_n$ via $\varepsilon = \|\delta\mathbf{p}\|/\|\mathbf{p}_n\| \leqslant e$; the control microphone configuration with its weighting matrix $\mathscr{D}$—the role of which in the reference situation is to give $\mathbf{v}_n = \boldsymbol{\phi}_n$—leading to $\mathbf{v} = -\mathbf{H}_c^{-1}.\mathbf{G}_c^*.\mathscr{D}.\mathscr{P}.\mathbf{p}$ (equation (5)). For the time being only the case of diagonal $\mathscr{D}$ is considered.

The main lines of the approach will now be described and Figure 6 illustrates the approach from a geometrical viewpoint. The demonstrations of each assertion are reported in Appendix B.

A property largely used in what follows is a prelude to the approach. Let $\boldsymbol{\beta}$ be the optimal control vector of optimal driving signals that best attenuates the sound level $\mathbf{I}(\mathbf{q})$ associated with the primary acoustic pressure field $\mathbf{q}$. Any control $\boldsymbol{\alpha}$, non-optimal, will attenuate $\mathbf{I}(\mathbf{q})$—more precisely will not amplify—if and only if $\|\boldsymbol{\alpha} - \boldsymbol{\beta}\| \leqslant \|\boldsymbol{\beta}\|$; in other terms if and only if $\boldsymbol{\alpha}$ belongs to the bowl $\mathscr{B}_\beta$ located in the control space with N_s dimensions (with N_s secondary sources, there are N_s driving signals) and with a distance defined by the H-norm (see section 2.1). The bowl $\mathscr{B}_\beta$ is centered at $\boldsymbol{\beta}$ and of radius $\|\boldsymbol{\beta}\|$.

One can now envisage the control $\mathbf{v}(\mathbf{p})$. It attenuates $J(\mathbf{p})$ if $\mathbf{v} \in \mathscr{B}$, bowl centered at $\boldsymbol{\phi}$ and of radius $\|\boldsymbol{\phi}\|$, where $\boldsymbol{\phi}$ is the optimal control to attenuate $J(\mathbf{p})$. The control $\mathbf{v}(\mathbf{p})$ attenuates $J(\mathbf{p}_n)$ if $\mathbf{v} \in \mathscr{B}_n$, bowl centered at $\boldsymbol{\phi}_n$ and of radius $\|\boldsymbol{\phi}_n\|$, where $\boldsymbol{\phi}_n$ is the optimal control to attenuate $J(\mathbf{p}_n)$. The control $\mathbf{v}(\mathbf{p})$ satisfies both attenuations at the intersection of the two bowls but one has not managed to eliminate in a simple manner $\boldsymbol{\phi}$ which is not included in the knowledge on which the approach is based. That is why only sufficient conditions have been sought.

The sufficient condition found for $\mathbf{v} \in \mathscr{B}$ is $\|\mathbf{v} - \boldsymbol{\phi}\| \leqslant \|\mathbf{v} - \boldsymbol{\phi}_n\| + (e/\sqrt{R_n})\,\|\boldsymbol{\phi}_n\|$. This information is obtained after having demonstrated that $\boldsymbol{\phi} \in \mathscr{B}_e$ where $\mathscr{B}_e$ is the bowl centered at $\boldsymbol{\phi}_n$, the radius of which is $(e/\sqrt{R_n})\,\|\boldsymbol{\phi}_n\|$.

The sufficient condition found for $\mathbf{v} \in \mathscr{B}_n$ is $\|\mathbf{v} - \boldsymbol{\phi}\| \leqslant (1 - e/\sqrt{R_n})\,\|\boldsymbol{\phi}_n\|$, thanks to a geometrical observation.

At this stage it is possible to eliminate $\boldsymbol{\phi}$ between the above two sufficient conditions to obtain $\|\mathbf{v}-\boldsymbol{\phi}_n\| \leqslant (1-2e/\sqrt{R_n})\,\|\boldsymbol{\phi}_n\|$. But $\|\mathbf{v}-\boldsymbol{\phi}_n\|^2 = (\mathbf{v}-\boldsymbol{\phi}_n)^*.\mathbf{H}.(\mathbf{v}-\boldsymbol{\phi}_n)$ may be expressed from $\delta\mathbf{p}$ and from a matrix $\mathscr{C}$ associated with the sensor configuration and its $\mathscr{D}$ matrix to keep it efficient.

It results in

$$\frac{\|\delta\mathbf{p}\|}{\|\mathbf{p}_n\|} \leqslant \frac{\sqrt{R_n}}{\sqrt{\|\mathscr{C}\|+2}} = e_{\mathscr{A}\mathscr{C}}. \tag{11}$$

This shows, therefore, that the set E_c of perturbed primary fields, such that $\mathbf{v}(\mathbf{p})$ surely attenuates $J(\mathbf{p})$ as well as $J(\mathbf{p}_n)$ is all the larger the higher the $e_{\mathscr{A}\mathscr{C}}$. With $\mathscr{D}$ diagonal, unique for each sensor configuration, is the configuration all the more robust the higher the $e_{\mathscr{A}\mathscr{C}}$, where the robustness is as defined in section 4? In other words, does there exist a good relation of order between $e_{\mathscr{A}\mathscr{C}}$ and $A_c^{min}(e_{min})$ that could lead to considering $e_{\mathscr{A}\mathscr{C}}$ as a measurement? Notwithstanding the fact that the trends are encouraging, some results from the following experiment will prevent one from reaching a definitive conclusion.

5.1.2. *Numerical tests and comparison between the relation of order of the indicator $e_{\mathscr{A}\mathscr{C}}$ and the robustness $A_c^{min}(e_{min})$*

A first set of sensor configurations is tested. They are extracted from Ω made up of four sensors of which two are control microphones. There are $C_4^2 = 6$ sensor configurations. The curves $A_c^{min}(e_{min})$ are obtained here by exhaustive means: given one secondary source, the arbitrary transfer functions and a primary field of reference, the six control configurations are working with a great number of primary fields, the spatial distributions of which vary. The minimized relative error and the attenuation accompany each primary field. The error-attenuation couples form a cloud of points that is located above the minimum attenuation. An analytical formula is used to plot the minimum attenuation when all four sensors constitute the whole acoustic area [1].

Each of the six configurations is provided with a single diagonal matrix $\mathscr{D}$ and thus with the coefficient $e_{\mathscr{A}\mathscr{C}}$. To put this coefficient to the test, one has to see if the relation of order between the indicators $e_{\mathscr{A}\mathscr{C}}$, corresponds to the relation of order between the configurations' minimum guaranteed attenuations: i.e., in the curves $A_c^{min}(e_{min})$ of the configurations, defined as robustness in section 4. Figure 7 shows the correspondence between the value of the

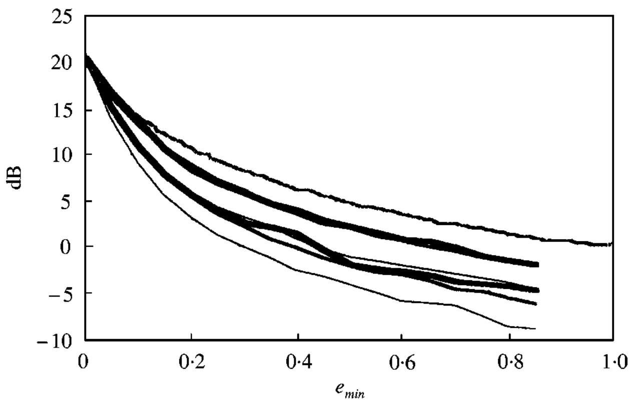

Figure 7. Comparison between the relations of order originating from the index $e_{\mathscr{A}\mathscr{C}}$ and from the minimal attenuation $A_c^{min}(e_{min})$ in dB obtained exhaustively (numerical simulations). ▬▬, 0·2804; ——, 0·2691; ——, 0·2182; ▬▬, 0·2102; ——, 0·2098; ——, 0·1805.

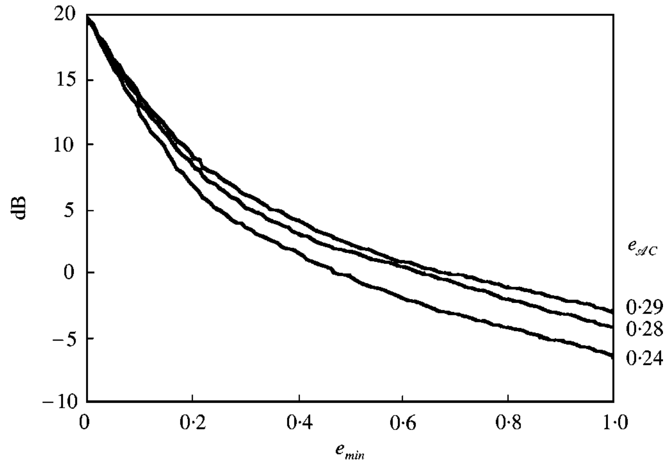

Figure 8. Comparison between the relations of order originating from the index $e_{\mathscr{AC}}$ and from the minimum attenuation $A_c^{min}(e_{min})$ in dB obtained by a natural method (numerical simulations).

indicator $e_{\mathscr{AC}}$ and the curve of minimum attenuation $A_c^{min}(e_{min})$ associated with each sensor configuration. The highest minimum attenuation corresponds to the set of four sensors: i.e., the whole domain Ω. Here, the observed relation of order arising from the indicators $e_{\mathscr{AC}}$ for the six configurations of two sensors weighted by a diagonal matrix is globally the same as the relation of order given by the minimum attenuations. However, the figure also shows what prevents one from reaching a definitive conclusion: the highest curve of $A_c^{min}(e_{min})$ may not correspond exactly to the highest value of $e_{\mathscr{AC}} = 0{\cdot}2804$ but to the value $e_{\mathscr{AC}} = 0{\cdot}2691$, slightly below.

The second numerical test is carried out with two control microphones extracted from Ω, made up of three sensors. There are $C_3^2 = 3$ sensor configurations. The same procedure as before is applied except that the minimum attenuation arises here from a natural method. Figure 8 shows here a relation of order arising from $e_{\mathscr{AC}}$, in close agreement with the one obtained from $A_c^{min}(e_{min})$ (the minimum attenuation for the whole domain Ω is not given here).

In these purely numerical tests, where all sound pressure fields of a given relative error may have any form, the relations of order due to $e_{\mathscr{AC}}$ and $A_c^{min}(e_{min})$ are in satisfactory agreement. This has been confirmed by other tests.

5.1.3. *Experimental tests*

Does the previous conclusion hold with experimental tests where the various primary fields inevitably satisfy the particular boundary conditions of the set-up?

The goal is the same as before: to compare the relations of order due to $e_{\mathscr{AC}}$ and $A_c^{min}(e_{min})$. To this end various sensor configurations measure a primary field submitted to spatial distribution variations. The experiments take place in a rectangular acoustic cavity. Figure 9 shows the transducer locations. Two sets of microphones are taken as two different acoustic areas. Each is made up of six microphones. The sensor configurations work with two microphones, leading to $C_6^2 = 15$ sensor configurations for each set. The frequency is 200 Hz, located between the two eigenfrequencies corresponding to modes 010 and 110 of the cavity. Two primary sources allow one to generate the spatial distribution of the primary field: one radiates a field of constant amplitude and phase, while the other radiates

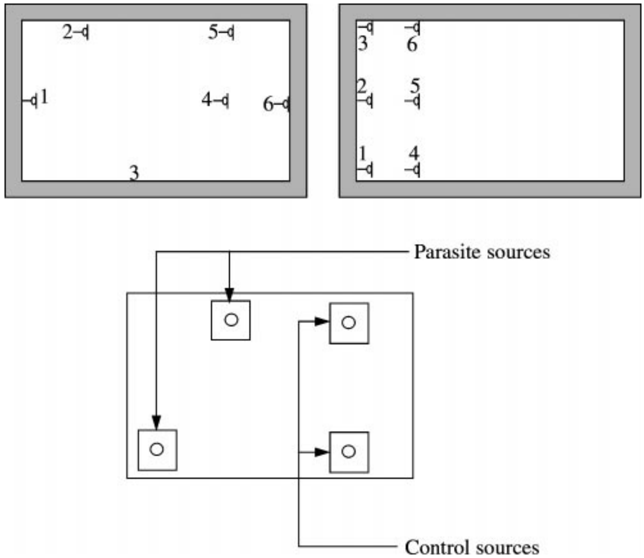

Figure 9. Transducers in the cavity. Top: seen from above, two types of sensor locations 30 cm above the floor of the cavity. Below: seen from above, primary and secondary sources set on the ceiling of the cavity.

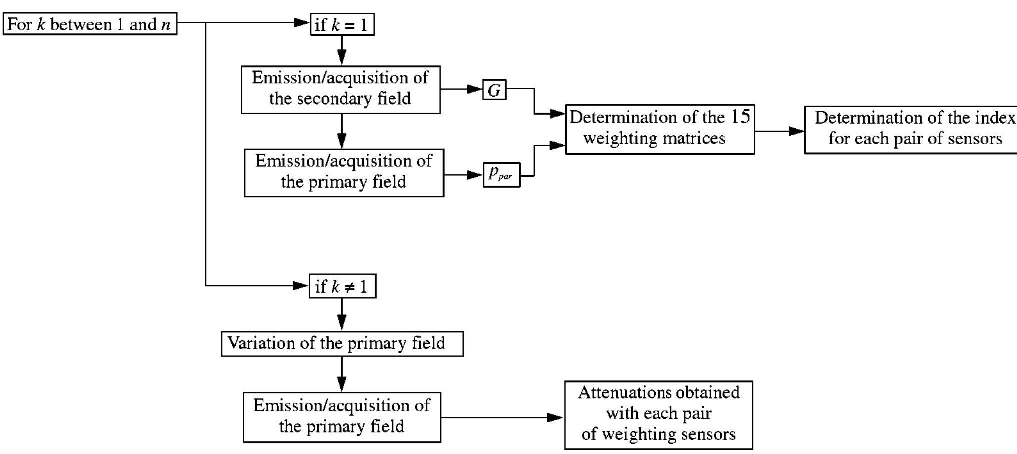

Figure 10. Flow chart of the experimental procedure.

a field the amplitude and the phase of which vary, thus leading to the spatial variation of the primary field. The other two sources are the secondary or control sources. A thousand primary fields are radiated. For each of them, the procedure is given in Figure 10.

Figure 11 shows, in the same way as before, the curves of the minimum attenuation for each sensor configuration obtained exhaustively—in fact five from among the 15 are shown—and the corresponding value of the indicator $e_{\mathscr{AC}}$. On the one hand, it is visible here that the highest value of $e_{\mathscr{AC}}$ corresponds to the highest curve of $A_c^{min}(e_{min})$. This remark may be generalized if a slight precaution is taken: whatever the experiments carried out, it is true that the highest curve of $A_c^{min}(e_{min})$ always corresponds to the highest value of $e_{\mathscr{AC}}$ as long as the value of $e_{\mathscr{AC}}$ is high. We have never noticed a high curve of $A_c^{min}(e_{min})$ corresponding to a weak value of the indicator $e_{\mathscr{AC}}$. On the other hand, the very poor correspondence of the

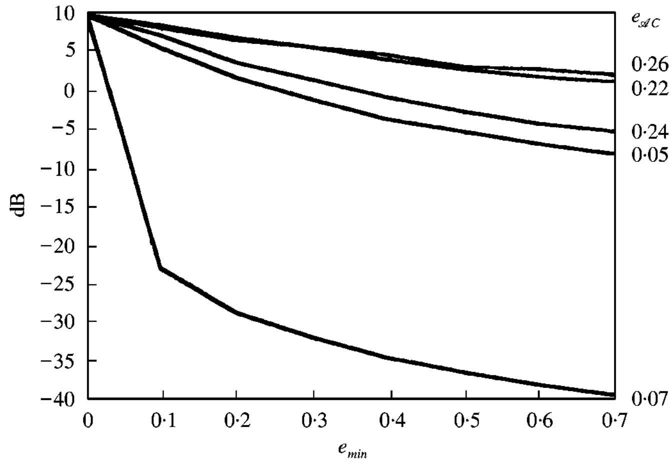

Figure 11. Comparison between the relations of order originating from the index $e_{\mathscr{AC}}$ and from the minimum attenuation $A_c^{min}(e_{min})$ in dB obtained exhaustively (experimental investigations).

relations of order between $e_{\mathscr{AC}}$ and $A_c^{min}(e_{min})$ appears here. This has always been the case in the experiments conducted and no conclusion can be drawn regarding the relations of order.

So the purely numerical and experimental tests show that if, among sensor configurations weighted with a single diagonal matrix, there exists one associated with an indicator of high value, it will provide an interesting curve of minimum attenuation. But one cannot begin to draw any conclusions about the relation of order for lower values of $e_{\mathscr{AC}}$. This would surely be the cause of difficulties if one were to find the full matrix $\mathscr{D}$ to maximize the value of $e_{\mathscr{AC}}$. Indeed, for any given configuration, a multitude of matrices $\mathscr{D}$ may result in an efficient sensor configuration, and one could hope to find the best matrix $\mathscr{D}$. But we were never able to improve the robustness $A_c^{min}(e_{min})$ by maximizing $e_{\mathscr{AC}}$.

5.2. SIMILARITY AS AN INDICATOR OF ROBUSTNESS

The remarks made at the end of sections 4.3 and 5.1.3 are at the origin of the search for another indicator of robustness.

It has been mentioned that a matrix $\mathscr{A}_c$ is associated with each sensor configuration. Matrix $\mathscr{A}_c$ is built with weighting matrix $\mathscr{D}$, the role of which is to keep with N_c sensors the efficiency reached with N sensors, concerning the attenuation of a reference primary field. Equation (6) then gives the attenuation $A_c(\mathbf{p})$ from which the robustness $A_c^{min}(e_{min})$ is deduced. Note that if $\mathscr{A}_c \equiv \mathbf{0}$, then $A_c(\mathbf{p}) = A(\mathbf{p})$ and the robustness is $A_c^{min}(e_{min})$, the highest that can be obtained. Is it possible to reduce the importance of $\mathscr{A}_c$ by playing with $\mathscr{D}$ to increase the robustness $A_c^{min}(e_{min})$? The indicator δ, called here of similarity, stems from this idea.

5.2.1. *The indicator of similarity*

For each control microphone configuration, it has been shown that one can consider either one single diagonal matrix or an infinity of full matrices $\mathscr{D}$. For diagonal $\mathscr{D}$, it has been shown that $A_c^{min}(e_{min})$ does not always depend on $e_{\mathscr{AC}}$ in a monotonic manner and it will probably be the same for full $\mathscr{D}$. This is most certainly the reason why the attempt for maximize $e_{\mathscr{AC}}$, thanks to the possible choice of full $\mathscr{D}$, failed. To complete the investigation about robustness, it was thus necessary to find an indicator, the minimization or maximization of which could lead to maximizing $A_c^{min}(e_{min})$. The indicator found consists of the distance δ between two matrices, expressed by

$$\delta = \sum_{ij} |[\mathscr{A} - \mathbf{G}.\mathbf{H}_c^{-1}.\mathbf{G}_c^*.\mathscr{D}.\mathscr{P}]_{ij}|, \tag{12}$$

distance which reveals the degree of similarity between the matrices $\mathscr{A}$ and $\mathscr{F} = \mathbf{G}.\mathbf{H}_c^{-1}.\mathbf{G}_c^*.\mathscr{D}.\mathscr{P}$. By realizing that $(\mathscr{A} - \mathscr{F})^*(\mathscr{A} - \mathscr{F}) = \mathscr{A}_c$, it is clear that $\mathscr{A}_c$ is totally related to δ (see the definition of $\mathscr{A}_c$ in section 3.1.)

At this level it is worth noting that the choice of a robust configuration for diagonal weighting in the purely numerical tests by means of the greater $e_{\mathscr{AC}}$ or the smaller distance δ, has always been the same. But, in contrast to the first indicator, the second made it possible to reach the greatest robustness after optimization. As has been suggested above, this success most likely stems from the fact that the minimum attenuation increases as the distance decreases: i.e., there is a character of monotonicity between both functions of $\mathscr{D}$.

5.2.2. *Natural method to optimize a full weighting matrix and achieve robustness; numerical tests*

With N sensors, the optimal control $\boldsymbol{\phi}_n$ that reduces at best the level associated to $\mathbf{p}_n$ minimizes $J_{res}(\boldsymbol{\phi}) = \|\mathbf{G}.\boldsymbol{\phi} + \mathbf{p}_n\|^2$ and $\boldsymbol{\phi}_n = -\mathbf{H}^{-1}.\mathbf{G}^*.\mathbf{p}_n$. With N_c sensors taken from among the N, the matrix $\mathscr{D}$ has been introduced in order that the optimal control $\mathbf{v}_n$ that reduces the level associated to $\mathscr{D}$. $\mathbf{p}_n^c$ be equal to $\boldsymbol{\phi}_n$. The control $\mathbf{v}_n$ minimizes $\tilde{J}_{res}(\mathbf{v}) = \|\mathbf{G}_c.\mathbf{v} + \mathscr{D}.\mathbf{p}_n^c\|^2$ and $\mathbf{v}_n = -\mathbf{H}_c^{-1}.\mathbf{G}_c^*.\mathscr{D}.\mathbf{p}_n^c$. The equality $\mathbf{v}_n = \boldsymbol{\phi}_n$ leads to

$$\mathbf{G}_c^*.\mathscr{D}.\mathbf{p}_n^c = \mathbf{H}_c.\mathbf{H}^{-1}.\mathbf{G}^*.\mathbf{p}_n. \tag{13}$$

When it was required that $\tilde{J}_{res}^{min}$ be zero, one obtained equation (4) which is a particular case of this more general equation (13).

There are $N_c \times N_c$ unknowns $(\mathscr{D})_{ik}$ while the system has only N_s ($N_s \ll N_c \times N_c$) equations. To obtain $\mathscr{D}$, an arbitrary choice of $N_c \times N_c - N_s$ elements with arbitrary values is made and the other N_s elements with their values are deduced to satisfy system (13). To this end, the system is now written (cf. Appendix C) $\mathbf{Fd} = \mathbf{b}$ where matrix $\mathbf{F}$ has the dimensions $N_s \times N_s$ and the vectors $\mathbf{d}$ and $\mathbf{b}$ the dimensions $N_s \times 1$. Provided matrix $\mathbf{F}$ is non-singular, the other N_s elements of $\mathbf{d}$ satisfy the matrix equation

$$\mathbf{d} = \mathbf{F}^{-1}\mathbf{b}. \tag{14}$$

The aim is now to increase the robustness of the sensor configuration associated with its weighting matrix by the help of a genetic algorithm, the main lines of which are given in Appendix A.

The individuals of the considered population are square matrices $\mathscr{D}$ satisfying equation (13). To this end, for an initial population, random values are given to $((N_c \times N_c) - N_s)$ random elements and the N_s last values are deduced via equation (14). The next generations have a population of crossed and/or mutated individuals with, at most, $((N_c \times N_c) - N_s)$ chromosomes, N_s elements being kept to satisfy constraint (13). The adaptability of an individual to its environment is evaluated by the expression $\sum_{ij}|[\mathscr{A} - \mathbf{G}.\mathbf{H}_c^{-1}.\mathbf{G}_c^*.\mathscr{D}.\mathscr{P}]_{ij}|$: the smaller this value, the better the adaptability.

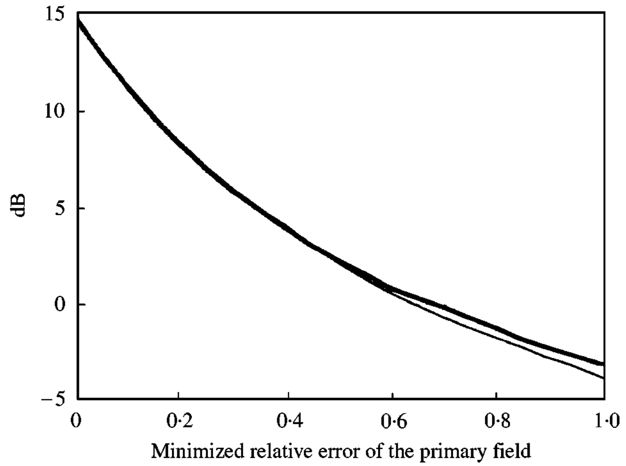

Figure 12. Minimum attenuation before and after optimization of the weighting matrix. The improvement is almost zero in this case where the diagonal matrix associated with the sensor configuration had previously given good robustness. ▬▬, Weighting matrix improved by genetic algorithm; ——, Diagonal weighting matrix.

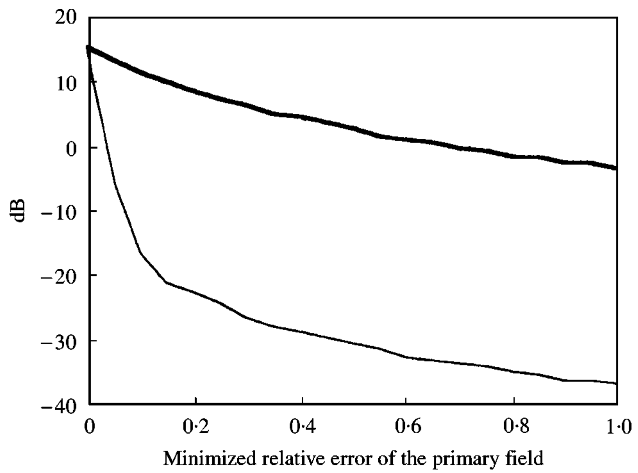

Figure 13. Minimum attenuation before and after optimization of the weighting matrix. The improvement is significant in this case where the diagonal matrix associated with the sensor configuration had previously given poor robustness: ▬▬, Weighting matrix improved by genetic algorithm; ——, Diagonal weighting matrix.

For the numerical tests, the control set is made up of three sensors in charge of controlling the pressure at six microphones. From among the $C_6^3 = 20$ possible sensor configurations, those selected were those where the weighting matrices were single and diagonal for each configuration, and those for which the greatest and weakest robustness indicators $e_{\mathscr{A}\mathscr{C}}$ were obtained. These were then submitted to the optimization procedure to try to increase the similarity between the two matrices. The optimization process takes about 5 h here. At the end of the procedure, the full weighting matrix obtained is an input for the calculation of the minimum attenuation. Figures 12 and 13, respectively, show the curves of minimum attenuation for the initially robust and non-robust sensor configurations. The improvement is quasi-zero when the configuration is already robust (Figure 12). At the other extreme, for an initially poor configuration, as far as robustness is

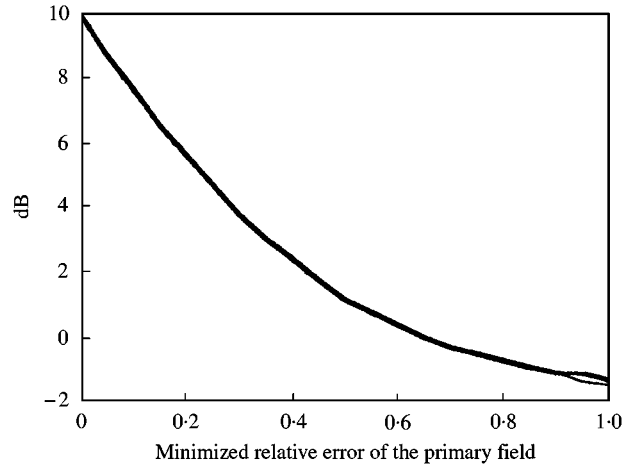

Figure 14. As Figure 12 for another already robust control microphone configuration indentified with its first indicator when the weighting matrix is diagonal: ——, Weighting matrix improved by genetic algorithm; —, Diagonal weighting matrix.

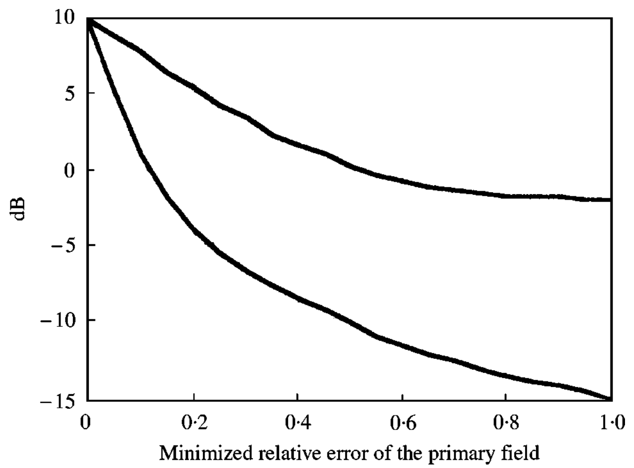

Figure 15. As Figure 13 for another control microphone configuration indentified as poorly robust through its first indicator when the weighting matrix is diagonal: ——, Weighting matrix improved by genetic algorithm; —, Diagonal weighting matrix.

concerned, the improvement is spectacular. Furthermore, the new robustness acquired is of the same order of magnitude as that of the configuration which was initially the best. These facts have always been observed and Figures 14 and 15, resulting from other cases, are given as further evidence to convince the reader.

6. CONCLUSION

Within the framework of adaptive active reduction of harmonic sound fields, the paper presented has focused on efficiency *and* robustness of sensor configurations, the robustness being against spatial perturbations of the primary field.

Concerning efficiency, the work has demonstrated that a limited number of control microphones can be transformed to achieve a result similar to the minimization of sound fields at a larger number of monitoring microphones. The transformation consists in filtering, or weighting, the primary field at the small number of control microphone locations. Such weighting applied to each sensor individually, called 'diagonal' weighting, can be unique. On the contrary, it is possible to use a full matrix which acts on all the sensors and couples all the primary pressures. There are an infinite number of such filters capable of making the sensor configuration efficient. From the informative point of view, the weighting procedure does not bring any new information as it simply transforms information used as inputs. But the weighting is at the center of determining or improving the robustness of sensor configurations.

As to robustness, its definition is chosen from among various possible concepts. Robustness is, here, the minimum attenuation $A_c^{min}(e_{min})$ guaranteed by a sensor configuration when the original, or 'reference' primary field, fluctuates. Given the relative error between the perturbed primary and reference fields, the higher the minimum attenuation, the more robust the sensor configuration. The minimum attenuation may be obtained either by an exhaustive method or, better, by using a natural algorithm to solve an optimization problem. The latter has been written without any type of constraint concerning the sound pressure fields involved. In this case, purely numerical tests lead to a good verification of the computed minimum attenuation. For sound fields stemming from experiments, they inevitably satisfy the boundary conditions present in the experiment and the minimum attenuation will be more precisely approached by exhaustive means. It turns out that the quantitative value of the definition is not always easily obtained, at least until now, and more accessible indicators of robustness have to be found. Two have been conceived from the starting point of other concepts of robustness.

A first indicator called $e_{\mathscr{AC}}$ stems from sufficient conditions only. A sensor configuration, the primary pressures of which are weighted with a unique diagonal matrix associated with an indicator of high value, is designated by the experiments as almost really robust. But below the high values of $e_{\mathscr{AC}}$, one cannot say that the higher the value, the more robust the configuration. This type of drawback is probably at the origin of the lack of success in trying to improve the robustness by increasing the value of the indicator $e_{\mathscr{AC}}$ thanks to optimization of a full weighting matrix $\mathscr{D}$, while still keeping the efficiency. One is thus informed at this level, that among various sensor configurations weighted each by a unique diagonal matrix, if one presents a high indicator value (of possible values), it is likely to be worth selecting. However, in one and only one of the numerous experimental tests carried out, the highest indicator $e_{\mathscr{AC}}$ was not accompanied by the highest curve of minimum attenuation $A_c^{min}(e_{min})$. In view of this, it may be wise, insofar as no rigorous way has yet demonstrated that the highest value of robustness indicator occurs simultaneolusly with the highest minimum attenuation, to ensure that optimizing the weighting matrix $\mathscr{D}$ cannot improve the robustness obtained.

Contrary to indicator $e_{\mathscr{AC}}$, the similarity has proved to be successful in optimizing a full matrix $\mathscr{D}$ in order to improve robustness. Indeed, the minimization of the distance between two matrices, one representing the situation with N sensors, another the configuration with N_c sensors ($N_c < N$), results in the optimal matrix $\mathscr{D}$ that leads to the best robustness. Besides, an already robust configuration with its own diagonal weighting is not improved significantly by optimizing a full matrix, while a non-robust configuration with diagonal weighting becomes much more robust by optimizing a full matrix. The fact that, very often, it has been possible with a full matrix to achieve robustness of the same order of magnitude as the best observed in the case of a diagonal weighting matrix, could suggest that all configurations are potentially almost equally robust.

This research consisted in finding an approach to address robustness of a sensor configuration, robustness against spatial perturbations of the primary field. The main theoretical, or technical, progress made is probably in determining the weighting matrix which enables the improvement of robustness by working on similarity, while keeping efficiency. From the practical point of view, there now exists a way of choosing or modifying a sensor configuration to make it both efficient and robust.

Regarding the physical interpretation of what has been found, some possible directions have been given. Matrix $\mathscr{D}$ is related to the objective that can actually be reached with the secondary sources. Minimum attenuation $A_c^{min}(e_{min})$ defines quite naturally the robustness. These curves show a value e_{min}^{max} beyond which one cannot guarantee the absence of amplification for some of the perturbed primary fields. The first indicator $e_{\mathscr{A}\mathscr{C}}$ is intuitively related to this property. As for the minimization of the second indicator δ, it results in the configuration with N_c sensors having a behavior similar to that of the configuration with N sensors, whatever the primary field, leading to good robustness.

Nevertheless, the work done has its limitations, the major one lying probably in the fact that the fluctuations of pressure around the field of reference are without constraints while, in cavities for example, they have to satisfy the boundary conditions.

It is worthwhile to emphasize the capacity of the method that optimizes the efficient weighting matrix to obtain a robust sensor configuration. Would this method still be helpful when dealing with other types of errors, for example those arising from transfer functions? According to the results obtained so far from the efforts made in this direction, it seems probable.

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APPENDIX A: MAIN FEATURES OF GENETIC ALGORITHMS

Genetic algorithms follow the laws of Darwin on the evolution of the species: individuals in a species who are best adapted to their surroundings have the greatest chances of survival and, thus, of reproducing themselves. As time goes by, the species increases its strength with regard to its environment. Reproduction consists in crossing and/or mutating the chromosomes of the parents. The construction of genetic algorithm follows the general pattern: (a) create an initial population which satisfies the constraints; (b) classify the individuals according to their value; (c) select the individuals who are going to play the role of parents by a procedure akin to the one known as the "roulette method"; (d) cross and/or mutate the parents' chromosomes and in so doing, create the new generation which must also satisfy the constraints; (e) classify the individuals of the new generation according to their value; (f) if the quality sought is sufficient, stop the algorithm or, if not, return to step (c).

APPENDIX B: DETAILS OF CALCULATION OF THE INDICATOR $e_{\mathscr{AC}}$

Knowing that $\boldsymbol{\phi}(\mathbf{p}) = -\mathbf{H}^{-1}.\mathbf{G}^*.\mathbf{p}$ (see section 2.1), it follows that $\delta\boldsymbol{\phi} = -\mathbf{H}^{-1}.\mathbf{G}^*.\delta\mathbf{p}$. One also has $\|\delta\boldsymbol{\phi}\|_{\mathrm{H}} = (\delta\mathbf{p}^*.\mathscr{A}.\delta\mathbf{p})^{1/2} \leqslant \|\delta\mathbf{p}\| = e\|\mathbf{p}_n\|$ where $\mathscr{A}$ has been defined in equation (2).

The definition of R_n leads to the equality $\|\boldsymbol{\phi}_n\| = \|\mathbf{p}_n\|.\sqrt{R_n}$ resulting in $\|\delta\boldsymbol{\phi}\| \leqslant (e/\sqrt{R_n}).\|\boldsymbol{\phi}_n\|$: i.e., with geometrical terms in the control space $\boldsymbol{\phi} \in \mathscr{B}_e$ where $\mathscr{B}_e$ is the bowl centered at $\boldsymbol{\phi}_n$, the radius of which is $(e/\sqrt{R_n}).\|\boldsymbol{\phi}_n\|$.

Besides, it is known that $\|\delta\boldsymbol{\phi}\|$ must remain less than $\|\boldsymbol{\phi}_n\|$ to prevent amplification of $J(\mathbf{p}_n)$. According to the latter information, a sufficient condition to avoid amplification of $J(\mathbf{p}_n)$ is $(e/\sqrt{R_n}).\|\boldsymbol{\phi}_n\| \leqslant \|\boldsymbol{\phi}_n\|$: that is, $e \leqslant \sqrt{R_n}$.

Thus the equality $\mathbf{p} = \mathbf{p}_n + \delta\mathbf{p}$ with $\|\delta\mathbf{p}\|/\|\mathbf{p}\| \leqslant e$ leads to (only in that direction) $\boldsymbol{\phi} \in \mathscr{B}_e \subset \mathscr{B}_n$ where $\mathscr{B}_n$ is the bowl centered at $\boldsymbol{\phi}_n$, of radius $\|\boldsymbol{\phi}_n\|$. This ends the first step.

In the presence of a sensor configuration of N_c microphones, to the primary field vector $\mathbf{p}$ with its N components corresponds the vector $\mathbf{p}_c$ with its N_c components where $N_c < N$. For this primary field $\mathbf{p}_c$, the configuration with its N_c control sensors provides the control $\mathbf{v}(\mathbf{p})$.

As $\boldsymbol{\phi}(\mathbf{p})$ is the optimal control to attenuate $J(\mathbf{p})$ at the N points, $\mathbf{v}(\mathbf{p})$ does not amplify $J(\mathbf{p})$ if $\mathbf{v} \in \mathscr{B}$ centered at $\boldsymbol{\phi}$ with a radius $\|\boldsymbol{\phi}\|$; in other words if $\|\mathbf{v} - \boldsymbol{\phi}\| \leqslant \|\boldsymbol{\phi}\|$.

But $\|\mathbf{v} - \boldsymbol{\phi}\| \leqslant \|\mathbf{v} - \boldsymbol{\phi}_n\| + \|\boldsymbol{\phi}_n - \boldsymbol{\phi}\| \leqslant \|\mathbf{v} - \boldsymbol{\phi}_n\| + (e/\sqrt{R_n})\,\|\boldsymbol{\phi}_n\|$. In Figure 6, it is noticeable that, at the end of this second step, nothing prevents $\mathbf{v}(\mathbf{p})$ from amplifying $J(\mathbf{p}_n)$.

The geometry of Figure 6 shows that there exists a control of $\mathscr{B}_e$, the norm of which is minimal and of value $\|\boldsymbol{\phi}\|_{min} = (1 - (e/\sqrt{R_n})\,\|\boldsymbol{\phi}_n\|)$. It tells one that $\forall\boldsymbol{\phi} \in \mathscr{B}_e$, if $\|\mathbf{v} - \boldsymbol{\phi}\| \leqslant \|\boldsymbol{\phi}\|_{min}$ then $\mathbf{v}(\mathbf{p})$ does not amplify $J(\mathbf{p}_n)$.

The fourth step consists of the chain of conditions sufficient to satisfy the initial objectives: $\|\mathbf{v} - \boldsymbol{\phi}\| \leqslant \|\mathbf{v} - \boldsymbol{\phi}_n\| + (e/\sqrt{R_n})\,\|\boldsymbol{\phi}_n\| \leqslant \|\boldsymbol{\phi}\|_{min}$ or $\|\mathbf{v} - \boldsymbol{\phi}_n\| \leqslant \{1 - (2e/\sqrt{R_n})\}\,\|\boldsymbol{\phi}_n\|$ leading now to $e \leqslant \sqrt{R_n}/2$.

But $\|\mathbf{v} - \boldsymbol{\phi}_n\|^2 = (\mathbf{v} - \boldsymbol{\phi}_n)^*.\mathbf{H}.(\mathbf{v} - \boldsymbol{\phi}_n)$ and $(\mathbf{v} - \boldsymbol{\phi}_n) = -\mathbf{H}_c^{-1}.\mathbf{G}_c^*.\mathscr{D}.\mathscr{P}.\mathbf{p} - \mathbf{H}^{-1}.\mathbf{G}^*.\mathbf{p}_n$. It is also known that $\boldsymbol{\phi}_n = -\mathbf{H}^{-1}.\mathbf{G}^*.\mathbf{p}_n = \mathbf{v}_n = -\mathbf{H}_c^{-1}.\mathbf{G}_c^*.\mathscr{D}.\mathscr{P}.\mathbf{p}_n$ (see section 2.1 and equation (5)) resulting in $(\mathbf{v} - \boldsymbol{\phi}_n) = -\mathbf{H}_c^{-1}.\mathbf{G}_c^*.\mathscr{D}.\mathscr{P}.(\mathbf{p} - \mathbf{p}_n)$. By writing $\mathscr{C} = \mathscr{P}^*.\mathscr{D}^*.\mathbf{G}_c.\mathbf{H}_c^{-1,*}.\mathbf{H}.\mathbf{H}_c^{-1}.\mathbf{G}_c^*.\mathscr{D}.\mathscr{P}$, it follows that $\|\mathbf{v} - \boldsymbol{\phi}_n\| = \sqrt{(\mathbf{p} - \mathbf{p}_n)^*.\mathscr{C}.(\mathbf{p} - \mathbf{p}_n)} \leqslant \{1 - (2e/\sqrt{R_n})\}\,\|\boldsymbol{\phi}_n\|$.

Another sufficient condition leads to $\|\delta\mathbf{p}\|\sqrt{\|\mathscr{C}\|} \leqslant \{1 - (2e/\sqrt{R_n})\}\|\boldsymbol{\phi}_n\|$ making it possible, at the end of this fifth step, to obtain finally, with the help of $\|\boldsymbol{\phi}_n\| = \|\mathbf{p}_n\|.\sqrt{R_n}$:

$$\frac{\|\delta\mathbf{p}\|}{\|\mathbf{p}_n\|} \leqslant \frac{\sqrt{R_n}}{\sqrt{\|\mathscr{C}\|} + 2} = e_{\mathscr{AC}}.$$

APPENDIX C: DETAILS OF CALCULATION TO OBTAIN EQUATION (14)

System (13) of N_s equations is more precisely written as

$$\sum_{i=1}^{N_c} (\mathbf{G}_c^*)_{ji} \sum_{k=1}^{N_c} (\mathscr{D})_{ik} (\mathbf{p}_n^c)_k = (\mathbf{H}_c.\mathbf{H}^{-1}.\mathbf{G}^*.\mathbf{p}_n)_j \quad \text{for } j = 1, N_s,$$

or

$$\sum_{i=1}^{N_c} \sum_{k=1}^{N_c} (\mathbf{G}_c^*)_{ji} (\mathbf{p}_n^c)_k (\mathscr{D})_{ik} = (\mathbf{H}_c.\mathbf{H}^{-1}.\mathbf{G}^*.\mathbf{p}_n)_j \quad \text{for } j = 1, N_s$$

where there are $N_c \times N_c$ unknowns $(\mathscr{D})_{ik}$ while the system has only $N_s (N_s \ll N_c \times N_c)$ equations. To obtain $\mathscr{D}$, an arbitrary choice of $N_c \times N_c - N_s$ elements with arbitrary values is made and the other N_s elements with their values are deduced to satisfy system (13). To this end, the system is now written as

$$\sum_{m=1}^{N_s} (\mathbf{G}_c^*)_{j,i(m)} (\mathbf{p}_n^c)_{k(m)} (\mathscr{D})_{i(m)k(m)} = (\mathbf{H}_c.\mathbf{H}^{-1}.\mathbf{G}^*.\mathbf{p}_n)_j - \sum_{i \neq i(m),\, k \neq k(m)} \sum (\mathbf{G}_c^*)_{ji} (\mathbf{p}_n^c)_k (\mathscr{D})_{ik}$$

$$\text{for } j = 1, N_s.$$

On the left-hand side, there are the N_s unknowns $(\mathscr{D})_{i(m)k(m)}$ and on the right-hand side one finds all the other $N_c \times N_c - N_s$ arbitrary elements and their values $(\mathscr{D})_{ik}$ with $i \neq i(m)$ and $k \neq k(m)$.

Define $(\mathscr{D})_{i(m)k(m)} = (\mathbf{d})_m$, the mth element of vector $\mathbf{d}$ with N_s components, $(\mathbf{G}_c^*)_{j,i(m)} (\mathbf{p}_n^c)_{k(m)} = (\mathbf{F})_{j,m}$, the ($j$th, mth) element of matrix $\mathbf{F}$ of dimensions $N_s \times N_s$, $(\mathbf{H}_c.\mathbf{H}^{-1}.\mathbf{G}^*.\mathbf{p}_n)_j - \sum_{i \neq i(m)} \sum_{k \neq k(m)} (\mathbf{G}_c^*)_{ji} (\mathbf{p}_n^c)_k (\mathscr{D})_{ik} = (\mathbf{b})_j$, the jth element of vector $\mathbf{b}$ with N_s components.

Provided matrix $\mathbf{F}$ is non-singular, the other N_s elements of $\mathbf{d}$ satisfy the matrix equation

$$\mathbf{d} = \mathbf{F}^{-1}\mathbf{b}. \tag{14}$$